IMO 2012 Problem I
An annotated worksheet for classroom use.

(The problem statement is slightly different from the `official' statement: it is adjusted a bit to American usage)

Given triangle ABC, point J is the excenter (center of the escribed circle) opposite vertex A. This escribed circle is tangent to side BC at M, and to lines AC, AB at K and L respectively. Lines LM and BJ meet at F, and lines KM and CJ meet at G. Lines AF and BC (extended) meet at S, and lines AG and BC (extended) meet at T. Show that M is the midpoint of ST.

Problem 1: How many circles are there which are tangent to three non-concurrent lines?

Problem 2: In a triangle ABC, draw the interior angle bisector at A and also the exterior angle bisector at A. What angle do they form?

Problem 3: ("Ice Cream Cone" theorem): Show that the two tangents to a circle from a point outside are equal.

Problem 4: A circle O is given, with center O, and a point A outside the circle. Tangents AK and AL are drawn to the circle. Show that:

(i) OP bisects angle KAL;
(ii) OP is the perpendicular bisector of KL.
(iii) Triangles OKL, PKL are both isosceles.

Problem 5: Figure (i) shows a triangle ABC. Point J is the center of the escribed circle opposite vertex A.

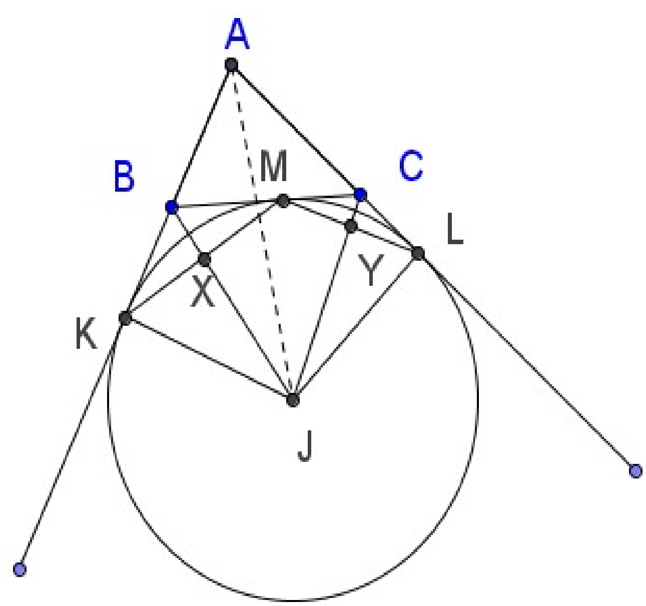

Figure (i)

(i) How may cases of theorem 3 can you find in figure (i)? How many 'ice cream cones'?
(ii) How many cases of the theorems in problem 4 can you find in figure (i)?

Problem 6: Figure (ii) shows the same triangle as before, with interior angle bisectors BI, CI drawn in.

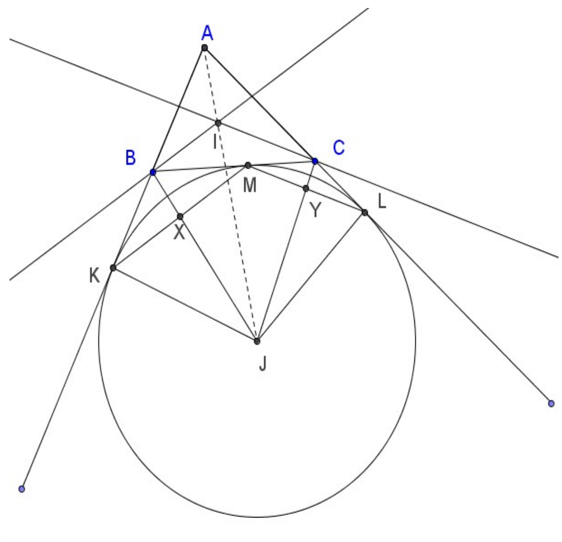

figure (ii)

(a) Why must their intersection I lie on line AJ?
(b) If <ABC = 46 degrees, how big is <BMX?
(c) If <ACB = 36 degrees, how big is <CMY?

In general, what is the relationship between <ABC and <BMX? Between <ACB and <CMY?

7. ("Angle chase") In the original triangle, let <BAC = α, <ABC = β, <BCA = γ. Express the measures of the following angles in terms of α, β, and γ:

a) < BMX	c) <CMY	e) <KBX	g) <YCM	i) <BJC	k) <ALJ
b) <BKX	d) <CLY	f) <MBX	h) <LCM	j) <AKJ	l) <BJC

(Questions (j) and (k) are, in some sense, 'trick questions'. But they will help you to answer (l).)

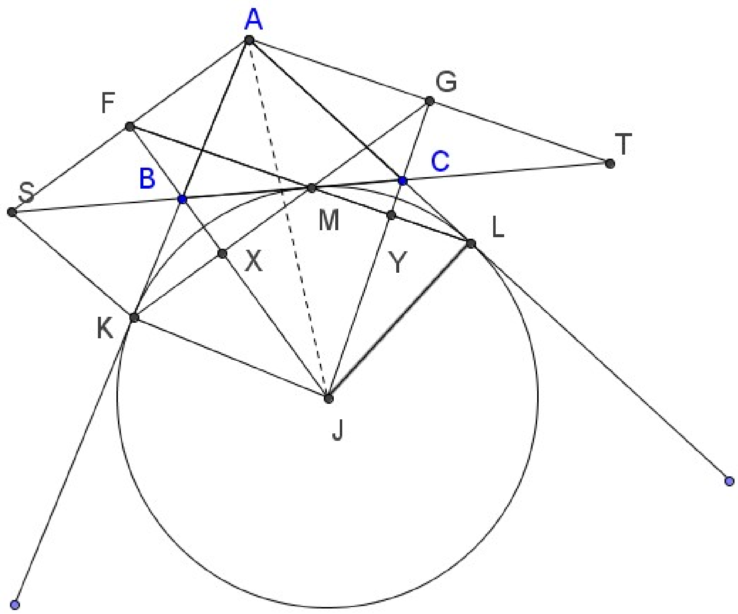

figure (iii)

Figure (iii) shows the same triangle ABC, with the same escribed circle, and a few more lines added. Lines LM and JB are extended to meet at F, and lines KM and JC are extended to meet at G. Then AF and AG are extended to meet line BC (again, extended) at S and T.

We need to prove that SM = MT.

8. (More angle chase): Again, let <BAC = α, <ABC = β, <BCA = γ. Express the measures of the following angles in terms of α, β, and γ:

a) <FBA	c) <ACG	e) <FMX	g)<JFL	h) <GMY	j) <MGY
b) <SBF	d) <TCG	f) <FXM		i) < GYM	

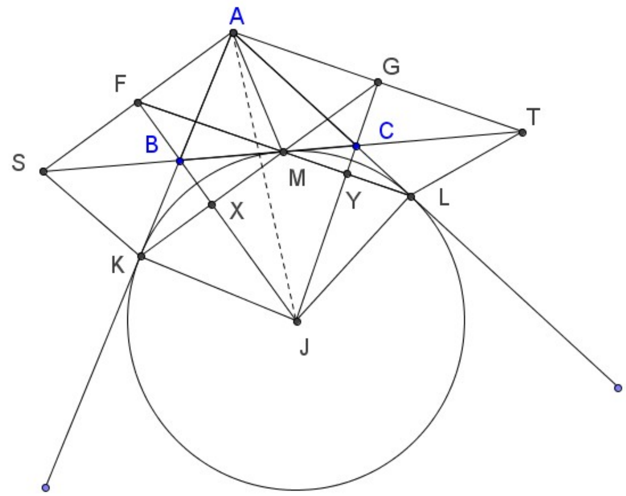

figure (iv)

Figure (iv) is the same as figure (iii), with segments AM, TL added. Rem ember that we need to prove that SM = MT.

9. Before we go ahead, let's work backwards a bit. Show that we can prove that SM = MT....

(a) IF we can show that SB = AB and CT = AC
(b) IF we can show that AS || KM and AT || LM
(c) IF we can show that quadrilaterals ASKM, ATLM are isosceles trapezoids.
(d) IF we can show that triangles SBA, TCA are isosceles
(e) IF we can show that JF is perpendicular to AS and JG is perpendicular to AT.

Can you show any of these, and complete the problem? (If not, look at the next page.)

10. (Completion of the analysis) We can show (e) by looking at cyclic quadrilaterals. (These are quadrilaterals whose four vertices all lie on one circle.

Before we go on, make sure you remember the following:

a) Not every quadrilateral is cyclic; that is, not every set of four points lies on the same circle. (Although any three non-collinear points do lie on a circle.)
b) A quadrilateral is cyclic if and only if its opposite angles are supplementary;
c) An angle inscribed in a semicircle is a right angle; and, in fact, an angle inscribed in a circle is always half its intercepted arc.
d) The converse of the statement in (c) isalso true.
e) A tangent to a circle is perpendicular to the radius at the point of contact.

NOW: Show that quadrilateral AKJL is cyclic. What is the diameter of the circle through A, K, J, and L?

11. Show that point F lies on the circle through A, K, J, and L. (Hint: angle chase.)

12. Show that quadrilateral AFJL is cyclic. What is the diameter of the circle through A, F, J, and L?

13. Show that JF is perpendicular to AS.

14. Write a complete solution of the problem, without including 'extra' information, and without going 'back and forth' in your logic.