Largedeviationsforzerosofrandompolynomials withi.i.d.exponentialcoefficients

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DedicatedtothememoryofWenboLi

Abstract

Wederivealargedeviationprincipleforthe empiricalmeasureofzerosof therandompolynomial $P_n(z)=\sum_{j=0}^{n}\xi_j z^j$,wherethecoefficients $\{\xi_j\}_{j\geq 0}$ form ani.i.d.sequenceofexponentialrandomvariables.

1Introduction

Thestudyofthezeroset $\{z_1,...,z_n\}$ ofrandompolynomials

$$P_n(z)=\sum_{j=0}^{n}\xi_j z^j \tag{1}$$

withi.i.d.coefficients $\{\xi_j\}_{j\geq 0}$ hasalongandrichhistory,whichwewillnotreview here;see[BRS86]foraclassicalaccountand[TV13]forthemostrecentresults.Under mildconditions,theconvergenceoftheempiricalmeasure $L_n=\frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}$ ofzerosof P_n.totheuniformmeasureontheunitcirclegoesbackatleastto[SS62]and[ET50]; scaledversionofthisconvergencecanbefoundin[SV95](fortheGaussiancase)and [IZ95](formoregenerali.i.d.coefficientsinthedomainofattractionofstablelaws).

Weareinterestedinthelargedeviationsfortheempiricalmeasure L_n.Inthecase ofGaussiancoefficients,thishasbeenstudiedbefore[ZZ10],[Be08],[Bl11],exploiting

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methodsrelatedtothoseusedinthestudyofrandommatricesfromtheclassical β-ensembles[BAG97,BAZ98,SS12].Likeinthecaseofrandommatrices,whenone venturesawayfromtheGaussiansetup(withi.i.d.coefficients),notmuchisknown concerninglargedeviations.

Ourgoalinthispaperistoexhibitanewclassofcoefficients,forwhichalarge deviationprincipleforthee mpiricalmeasurecanbeproved,namelytheclassofi.i.d. exponentialcoefficients,whichforconcretenesswenormalizetohaveparameter1.To ourknowledge,thefirsttoconsiderexplicitlyasymptoticsforthisclasswasWenbo Li[Li11],whousedgeneralformulaeofZaporozhets[Za04]inordertocomputethe probabilitythatallrootsinsuchapolynomialarereal.Werelateourresulttoli's computationinTheorem1.2below.

Inordertostateourresults,weintroducesomenotation.Intherestofthepaper, P_n denotesarandompolynomialasin(1),withi.i.d.exponential(ofparameter1) coefficients $\{\xi_i\}$ andassociatedempiricalmeasureofzeros L_n.Foranypolishspace X,let $\mathcal{M}_1(X)$denotethespaceofprobabilitymeasureson X,equippedwiththe topologyofweakconvergence.Letpol_+denotethecollectionofpolynomials(over $\mathbb{C}$)withcoefficientsthatarerealpositive.For $p \in \mathrm{pol}_+$,let $\mu_p \in \mathcal{M}_1(\mathbb{C})$denote theempiricalmeasureofzerosof p.Notethat μ_p dependsonthesetofzerosand notonaparticularlabelingofthezeros,that μ_p issymmetricwithrespecttothe transformation $z \mapsto z^*$,andthat $\mu_p(\mathbb{R}_+)=0$.(Hereandinthesequel,weuse $\mathbb{R}_+$ to denotetheinterval$(0,\infty)$.)

Weintroducetheclosureofthecollectionofempiricalmeasuresofpolynomialswith positivecoefficients

$$\mathcal{P} = \overline{\{\mu_p : p \in \mathrm{pol}_+\}} \subset \mathcal{M}_1(\mathbb{C})\,. \tag{2}$$

Obviously, $L_n \in \mathcal{P}$.

Definition1. *Foranymeasure* $\mu \in \mathcal{M}_1(\mathbb{C})$, *definethelogarithmicpotentialfunction tobe*

$$\mathbb{L}_\mu(z) = \int \log|z-w|\,d\mu(w)$$

andthelogarithmicenergytobe

$$\Sigma(\mu) = \iint \log|z-w|\,\mu(z)\mu(w).$$

Definition2. *Definethefunction* $I : \mathcal{M}_1(\mathbb{C}) \to \mathbb{R}_+$ *by*

$$I(\mu) = \begin{cases} \int \log|1-z|\,d\mu(z) - \frac{1}{2}\iint \log|z-w|\,d\mu(z)d\mu(w), & \text{if } \mu \in \mathcal{P}\,, \\ \infty, & \text{if } \mu \notin \mathcal{P} \end{cases}$$

WewillseeinSection3.1that I iswelldefined(for $\mu \in \mathcal{P}$,astheintegralwith respectto $\mu \times \mu$ ofthefunction $f(z,w)=\log|1-z|+\log|1-w|-\log|z-w|$)and non-negative(thelatterfactisimmediatefromthelowerboundinLemma3.7) [1].

[1] A.Eremenkoshowedusadirectproofofthenon-negativityof I,thatbypassestheuseofthe lowerboundfromLemma3.7.Sinceweneedthelatterlemmaforotherreasons,wedonotreproduce hisproofhere

Our main result concerning large deviations of L_n is the following.

Theorem 1.1. *The random measures L_n satisfy a large deviation principle in the space $\mathcal{M}_1(\mathbb{C})$ with speed n^2 and good rate function I. Explicitly, we have:*

(i) The function $I : \mathcal{M}_1(\mathbb{C}) \to [0, \infty]$ has compact level sets, i.e. the sets $\{\mu : I(\mu) \leq M\}$ are compact subsets of $\mathcal{M}_1(\mathbb{C})$ for each $M \in \mathbb{R}$.

(ii) For each open set $O \subset \mathcal{M}_1(\mathbb{C})$, we have

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in O) \geq - \inf_{\mu \in O} I(\mu).$$

(iii) For each closed set $F \subset \mathcal{M}_1(\mathbb{C})$, we have

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in F) \leq - \inf_{\mu \in F} I(\mu).$$

Comparing the statement of Theorem 1.1 with the main results in [BAZ98] and [ZZ10], one sees that in spite of the fact that we are dealing with zeros of random polynomials, the rate function is closer to a random matrix theory rate function than to the one appearing in the Gaussian case. This is due to the expression for the joint distribution of zeros, see Section 2.1 below. We also note that because I is a good rate function, any minimizer μ of $I(\cdot)$ in $\mathcal{M}_1(\mathbb{C})$ must satisfy that $I(\mu)=0$; in particular, the uniform measure on the unit circle is a minimizer, as one expects from the limit results in [ET50], [SS62]. The strict convexity of I (which follows from the same argument as in [BAG97]) shows that it is the unique minimizer.

As mentioned above, we tie our results to Li's computation in [Li11]. Toward this end, let $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+$ and define $\mu_R \in \mathcal{M}_1(\mathbb{R}_-) \subset \mathcal{M}_1(\mathbb{C})$ to be such that $I(\mu_R) = \inf_{\mu \in \mathcal{M}_1(\mathbb{R}_-)} I(\mu) =: I_R$. (Such a minimizer exists due to the lower semicontinuity of I.)

Theorem 1.2. *Conditioned on $L_n \in \mathcal{M}_1(\mathbb{R}_-)$, the sequence of random empirical measures L_n satisfy the large deviation principle in $\mathcal{M}_1(\mathbb{R}_-)$ with speed n^2 and rate function $I_R(\mu) = I(\mu) - I_R$. In particular, conditioned on $L_n \in M_1(\mathbb{R})$, the sequence L_n converges weakly to $\mu_R \in M_1(\mathbb{R}_-)$.*

A characterization of μ_R is given in the next theorem, due to J. Baik.

Theorem 1.3. *The minimizer μ_R has density with respect of Lebesgue measure on $\mathbb{R}_-$ equal to*

$$\phi(x) = \frac{1}{\pi(|x| + 1)\sqrt{|x|}} \mathbf{1}_{\{x<0\}}. \tag{3}$$

Aninterestingfeatureoftheminimizer μ_R isthatitisnotcompactlysupported.We discussTheorems1.2and1.3inSection6.

HistoryandAcknowledgements: Ourinterestinthisproblemstartedwhenone ofus(O.Z.)attendedatalkbyWenboLion[Li11];thattalksuggestedthatan underlyinglargedeviationprincipleshouldexistintherealcase,andJ.Baikcomputed itsequilibriummeasure,repeatedhereasTheorem1.3.WenboLi'suntimelydeath promptedS.G.andO.Z.torevisittheproblem,andtheimportantroleoftheclass $\mathcal{P}$ inthecomplexcaseemerged.Wepostedthequestionconcerningthecharacterization of $\mathcal{P}$ onMathOverflow[MO13],andthequestionwasansweredin[BES13].

WeareindebtedtoJ.Baikforallowingustousehisproofof Theorem1.3,and toA.Eremenkoformaking[BES13]availabletousasapreprint,forhispatiencein answeringourquestions,andforhiscommentsonapreliminarydraftofthispaper.

2Preliminaries

Wediscussinthissectionseveralpreliminaries.Wefirstintroducethejointdistribu- tionofzerosandthenwedescribepropertiesof $\mathcal{P}$.

2.1Thejointdistributionofzeros

Let $p \in \mathrm{pol}_+$ beofdegree n with $n-2k$ realzeros, $k = 0, 1, \ldots, \lfloor n/2 \rfloor$.Weconsiderthe zerosof p asavector$(z_1, \cdots, z_n)$withtheconventionthat $z_1, \cdots, z_k$ arethenon-real zeroswithpositiveimaginarypart, $z_{k+1} = \overline{z}_1, \cdots, z_{2k} = \overline{z}_k$ and $z_{2k+1}, \cdots, z_n$ denote the $n-2k$ realzeros.Inthisnotation,for k fixed,asetofzerosisgenericallymapped to $k!(n-2k)!$distinctpointsin $A_{n,k}^+ = \mathbb{C}_+^k \times \mathbb{C}_-^k \times \mathbb{R}^{n-2k}$,and $A_{n,k}^+$ isparametrized by $\mathbb{C}_+^k \times \mathbb{R}^{n-2k}$.

Performingthechangeofvariablesfrom$(\xi_0, \ldots, \xi_n)$to$(z_1, \ldots, z_k, z_{n-2k+1}, \ldots, z_n, \xi_n)$, countingmultiplicities,usingtheformoftheexponentialdensityandintegratingover thedensityof ξ_n (see[Za04]forasimilarcomputation),onehasthattherandom polynomial P_n inducesthefollowingmeasureon $\mathbb{C}^n$:

$$d\mathbb{P}_n(z_1, \cdots, z_n) = \tag{4}$$
$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^k}{k!(n-2k)!} \frac{\prod_{1 \le i < j \le n} |z_i - z_j|}{\prod_{j=1}^n |1 - z_j|^{n+1}} 1_{B_{n,k}}(z_1, \cdots, z_n) d\mathcal{L}(z_1) \cdots d\mathcal{L}(z_k) d\ell(z_{2k+1}) \cdots d\ell(z_n).$$

Here $\mathcal{L}$ istheLebesguemeasureon $\mathbb{C}$, ℓ istheLebesguemeasureon $\mathbb{R}$,and $B_{n,k}$ consistsofthe n-tuples$(z_1, \ldots, z_n) \subset A_{n,k}$ thatcanbeobtainedasthezeroset(with

$n-2k$ realzeros)ofapolynomialofdegree n withpositivecoefficients.Inparticular, letting $A_{n,k}=(\mathbb{C}\setminus\mathbb{R})^{2k}\times\mathbb{R}^{n-2k}\subset\mathbb{C}^n$,weseethatthedensityof $\mathbb{P}_n$ onanyfixed $A_{n,k}$ is

$$\frac{1}{\mathcal{Z}_{n,k}}1_{B_{n,k}}(z_1,\cdots,z_n)\exp\left(\sum_{1\le i<j\le n}\log|z_i-z_j|-(n+1)\sum_{j=1}^{n}\log|1-z_j|\right). \quad (5)$$

whereretheconstants $\mathcal{Z}_{n,k}$ satisfythat

$$\lim_{n\to\infty}\frac{1}{n^2}\log\max_{k=1}^{\lfloor n/2\rfloor}\mathcal{Z}_{n,k}=\lim_{n\to\infty}\frac{1}{n^2}\log\min_{k=1}^{\lfloor n/2\rfloor}\mathcal{Z}_{n,k}=0\ . \quad (6)$$

Therepresentation(5)with(6)isparticularlysuitedforLDPanalysis.

2.2Propertiesoftheclass $\mathcal{P}$ ofmeasures

Obviously,forany $p\in\mathrm{pol}_p$ with μ_p itsempiricalmeasureofzeros,wehavethat $\mu_p(\mathbb{R}_+)$=0.However,thatpropertyisnotpreservedbyweakconvergence,andhence a-prioriitisnotclearthatallmeasuresin $\mathcal{P}$ satisfyit(althoughwewillsee,asa consequenceofObrechkoff'stheorembelow,thatinfacttheydo).Inthissubsection, wediscussthisandotherpropertiesoftheclass $\mathcal{P}$.

2.2.1Obrechkoff'sTheorem

Astartingpointforthedescriptionof $\mathcal{P}$ isthefollowingclassicaltheorem.

Theorem2.1 (Obrechkoff). *Let $p\in\mathrm{pol}_+$ andlet*

$$C_\alpha=\{z\in\mathbb{C}:|\arg z|\le\alpha\}$$

denotethesymmetric(aroundthepositiverealline)conein $\mathbb{C}$ withapexattheorigin andangle 2α. Then, $\mu_p(C_\alpha)\le 2\alpha/\pi$.

Theproof,givenin[Ob23],usestheargumentprinciple.Forourneeds,notethat Obrechkoff'sTheoremimpliesthat $\mu(C_\alpha)\le 2\alpha/\pi$ forany $\mu\in\mathcal{P}$ .Inparticular, $\mu(\mathbb{R}_+)$=0forsuch μ.

Obrechkoff'sTheoremleadstothefollowinglemmaontheintegrabilityofthelogarithmnear1for $\mu\in\mathcal{P}$.

Lemma2.2. *Let $M>0$ andset $A_M=\{z:\log|1-z|\le -M\}$. Thenthereisa positivequantity $C(M)$ satisfying $\lim_{M\to\infty}C(M)=0$ suchthatforany $\mu\in\mathcal{P}$,*

$$\max\{\mu(A_M),\int_{A_M}|\log|1-z||\,d\mu(z)\}\le C(M). \quad (7)$$

Proof. We first consider $p \in \mathrm{pol}_+$ of degree N. For $M > 0$, let Z_M consists of all zeros z_i of p such that $\log|1 - z_i| < -M$. Let $N(p,M)$ be the cardinality of Z_M and $S(p,M) = \sum_{i:z_i \in Z_M} \log|1 - z_i|$. By Lemma 2.1, there exists a constant M_0 independent of p or N such that for $M > M_0$, $|Z_M| < 2e^{-M}N$. Thus, with $B_j := \{z : -(j+1) < \log|1 - z| \leq -j : j \geq M\}$, we get

$$\frac{1}{N}|S(p,M)| \leq \frac{1}{N} \sum_{j=M}^{\infty} \sum_{i:z_i \in B_j} |\log|1 - z_i|| \leq \sum_{j=M}^{\infty} j \cdot 4e^{-j} =: c(M)\,,$$

with $c(M) \to_{M\to\infty} 0$. Since $|S(p,M)| \geq N(p,M)$ if M_0 is chosen large enough, we obtain the same inequality for $N(p,M)/N$. Thus, (7) holds for μ_p, uniformly in p, N.

To obtain the same inequality for $\mu \in \mathcal{P}$, take an approximating sequence $\mu_{p_n} \to \mu$, and use that $\mu(A_M) \leq \limsup_{n\to\infty} \mu_{p_n}(A_{M-1})$ together with

$$\int_{A_M} |\log|1 - z| \vee -K| d\mu(z) \leq \limsup_{n\to\infty} \int_{A_{M-1}} |\log|1 - z| \vee -K| d\mu_{p_n}(z) \leq c(M-1)\,,$$

and then apply monotone convergence over K. One concludes that (7) holds with $C(M) = c(M-1)$. ∎

2.2.2 The Bergweiler-Eremenko-Sokal Theorem

For $\mu \in \mathcal{M}_1(\mathbb{C})$, let

$$\widehat{\mathbb{L}}_\mu(z) = \int_{|z-w|\leq 1} \log(|z - w|) d\mu(w) + \int_{|w|>1} \log(|1 - \frac{z}{w}|) d\mu(w)\,.$$

Whenever $C_\mu := \int \log_+ |w| \mu(dw) < \infty$, it holds that

$$\widehat{\mathbb{L}}_\mu(z) = \mathbb{L}_\mu(z) - C_\mu\,.$$

In a recent work [BES13], Bergweiler, Eremenko and Sokal proved the following.

Theorem 2.3 (Bergweiler-Eremenko-Sokal). *$\mu \in \mathcal{P}$ if and only if it is invariant with respect to conjugation and satisfies $\widehat{\mathbb{L}}_\mu(z) \leq \widehat{\mathbb{L}}_\mu(|z|)$ for all $z \in \mathbb{C}$.*

In the proof of Theorem 1.1, we will exploit this result, and in addition, its proof.

3 Proof of Theorem 1.1

We prove in this section Theorem 1.1. The proof is divided into sections. We first study, in section 3.1, properties of I and establish that it is well defined and lower-semicontinuous. In Section 3.2, we prove the exponential tightness of $\{\mathbb{P}_n\}$. Section 3.3 is devoted to the proof of the upper bound. Finally, Section 3.4 states and proves Lemma 3.7, which is the lower bound; the proof of Lemma 3.7 uses some some technical approximation lemmas whose proofs are postponed to Sections 4 and 5.

3.1 I iswelldefinedandhascompactlevelsets.

Weprovethefollowing.

Lemma3.1. *Thefunction I iswelldefinedon $\mathcal{M}_1(\mathbb{C})$ anditpossessescompactlevel sets.*

Lemma3.1almostshowsthat I isagoodratefunction;whatismissingisaproof that $I(\mu) \geq 0$for $\mu \in \mathcal{M}_1(\mathbb{C})$.ThisfactisaconsequenceofLemma3.7below.

Proof. Define $f(z,w)=\log |1-z| + \log |1-w| - \log|z-w|$.Wefirstshowthatone canchooseafunction $K(L) \to_{L\to\infty} \infty$ sothatthefollowinginclusionholdsforall L large:

$$\{(z,w): |z|>L, |w|>L\} \subset \{ f(z,w) \geq K(L)\}. \tag{8}$$

Indeed,setting $z' = 1 - z$ and $w' = 1 - w$,weget

$$f(z,w)=\log |z'| + \log |w'| - \log|z'-w'|.$$

But

$$\frac{|z'w'|}{|z'-w'|} \geq \frac{1}{\frac{1}{|z'|}+\frac{1}{|w'|}} \geq \frac{1}{2}\min\{|z'|,|w'|\}.$$

Clearly,thisimplies(8).Further,thelastinequalityalsoimpliesthat,with $A = \{(z,w) \in \mathbb{C}^2 : |1-z|>1/4, |1-w|>1/4\}^{\complement}$,

$$\inf_{A^{\complement}} f(z,w) \geq \frac{1}{8}. \tag{9}$$

Wenextshowthat $I(\mu)$iswelldefined.Forthatitisenoughtoconsider $\mu \in \mathcal{P}$. Since $f(z,w) \geq c + \log\min(|z-1|, |w-1|)$forsomeconstant C,anapplicationof Lemma2.2impliesthattheintegralof f iswelldefined(andboundedbelow).

Wenextshowthatthelevelsetsof I areprecompact.Choose L largeenoughso that $K(L) > 1$.Then,

$$\begin{aligned}
\mu(|z|>L)^2 &= \mu\otimes\mu(|z|>L, |w|>L) \qquad (10)\\
&\leq \mu(\{f(z,w)\geq K(L)\}\cap\{|1-z|>1/4, |1-w|>1/4\})\\
&\leq \frac{1}{K(L)-1/8}\iint_{A^{\complement}} (f(z,w)-1/8)d\mu(z)d\mu(w)\\
&\leq \frac{1}{K(L)-1/8}\left(\iint f(z,w)d\mu(z)d\mu(w) - \iint_A f(z,w)d\mu(z)d\mu(w)\right),
\end{aligned}$$

whereweused(9)inthesecondinequality.

Ournexttaskistoshowthat

$$-\iint_A f(z,w)\,d\mu(z)d\mu(w) \le c, \tag{11}$$

forsomeconstant c independentof $\mu \in \mathcal{P}$ .Tothisend,wewrite $A = A_1 \cup A_2 \cup A_3 \cup A_4$ with

$$A_1 := \{|1-z| \le 1/2, |1-w| \le 1/4\}, A_2 := \{|1-z| > 1/2, |1-w| \le 1/4\},$$
$$A_3 := \{|1-w| \in [1/4, 1/2], |1-z| \le 1/4\}, A_4 := \{|1-w| > 1/2, |1-z| \le 1/4\}.$$

Since $|z-w| \le 3/4$for$(z,w) \in A_1$,wehave

$$\begin{aligned} -\iint_{A_1} f(z,w)\,d\mu(z)d\mu(w) &\le \iint_{A_1} \log|z-w|\,d\mu(z)d\mu(w) - 2\int_{\{z:|1-z|\le 1/2\}} \log|1-z|\,d\mu(z) \\ &\le -2\int_{\{z:|1-z|\le 1/2\}} \log|1-z|\,d\mu(z) \le C(\log 2)\,, \end{aligned} \tag{12}$$

where $C(\log 2)$isgivenbyLemma2.2.Withthesameargument,wealsohave

$$-\iint_{A_3} f(z,w)\,d\mu(z)d\mu(w) \le C(\log 2)\,. \tag{13}$$

Fortheintegralovertheset A_2,wenotethat $|1-(1-w)/(1-z)| \in (1/2, 3/2)$for $(z,w) \in A_2$,andtherefore

$$\begin{aligned} -\iint_{A_2} f(z,w)\,d\mu(z)d\mu(w) &= -\iint_{A_2} \log\frac{|1-w|}{|1-\frac{1-w}{1-z}|}\,d\mu(z)d\mu(w) \quad (14) \\ \le \log(3/2) - \int_{\{w:|1-w|\le 1/4\}} \log|1-w|\,d\mu(w) &\le \log(3/2) + C(\log 4)\,, \end{aligned}$$

where $C(\log 4)$isagaingivenbyLemma2.2.

Since $\iint_{A_4} f(z,w)\,d\mu(z)d\mu(w) = \iint_{A_2} f(z,w)\,d\mu(z)d\mu(w)$,weobtainbycombining (12),(13)and(14)that(11)holds.

From(11)weobtainthatforany $M > 0$,

$$\sup_{\{\mu:I(\mu)\le M\}} \mu(|z| > L) \to_{L\to\infty} 0\,,$$

whichyieldsthepre-compactnessofthelevelsetsof I byanapplicationofProhorov's criterion.

Itremainstoshowthat I islowersemicontinuous.Since $\mathcal{P}$ isclosedin $\mathcal{M}_1(\mathbb{C})$,itis enoughtocheckthelowersemicontinuityin $\mathcal{P}$.Towardthisend,for $\epsilon, M > 0$define

$$f^{\epsilon,M}(z,w) = \left[\left(\log|1-z| \vee (-\frac{1}{\epsilon})\right) + \left(\log|1-w| \vee (-\frac{1}{\epsilon})\right) - (\log|z-w| \vee (-M))\right] \wedge M$$

and

$$f^{\epsilon}(z,w)=\log\ \ |1-z|\vee\ (-\frac{1}{\epsilon})+\log\ \ |1-w|\vee\ (-\frac{1}{\epsilon})-\log|z-w|.$$

Set

$$I^{\epsilon,M}:=\frac{1}{2}\iint f^{\epsilon,M}(z,w)d\mu(z)d\mu(w)$$

and

$$I^{\epsilon}:=\frac{1}{2}\iint f^{\epsilon}(z,w)d\mu(z)d\mu(w).$$

Notethatbymonotoneconvergence, $I^{\epsilon}=\sup_{M>0} I^{\epsilon,M}$,andsince $I^{\epsilon,M}:\mathcal{M}_1(\mathbb{C})\to\mathbb{R}$ iscontinuous,wehavethat I^{ϵ} islowersemicontinuouson $\mathcal{M}_1(\mathbb{C})$,andthereforeon $\mathcal{P}$.Ontheotherhand, I^{ϵ} convergesuniformlyto I on $\mathcal{P}$ byLemma2.2.Itfollows that I isalsolowersemicontinuouson $\mathcal{P}$,completingtheproofofthelemma. ■

3.2ExponentialTightnessof $\{\mathbb{P}_n\}$.

Weproveinthissubsectiontheexponentialtightnessofthefamily $\{\mathbb{P}_n\}$.

Lemma3.2. *Thefamily* $\{\mathbb{P}_n\}$ *isexponentiallytight.Thatis,with* $T>0$ *thereexist compactsets* $K_T\subset\mathcal{P}$ *sothat*

$$\limsup_{n\to\infty}\frac{1}{n^2}\log P(L_n\in K_T^{\complement})\leq -T.$$

Proof. Introducethefunction $g(z,w)=\log\ \ |1-z|+\log\ |1-w|-\ \log_+(|z-w|)$,and definethefunction J on $\mathcal{P}$ by

$$J(\mu)=\iint g(z,w)d\mu(z)d\mu(w)\,.$$

UsingLemma2.2andarguingasinsubsection3.1,oneseesthatthesets

$$K_B:=\{\mu\in\mathcal{P}\ :J(\mu)\leq 5B\}$$

arecompactin $\mathcal{M}_1(\mathbb{C})$for B large.

Weneedthustoestimate $P(L_n\in K_B^{\complement})$.Introducetherandomvariables

$$X_n=\frac{1}{n^2}\sum_{i=1}^{n}\log|1-z_i|=\frac{1}{n^2}\log\frac{P_n(1)}{\xi_n}=\frac{1}{n^2}\log\frac{\xi_0+\cdots+\xi_n}{\xi_n}$$

and

$$Y_n=\frac{1}{n}\sum_{\{i:|1-z_i|<1\}}|\log|1-z_i||\,.$$

Weneedthefollowingestimate,whoseproofispostponedtotheendofthesubsection.

Lemma3.3. *Thereexistsaconstant* $c>0$ *suchthatforall* n *large,*

$$\mathbb{P}_n(|X_n|>B)\leq 20ne^{-Bn^2} \tag{15}$$

and

$$\mathbb{P}_n(Y_n>c)=0 \quad . \tag{16}$$

ContinuingwiththeproofofLemma3.2,wehave

$$\begin{aligned}\mathbb{P}_n(L_n\in K_B^{\complement}) &\leq \mathbb{P}_n\left(\{L_n\in K_B^{\complement}\}\cap\{|X_n|\leq B\}\right)+\mathbb{P}_n\left(\{|X_n|>B\}\right)\\ &\leq \sum_{k=0}^{\lfloor n/2\rfloor}\mathbb{P}_n\left(\{L_n\in K_B^{\complement}\}\cap\{|X_n|\leq B\}\cap A_{n,k}\right)+20\,ne^{-Bn^2}, \end{aligned}\tag{17}$$

see(5)forthedefinitionof $A_{n,k}$.

Wenextconsiderthedensityof $\mathbb{P}_n$ on $A_{n,k}$,see(5),whichwewriteas

$$\begin{aligned}&f_{k,n}(z_1,\cdots,z_N)=\\ &\frac{1}{\mathcal{Z}_{n,k}}\exp\left(\frac{n^2}{2}\left(\frac{1}{n^2}\sum_{i\neq j}\log|z_i-z_j|-\frac{2}{n}\sum_i\log|1-z_i|+\frac{4}{n^2}\sum_i\log|1-z_i|\right)\right)\\ &\times\exp\left(-3\sum_i\log|1-z_i|\right)\mathbf{1}_{B_{n,k}}(z_1,\ldots,z_n).\end{aligned}$$

Notethat

$$\frac{1}{n^2}\sum_{i\neq j}\log|z_i-z_j|\leq\frac{1}{n^2}\sum_{i\neq j}(\log_+|z_i-z_j|)=\iint\log_+|z-w|dL_n(z)dL_n(w).$$

Thus,ontheevent $\{L_n\in K_B^{\complement}\}\cap\{|X_n|\leq B\}\cap A_{n,k}$,wehavethat

$$\frac{1}{n^2}\sum_{i\neq j}\log|z_i-z_j|-\frac{2}{n}\sum_i\log|1-z_i|+\frac{4}{n^2}\sum_i\log|1-z_i|\leq -5B+4B=-B$$

andthereforeonthisevent,

$$f_{k,n}(z_1,\cdots,z_N)\leq\frac{1}{\mathcal{Z}_{n,k}}e^{-n^2B/2}\exp\left(-3\sum_i\log|1-z_i|\right)\mathbf{1}_{B_{n,k}}(z_1,\ldots,z_n).$$

Thus,using(16)andtheconstant c inthestatementofthelemma,

$$\begin{aligned}&\mathbb{P}_n\left(\{L_n\in K_B^{\complement}\}\cap\{|X_n|\leq B\}\cap A_{n,k}\right)\\ \leq\ &\frac{1}{\mathcal{Z}_{n,k}}e^{-n^2B/2}\int\cdots\int\left(\left[\prod_{i=0}^{n}\frac{1}{|1-z_i|^3}\right]\wedge e^{3cn}\right)d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n).\end{aligned}$$

Lemma3.2followsfromsubstitutingthelastdisplayin(17)andperformingthe integration. ∎

ProofofLemma3.3. Bytheargumentintheproofof Lemma2.2,wehavethatfor any j non-negativeinteger,

$$\frac{1}{n}\sum_{z_i:|1-z_i|\in[2^{-j+1},2^{-j}]}|\log|1-z_i||\leq\ j\cdot 2^{-j}\,.$$

Thus,

$$Y_n=\frac{1}{n}\sum_{z_i:|1-z_i|\leq 1}|\log|1-z_i||\leq\ \sum_{j=0}^{\infty}j2^{-j}\,. \tag{18}$$

Inparticular,forall n large,

$$\mathbb{P}_n(X_n\leq -1)=0\quad. \tag{19}$$

Next,wecontroltheuppertailof X_n.Wehave

$$\begin{aligned}P(X_n>B\,)=&\quad P(\sum_{i=0}^{n-1}\xi_i>(e^{Bn^2}-1)\xi_n)\leq P(\sum_{i=1}^{n-1}\xi_i>\frac{1}{2}e^{Bn^2}\xi_n)\\ =&\int_0^{\infty}e^{-x}P(\sum_{i=1}^{n-1}\xi_i>\frac{1}{2}e^{Bn^2}x)dx.\end{aligned} \tag{20}$$

UsingChebycheff'sinequality,wehave

$$P(\sum_{i=1}^{n-1}\xi_i>\frac{1}{2}e^{Bn^2}x)\leq e^{-\lambda e^{Bn^2}x/2}E(e^{\lambda\xi_1})^n\leq\frac{e^{-\lambda e^{Bn^2}x/2}}{(1-\lambda)^n}\,.$$

Chooising $\lambda=1/n$ andsubstitutingin(20)gives

$$P(X_n>B\,)\leq e\cdot\int_0^{\infty}e^{-x(1+e^{Bn^2}/2n)}dx\leq 4e\cdot ne^{-Bn^2}\,.$$

Combiningthelastdisplaywith(19)completestheproof. ■

3.3TheUpperBound

Recallthenotation $f(z,w\,)=\log\ |1-z|+\log\ |1-w|-\ \log|z-w|$.Weproveinthis subsectionthefollowing.

Lemma3.4. *Forany* $\mu\in\mathcal{P}$ *,*

$$\lim_{\epsilon\to 0}\overline{\lim_{n\to\infty}}\frac{1}{n^2}\log\,\mathbb{P}_n\left(d(L_n,\mu\,)\leq\epsilon\right)\leq-\frac{1}{2}\iint f(z,w\,)d\mu(z)d\mu(w)\,. \tag{21}$$

Here, $d(\cdot,\cdot)$isanarbitrarymetricon $\mathcal{M}_1(\mathbb{C})$whichiscompatiblewiththeweak topology,e.g.theL´evymetric.

Proof. Define the set of measures

$$E_n := \{\nu \in \mathcal{P} \,:\, \frac{1}{n}\int \log|1-z|d\nu(z) \geq \frac{1}{2}\iint f(z,w)d\nu(z)d\nu(w)\}.$$

The set E_n corresponds to a subset of $\cup_{k=0}^{\lfloor n/2\rfloor} A_{n,k}$ which gives rise to empirical measures ν as described in Section 1. By abuse of notation, we denote this set by E_n as well.

An application of (15) of Lemma 3.3 gives the following.

Proposition 3.5. *With notation as above,* $\mathbb{P}_n(E_n) \leq 20n\exp\left(-\frac{1}{2}n^2\iint f(z,w)d\mu(z)d\mu(w)\right)$.

Now,

$$\mathbb{P}_n\left(d(L_n,\mu) \leq \epsilon\right) \leq \mathbb{P}_n\left(E_n^{\complement} \cap \{\, d(L_n,\mu) \leq \epsilon\}\right) + \mathbb{P}_n(E_n).$$

Therefore,

$$\overline{\lim_{n\to\infty}} \frac{1}{n^2}\mathbb{P}_n\left(d(L_n,\mu) \leq \epsilon\right) = \text{Max}\ \{\overline{\lim_{n\to\infty}} \frac{1}{n^2}\mathbb{P}_n(E_n), \overline{\lim_{n\to\infty}} \frac{1}{n^2}\mathbb{P}_n\left(E_n^{\complement} \cap \{\, d(L_n,\mu) \leq \epsilon\}\right)\}.$$

Since $\overline{\lim}_{n\to\infty} \frac{1}{n^2}\log \mathbb{P}_n(E_n)$ is bounded above by the desired upper bound, it remains to deal with $\overline{\lim}_{n\to\infty} \frac{1}{n^2}\mathbb{P}_n\left(E_n^{\complement} \cap \{\, d(L_n,\mu) \leq \epsilon\}\right)$.

We begin with

$$\mathbb{P}_n(E_n^{\complement} \cap \{\, d(L_n,\mu) < \epsilon\,\}) = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{1}{\mathcal{Z}_{n,k}} I_{k,n}^{\epsilon},$$

where

$$I_{k,n}^{\epsilon} = \int_{\{E_n^{\complement}\cap A_{n,k}\cap B_{n,k}\cap\{d(W_n,\mu)\leq\epsilon\}\}} \exp\left(\sum_{1\leq i<j\leq n} \log|z_i - z_j| - (n+1)\sum_{j=1}^{n}\log|1-z_j|\right)$$

$$d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n)$$

where $W_n(z_1,\cdots,z_n)$ is the empirical measure $\frac{1}{n}\sum_{i=1}^{n}\delta_{z_i}$.

We will upperbound $\overline{\lim}_{n\to\infty}\frac{1}{n^2}\log I_{k,n}^{\epsilon}$ for each $0 \leq k \leq \lfloor n/2\rfloor$, uniformly in k; by summing over k, this (together with (6)) will be sufficient for the overall upper bound on $\mathbb{P}_n\left(E_n^{\complement} \cap \{\, d(L_n,\mu) \leq \epsilon\}\right)$.

For reasons similar to those encountered in the proof of exponential tightness, we write the integrand in $I_{k,N}^{\epsilon}$ as

$$\exp\left(\sum_{1\leq i<j\leq n} \log|z_i - z_j| - (n-1)\sum_{j=1}^{n}\log|1-z_j| + \epsilon\sum_{j=1}^{n}\log|1-z_j|\right)\prod_{j=0}^{n}\frac{1}{|1-z_j|^{2+\epsilon}}. \tag{22}$$

Notethattoupperboundtheexponentin(22),itsufficestotruncatelog $|z_i - z_j|$ frombelowandlog $|1 - z_i|$ fromabove.Tothisend,wefixabigpositivenumber M anddefinethetruncatedfunction

$$f_M(z,w) = f(z,w) \wedge M.$$

Theexponentin(22)is

$$\mathcal{E}_n(z_1, \cdots, z_N) \leq \frac{n^2}{2}\left(-\iint_{z\neq w} f_M(z,w)dL_n(z)dL_n(w) - \frac{2\epsilon}{n}\int \log|1-z|dL_n(z)\right) \tag{23}$$

and$\exp(\mathcal{E}_n(z_1, \cdots, z_n))$isintegrated,foreachfixed k,withrespecttothemeasure

$$\prod_{j=0}^{n} \frac{1}{|1-z_j|^{2+\epsilon}} \mathbf{1}_{B_{n,k}}(z_1,\ldots,z_n)d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n). \tag{24}$$

But

$$\iint_{z\neq w} f_M(z,w)dL_n(z)dL_n(w) = \iint f_M(z,w)dL_n(z)dL_n(w) - M/n$$

Intheaboveequality,the M/n termcomesfromthediagonaltermsinthediscrete sum $\iint f_M(z,w)dL_n(z)dL_n(w)$.

Wehandle(23)withthefollowingproposition,whoseproofisdeferedtotheendof thissection.

Proposition3.6. *Thereexist* $\delta_M(\epsilon) > 0$ *and* $c(M) > 0$ *suchthatforall* $\nu \in \mathcal{P}$ *such that* $d(\nu,\mu) < \epsilon$ *wehave*

$$\left|\iint f_M(z,w)d\nu(z)d\nu(w) - \iint f_M(z,w)d\mu(z)d\mu(w)\right| < \delta_M(\epsilon) + c(M).$$

where $\delta_M(\epsilon) \to 0$ *as* $\epsilon \to 0$ *foreachfixed* M *(biggerthansomeuniversalconstant) and* $c(M) \to 0$ *uniformlyin* $\nu \in \mathcal{P}$ *and* ϵ.

Continuingwiththeproofoftheupperbound,weuseProposition3.6in(23) togetherwith(16)towrite

$$\begin{aligned} I^{\epsilon}_{k,n} \leq\ & Cn\exp\left\{4n^2(\delta_M(\epsilon) + c(M) + n^{-1}M)\right\} \\ & \times \exp\left\{-\frac{n^2}{2}\left(\iint f(z,w)d\mu(z)d\mu(w) - \epsilon\iint f(z,w)d\mu(z)d\mu(w)\right)\right\} \\ & \int\cdots\int\left(\prod_{i=0}^{n}\frac{1}{|1-z_i|^{2+\epsilon}} \wedge c^{(2+\epsilon)n}\right) d\mathcal{L}(z_1)\cdots d\mathcal{L}(z_k)d\ell(z_{2k+1})\cdots d\ell(z_n). \end{aligned}$$

Thelastintegralisdominatedby $e^{C(\epsilon)n}$ forappropriate $C(\epsilon)$.Takinglogarithm, dividingby n^2 andletting $n \to \infty$, $\epsilon \to 0$and $M \to \infty$ (inthatorder)wegetthe desiredupperbound(21). ■

Proof of Proposition 3.6. The statement would follow immediately from the definitions if f_M was a bounded continuous functions. Although f_M is not a bounded continuous function, f_M is clearly bounded above. We introduce $g_M = f_M \vee (-M)$. Note that switching with $z' = 1 - z, w' = 1 - w$, we have $g_M(1 - z', 1 - w') = (-M) \vee M \wedge (-\log|\frac{1}{z'} - \frac{1}{w'}|)$.

Let A_M be the set

$$A_M := \{(z,w): \ f_M(z,w) < -M\}.$$

Clearly,

$$\iint f_M(z,w)d\nu(z)d\nu(w) = \iint g_M(z,w)d\nu(z)d\nu(w) + \iint_{A_M} (f_M(z,w)+M)d\nu(z)d\nu(w).$$

We consider integration over the domain $|w'| \leq |z'|$; by symmetry, the complementary domain can be handled similarly. Then on the set A_M, we have $f_M(z,w) = -\log|1 - \frac{w'}{z'}| + \log|w'|$. But since $|w'| \leq |z'|$, we have $-\log|1 - \frac{w'}{z'}| \geq -\log 2$ and therefore on the set A_M we have

$$\log|w'| \leq -M + \log 2$$

and

$$-\log 2 + \log|w'| \leq f_M(z,w) \leq -M. \tag{25}$$

Let $B_{M,\nu}$ be the event that for two i.i.d. variables (X,Y) sampled from ν, the minimum satisfies

$$\log(\min(|1 - X|, |1 - Y|)) \leq -M + 2.$$

Clearly, $A_M \subset B_{M,\nu}$. From Lemma 2.2, we deduce that

$$\nu(B_{M,\nu}) < c_1(M) \tag{26}$$

where $c_1(M) \to 0$ as $M \to \infty$ uniformly in ν. Furthermore, the same lemma implies that

$$\left| \int_{B_{M,\nu}} \log(\min(|1 - z|, |1 - w|)) \, d\nu(z)d\nu(w) \right| < c_2(M) \tag{27}$$

where $c_2(M) \to 0$ as $M \to \infty$ uniformly in ν.

Combining (25), (26) and (27) we get

$$\iint_{A_M} (f_M(z,w) + M)d\nu(z)d\nu(w) = c_3(M,\nu)$$

where $c_3(M,\nu) \to 0$ as $M \to \infty$ uniformly in $\nu \in \mathcal{P}$. In other words, we have

$$\left| \iint f_M(z,w)d\nu(z)d\nu(w) - \iint g_M(z,w)d\nu(z)d\nu(w) \right| \to 0$$

as $M \to \infty$, uniformly in $\nu \in \mathcal{P}$.

It remains to show that $\iint g_M d\nu \otimes d\nu \to \iint g_M d\mu \otimes d\mu$ as $\nu \to \mu$ for a fixed M. But this is true by definition since g_M is a bounded continuous function. ∎

3.4 The Lower Bound

Our goal in this subsection is to prove the following.

Lemma 3.7. *For any $\mu \in \mathcal{P}$ with $I(\mu) < \infty$,*

$$\lim_{\epsilon \to 0} \varliminf_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu) \leq \epsilon \right) \geq - I(\mu) = -\frac{1}{2} \iint f(z,w) d\mu(z) d\mu(w). \tag{28}$$

(Recall that $f(z,w) = \log |1-z| + \log |1-w| - \log |z-w|$.)

The proof of Lemma 3.7 proceeds by several approximation steps. Those are detailed in the rest of the subsection, with several technical propositions deferred to Sections 4 and 5.

3.4.1 A dense subclass $\mathcal{D} \subset \mathcal{P}$

We introduce a dense (in the metric of $\mathcal{M}_1(\mathbb{C})$) subset $\mathcal{D} \subset \mathcal{P}$ such that for any measure $\mu \in \mathcal{P}$ there is a sequence $\{\mu_m\}_{m=1}^{\infty}$ from $\mathcal{D}$ such that

(i) $\mu_m \to \mu$ as $m \to \infty$ (convergence in the weak topology of $\mathcal{M}_1(\mathbb{C})$),

(ii) $I(\mu_m) \to I(\mu)$ as $m \to \infty$,

(iii) For any $\nu \in \mathcal{D}$, the estimate (28) holds.

Once such a subset $\mathcal{D}$ is constructed, Lemma 3.7 follows at once.

Such a desirable dense subset $\mathcal{D} \subset \mathcal{P}$ will be obtained from the proof of the characterization Theorem 2.3 for $\mathcal{P}$. In [BES13, Proof of Theorem 2], the authors begin with any fixed measure $\mu \in \mathcal{P}$, and perform a sequence of approximation steps (steps 1-5 in their paper) to obtain a measure $\mu_\epsilon \in \mathcal{P}$, which has (roughly) the following support properties:

(a) supp(μ_ϵ) is contained in a compact annulus centred at the origin.

(b) supp(μ_ϵ) is disjoint from a cone with apex at the origin and axis the positive ray $\mathbb{R}_+$.

(c) $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) < \widehat{\mathbb{L}}_{\mu_\epsilon}(|z|)$ for each $z \in \mathbb{C} \setminus \bar{\mathbb{R}}_+$, while $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) = \widehat{\mathbb{L}}_{\mu_\epsilon}(0) + a \Re z + O(|z|^2)$ as $|z| \to 0$ and $\widehat{\mathbb{L}}_{\mu_\epsilon}(z) = \log |z| + b/\Re(z) + O(1/|z|^2)$ as $|z| \to \infty$, with both a and b being positive.

Inaddition, $\mu_\epsilon \to_{\epsilon\to 0} \mu$ (again,intheweaktopologyof $\mathcal{M}_1(\mathbb{C})$.

Wedefine $\mathcal{D}$ tobethesubsetof $\mathcal{M}_1(\mathbb{C})$consistingofprobabilitymeasures μ_ϵ satisfying(a)–(c).Wewillusethe[BES13]construction(withaslightmodificationof theirstep5),inordertoconstruct,forany $\mu \in \mathcal{P}$ with $I(\mu) < \infty$,asequence $\mu_\epsilon \in \mathcal{D}$ with $\mu_\epsilon \to \mu$ andinaddition $I(\mu_\epsilon) \to_{\epsilon\to 0} I(\mu)$.Forthesakeofcompleteness,wegivea completeaccountoftheconstruction,itsmodificationsandapproximationproperties inSection5.Thatis,weproveinSection5thefollowingproposition.

Proposition3.8. *Forany $\mu \in \mathcal{P}$ with $I(\mu) < \infty$,thereexistsasequence $\mu_\epsilon \in \mathcal{D}$ so that $\mu_\epsilon \to_{\epsilon\to 0} \mu$ and $I(\mu_\epsilon) \to_{\epsilon\to 0} I(\mu)$.*

EquippedwithProposition3.8andinviewofproperties(i)–(iii)above,Lemma3.7 isanimmediateconsequenceofthefollowingpropositionandthelocalnatureofthe largedeviationslowerbound.

Proposition3.9. *Thelowerbound* (28) *holdsforany $\mu \in \mathcal{D}$.*

TherestofthissectionisdevotedtotheproofofProposition3.9.

3.4.2ProofofProposition3.9.

Theproofproceedsinseveralapproximationsteps.Wefixthroughouta $\mu \in \mathcal{D}$ .We firstconstructinProposition3.10asequenceofpolynomialswithpositivecoefficients whoseempiricalmeasureofzerosapproximates μ andsothattheir(discrete)logarithmicenergiesapproximatethelogarithmicenergyof μ.WethenshowinProposition 3.11thatitisenoughtoprovethelowerboundforballscenteredattheempirical measure μ_k ofzerosoftheseapproximatingpolynomials.Theproofofthelaterlower boundisthenobtainedbyfirstconstructingappropriateneighborhoodsof μ_k,and thenlowerboundingtheprobabilitybylowerboundingthedensityof $\mathbb{P}_n$ onthese neighborhoods.

StageI:Reductiontoatomicmeasures
Weintroducethediscreteversionoflogarithmicenergyforatomicmeasureswith distinctatoms,asfollows.

Definition3. *Foranatomicmeasure μ withequalmass $\frac{1}{k}$ atthe k distinctatoms $\{z_i\}_{i=1}^k$,let*

$$\Sigma_{\mathrm{a}}(\mu) = \frac{1}{k^2} \sum_{i\neq j} \log|z_i - z_j|$$

denotethemodifiedlogarithmicenergy.

Byanabuseofnotation,wewillalsousethesamenotation$\Sigma_{\mathrm{a}}(P)$where P isa polynomial(withdistinctzeros);inthatcase,theatomicmeasurebeingconsidered istheempiricalmeasureofthezerosof P.Webeginwiththefollowingproposition.

Proposition3.10. *Foreach* $\mu \in \mathcal{D}$ *onemayfindasequenceofmonicpolynomials* $\{P_k\}$,*withempiricalmeasureofzeros* $\{\mu_k\}$,*satisfyingthefollowingproperties.*

(i) P_k *haspositivecoefficients.*

(ii) $\mu_k \in \mathcal{P}$, $\mu_k \to \mu$ *in* $\mathcal{M}_1(\mathbb{C})$ *and* $\Sigma_{\mathrm{a}}(\mu_k) \to \Sigma(\mu)$ *as* $k \to \infty$.

(iii) μ_k *has* $a(k)$ *distinctatoms,noneofwhichisreal,and* μ_k *putsequalmassof* $1/a(k)$ *oneachatom,with* $a(k) \to \infty$ *as* $k \to \infty$.*Inparticular,* $a(k)$ *iseven.*

(iv) P_k *isofdegree* $d(k) = a(k)n(k)$ *forsomepositiveinteger* $n(k)$.

TheproofofProposition3.10isgiveninSection4.

TheimportanceofthepolynomialsinProposition3.10liesinthefollowingproposition.

Proposition3.11. *Toobtain(28)forsome* $\mu \in \mathcal{D}$,*itsufficestoprovethatforall largeenough* k,*andwith* μ_k *asinProposition3.10,wehave*

$$\varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu_k) \leq \epsilon \right) \geq -\frac{1}{2} \left(\int \log|1-z| d\mu_k(z) + \int \log|1-z| d\mu_k(w) - \Sigma_{\mathrm{a}}(\mu_k) \right). \tag{29}$$

ProofofProposition3.11. Given $\epsilon > 0$,wehaveforall k largeenoughtheinclusion ofsets

$$\{\nu : d(\nu, \mu_k) < \epsilon/2\} \subset \{ \nu : d(\nu, \mu) < \epsilon \}.$$

Thisimpliesthatgiven $\epsilon > 0$,wehaveforalllargeenough k

$$\begin{aligned} &\varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu) \leq \epsilon \right) \geq \varliminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n, \mu_k) \leq \epsilon/2 \right) \\ &\geq -\frac{1}{2} \left(\int \log|1-z| d\mu_k(z) + \int \log|1-z| d\mu_k(w) - \Sigma_{\mathrm{a}}(\mu_k) \right). \end{aligned} \tag{30}$$

Now,thesupportofeach μ_k iscontainedinthe $1/a(k)$thickeningofthesupportof μ whichisacompactsetboundedawayfrom1.Hencethefunction $\log|1-z|$ is continuousonaclosedneighborhoodofthesupportof μ,andtherefore $\mu_k \to \mu$ in $\mathcal{M}_1(\mathbb{C})$ impliesthat $\int \log|1-z| d\mu_k(z) \to \int \log|1-z| d\mu(z)$.Moreover,byproperty (ii)of μ_k wehavethat $\Sigma_{\mathrm{a}}(\mu_k) \to \Sigma(\mu)$ as $k \to \infty$ .Lettingnow $k \to \infty$ firstandthen $\epsilon \to 0$ in(30)yields(28). ■

StageII:TheNeighbourhoods $\mathcal{N}_n(\epsilon, \delta)$ of μ_k

Inthisstagewewillfix k andconsiderapositivenumber $0 < \delta < 1$(wewill eventuallylet $\delta \to 0$).Wewilldefinesuitable n-dependentneighbourhoods $\mathcal{N}_n(\epsilon, \delta)$ of μ_k in $\mathcal{M}_1(\mathbb{C})$(dependingon N),whicharecontainedintheset $\{\nu : d(\nu, \mu_k) < \epsilon\}$.

We begin with several definitions. If $\{w_i\}_{i=1}^p$ is a collection of complex numbers (multiplicities allowed), we say that the collection of complex numbers $\{\widetilde{w}_i\}_{i=1}^p$ is *ρ-compatible* with $\{w_i\}_{i=1}^p$ if $|w_i - \widetilde{w}_i| < \rho$ for all i and $\widetilde{w}_i$ is real whenever w_i is real.

Definition 4. *Let P be a polynomial of degree p with positive coefficients and zero set $\{w_i\}_{i=1}^p$ (multiplicities allowed). Then ρ_P is defined as the largest $\rho > 0$ satisfying that for any $i \neq j$, either $w_i = w_j$ or $B(w_i, \rho) \cap B(w_j, \rho) = \emptyset$, and in addition any collection $\{\widetilde{w}_i\}_{i=1}^p$ which is symmetric under conjugation and ρ-compatible with $\{w_i\}_{i=1}^p$ is the zero set of a polynomial with positive coefficients.*

For any $\rho < \rho_P$, we denote by $S(P, \rho)$ the set of all empirical measures corresponding to such collections $\{\widetilde{w}_i\}$.

Let P_k be as in Proposition 3.10, with corresponding empirical measure μ_k and set of atoms $\{z_i\}_{i=1}^{a(k)}$. Recall that none of the zeros of P_k is real. We will consider the set of atomic measures $S(P_k, \rho(\delta))$ (where δ is a small parameter to be sent to 0 eventually) and choose $\rho(\delta) \to 0$ as $\delta \to 0$ depending on P_k so that:

- $d(\mu_k, \nu) < \epsilon/2$ for any $\nu \in S(P_k, \rho(\delta))$.
- $\left| \int \log|1 - z| d\mu_k - \int \log|1 - z| d\nu(z) \right| < \delta$ for each $\nu \in S(P_k, \rho(\delta))$.
- $\rho(\delta) < \delta \cdot \min_{i \neq j} |z_i - z_j|$.

Given a positive integer n, set $m = \lfloor n/d(k) \rfloor$ (where we recall that $d(k) = a(k)n(k)$ is the degree of P_k). Define the set of atomic measures

$$S^{(m)}(P_k, \rho(\delta)) := \left\{ \nu : \frac{1}{m} \sum_{i=1}^{m} \nu_i;\ \nu_i \in S(P_k, \rho(\delta)) \text{ and } \nu \text{ has } md(k) \text{ distinct atoms} \right\}.$$

Each $\nu \in S^{(m)}(P_k, \rho(\delta))$ has the following structure: it has $md(k)$ distinct atoms with equal mass at each atom, and the atoms can be grouped into $a(k)$ sets according to their nearest z_i; by the definition of $S(P_k, \rho(\delta))$, each atom is at distance at most $\rho(\delta)$ from the corresponding z_i. In particular, none of the atoms is real.

Fix a bounded interval $\mathcal{I}$ of length < 1 on the negative real line such that $\mathcal{I}$ is also bounded away from the support of μ_k by a distance ≥ 2. Define the set of measures

$$\Upsilon^{(m)} := \left\{ \frac{1}{n - md(k)} \sum_{i=1}^{n - md(k)} \delta_{\theta_i} : \theta_i \text{ are distinct numbers } \in \mathcal{I} \right\}.$$

Finally, define $\mathcal{N}_n(\epsilon, \delta)$ to be the set of measures

$$\mathcal{N}_n(\epsilon, \delta) := \left\{ \frac{md(k)}{n} \nu_1 + \left(1 - \frac{md(k)}{n}\right) \nu_2 : \nu_1 \in S^{(m)}(P_k, \rho(\delta)), \nu_2 \in \Upsilon^{(m)} \right\}.$$

Notethatallmeasuresin $\mathcal{N}_n(\epsilon,\delta)$possessprecisely$(n - md(k))$realzeros.Since $0 \leq n - md(k) \leq d(k)$(whichisfixedsincewareconsidering k tobefixed),forlarge enough n wehave $d(\mu_k,\nu) < \epsilon$ forallmeasures $\nu \in \mathcal{N}_n(\epsilon,\delta)$.Therefore,

$$\underline{\lim}_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(d(L_n,\mu_k) \leq \epsilon\right) \geq \underline{\lim}_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n \left(\mathcal{N}_n(\epsilon,\delta)\right). \tag{31}$$

Remark3.1. *Each $\nu \in \mathcal{N}_n(\epsilon,\delta)$ hasthefollowingstructure:*

- *ν has n distinctatoms,eachhavingequalmass $1/n$.*
- *Theatomsof ν arethedisjointunionof $(a(k)+1)$ subsetsasfollows:*
  - *$\Lambda_i(\nu) := \{w : w$ anatomof $\nu, |z_i - w| \leq \rho(\delta)\}, 1 \leq i \leq a(k)$,with $|\Lambda_i(\nu)| = d(k)m/a(k)$ foreach $i \in \{1, \cdots, a(k)\}$.*
  - *$\Lambda_0(\nu) := \{w : w$ anatomof $\nu, w \in \mathcal{I}\}$ with $|\Lambda_0(\nu)| = n - d(k)m$.*

Conversely,everycollectionof n pointssatisfyingtheabovestructurehastheproperty thatthecorrespondingempiricalmeasureisin $\mathcal{N}_n(\epsilon,\delta)$.

Foreach $\nu \in \mathcal{N}_n(\epsilon,\delta)$,wedefinetheatomicmeasure

$$\nu(i) := \frac{1}{|\Lambda_i(\nu)|} \sum_{w\in\Lambda_i(\nu)} \delta_w$$

for$0 \leq i \leq a(k)$.

StageIII:Agoodsubset $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$ andcompletionoftheproofof Proposition3.9.

Weintroduceasubset $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$,andestimateitsvolumeinProposition 3.12.Wethenusetheestimatetocompletetheproofof Proposition3.9.

Throughout,wefix $q = q(n) = md(k)/2$,recallingthat $m = \lfloor n/d(k) \rfloor$ andthatby construction, $d(k)$iseven.

Definition5. *For $1 \leq i \leq a(k)$,define U_i tobetheballofradius $\rho(\delta)$ centeredat z_i. Define U_0 tobetheinterval $\mathcal{I}$.*

Definethesetofatomicmeasures $\widetilde{\mathcal{N}}_n(\epsilon,\delta) \subset \mathcal{N}_n(\epsilon,\delta)$ asfollows:

$$\widetilde{\mathcal{N}}_n(\epsilon,\delta) := \{\nu \in \mathcal{N}_n(\epsilon,\delta) : \Sigma_{\mathrm{a}}(\nu(i)) > 2\Sigma(U_i) \text{ foreach } 0 \leq i \leq a(k)\}.$$

Byanabuseofnotation,wealsodenoteby $\widetilde{\mathcal{N}}_n(\epsilon,\delta)$ thesubsetof $\mathbb{C}^q \times \mathbb{R}^{n-2q}$ induced bytheatomsofmeasures $\nu \in \widetilde{\mathcal{N}}_n(\epsilon,\delta)$ inthemannerdescribedinSection1.

Notethatbythechoiceof $\rho(\delta)$, $U_i \cap \mathbb{R} = \emptyset$ for $i = 1, \ldots, a(k)$.

Withthisdefinition,wehavethefollowingproposition,whoseproofisgivenatthe endofthissubsection.

Proposition3.12. *Withnotationasabove,wehave*

(i) Volume$(\widetilde{\mathcal{N}}_n(\epsilon,\delta)) \geq \frac{1}{2^{a(k)}} \rho(\delta)^{d(k)m} |\mathcal{I}|^{d(k)}$.

(ii) Foreach $\nu \in \widetilde{\mathcal{N}}_n(\epsilon,\delta)$, *wehave*

$$\Sigma_{\mathrm{a}}(\nu) \geq \Sigma_{\mathrm{a}}(\mu_k) + \log |1 - 2\delta| + \frac{2}{a(k)} A(\delta) + \frac{C(k)}{n^2}$$

where $A(\delta)$ *and* $C(k)$ *arepositivequantitiessuchthat* $A(\delta) \to 0$ *as* $\delta \to 0$ *with* k *fixed.*

(iii) $\left| \int \log|1 - z| d\mu_k - \int \log|1 - z| d\nu(z) \right| < \delta$ *foreach* $\nu \in \widetilde{\mathcal{N}}_n(\epsilon,\delta)$.

Intheproposition,byVolumewemeantheEuclideanvolume.

Wecannowcompletethefollowingproof.

ProofofProposition3.9. Inwhatfollows,wewillusethenotation $\widetilde{\mathcal{N}}_n$ todenote $\widetilde{\mathcal{N}}_n(\epsilon,\delta)$,unlessexplicitlymentionedotherwise.Wehave

$$P_n\left(\widetilde{\mathcal{N}}_n\right) \geq \frac{1}{\mathcal{Z}_{n,q}} \times$$
$$\int_{\widetilde{\mathcal{N}}_n} \exp\left\{ \frac{n^2}{2} \left(\Sigma_{\mathrm{a}}(\mathcal{W}) - 2\frac{n+1}{n^2} \sum_{i=1}^{n} \log|1 - w_i| \right) \right\} \prod_{i=1}^{q} d\mathcal{L}(w_i) \prod_{j=0}^{n-2q-1} d\ell(w_{N-j}),$$

wherewerecallthateachmeasurein $\widetilde{\mathcal{N}}_N$ has $n-2q$ realatoms,and $\mathcal{W}$ istheempirical measurecorrespondingtothesetofatoms $\{w_i\}_{i=1}^{n}$.

ByProposition3.12,theexponentinthelastintegralislowerbounded(uniformly overallmeasuresin $\widetilde{\mathcal{N}}_n$)by

$$\mathcal{E}_n = \frac{n^2}{2} \left(\Sigma_{\mathrm{a}}(\mu_k) + \log |1 - 2\delta| + \frac{2}{a(k)} A(\delta) + \frac{C(k)}{n^2} - (1 + \frac{1}{n}) \cdot 2 \int \log|1 - z| d\mu_k(z) - 2(1 + \frac{1}{n})\delta \right).$$

Hence,usingalso(6),

$$\mathbb{P}_n(\widetilde{\mathcal{N}}_n) \geq \exp\left(o(n^2) + \mathcal{E}_n \right) \mathrm{Volume}(\widetilde{\mathcal{N}}_n).$$

FromProposition3.12itfollowsthat $\frac{1}{n^2} \log \mathrm{Volume}(\widetilde{\mathcal{N}}_n) \to 0$as $n \to \infty$.

Therefore,

$$\underline{\lim}_{n\to\infty}\frac{1}{n^2}\mathbb{P}_n(\widetilde{\mathcal{N}}_n(\epsilon,\delta\,))\geq\frac{1}{2}\left(\Sigma_{\mathrm{a}}(\mu_k)+\log\ \ |1-2\delta|+\frac{2}{a(k)}A(\delta)-2\int\log|1-z|d\mu_k(z)-2\delta\right).$$

Letting $\delta\to 0$,holding k fixed,weobtain(29).Proposition3.11nowimpliesthatthe proofof(28)iscomplete. ■

ProofofProposition3.12. Webeginwithpart(i).Notethatfor $i=1,\ldots,a\ \ (k)$, U_i doesnotintersecttherealaxis.Recallalsothat $|\Lambda_i(\nu)|=d(k)m/a(k)=:\ p.$

Letap-tuple $\gamma=(\gamma_1,\cdots,\gamma_p)$besampledwitheachco-ordinatedrawnindependentlyfromtheuniformdistributionon U_i;byabuseofnotation,westilluse γ to denotetheatomicmeasure $\frac{1}{p}\sum_j\delta_{\gamma_j}$.Then, $\mathbb{E}(\Sigma_{\mathrm{a}}(\gamma))=(1-\frac{1}{p})\Sigma(U_i)$.Noticehere thatsincetheradiusof U_i islessthan1,$\Sigma_{\mathrm{a}}(\gamma)and\Sigma(U_i)$bothhavenegativesigns. Therefore,byafirstmomentbound,

$$\text{Volume}\{\gamma\in U_i^p:|\Sigma_{\mathrm{a}}(\gamma)|=-\Sigma_{\mathrm{a}}(\gamma)\geq 2|\Sigma(U_i)|=-2\Sigma(U_i)\}\leq\ \frac{1}{2}\text{Volume}(U_i^p).$$

Thisimpliesthat

$$\text{Volume}\{\gamma\in U_i^p:\Sigma_{\mathrm{a}}(\gamma)>2\Sigma(U_i)\}>\frac{1}{2}\text{Volume}(U_i^p).$$

Notethatthequantity$\Sigma(\ \ \mu)$isinvariantundertranslationsofthemeasure μ,hence any$\Sigma(U_i)$with $i\geq 1$isequalto$\Sigma(\ \ U)$,where U isthediskofradius $\rho(\delta)$centeredat theorigin.Notethat$\Sigma(\ \ U)\to 0$as $\delta\to 0$.

Asimilarargumentwiththeatomsin$\Lambda_0(\nu)$for $\nu\in\mathcal{N}_n$ impliesthat

$$\text{Volume}\{\gamma\in\mathcal{I}^{\,n-d(k)m}:\Sigma_{\mathrm{a}}(\gamma)>2\Sigma(\mathcal{I})\}>\frac{1}{2}\text{Volume}(|\mathcal{I}|^{n-d(k)m}).$$

Definition5andRemark3.1nowimplypart(i)ofProposition3.12.

Wenextturntotheproofofpart(ii).For $\nu\in\mathcal{N}_n(\epsilon,\delta\,)$wehave

$$\Sigma_{\mathrm{a}}(\nu)=\sum_{i=1}^{a(k)}\frac{d(k)^2m^2}{a(k)^2n^2}\Sigma_{\mathrm{a}}(\nu(i))+\frac{(n-d(k)m)^2}{n^2}\Sigma_{\mathrm{a}}(\nu(0))+\frac{1}{n^2}\sum_{w_i,w_j\ \text{notinsame}\Lambda_{\ l}}\log|w_i-w_j|. \tag{32}$$

Recallthat $\rho(\delta)<\delta\cdot\min_{i\neq j}|z_i-z_j|$.Thisimpliesthatfor w_i,w_j from$\Lambda_\alpha,\Lambda_\beta$ respectivelywith $\alpha\neq\beta\neq 0$,wehave

$$(1-2\delta)|z_\alpha-z_\beta|\leq|\ w_i-w_j|\leq\ (1+2\ \delta)|z_\alpha-z_\beta|.$$

Ontheotherhand,if $w_i\in\Lambda_0$ and $w_j\notin\Lambda_0$,then

$$0\leq\log|w_i-w_j|\leq\ \log|1+\ |\mathcal{I}|+D+\rho(\delta)|,$$

where D isthediameterofthesupportof μ_k.Hence

$$\left|\frac{1}{n^2}\sum_{\text{Exactlyoneof } w_i,w_j\in\Lambda_0}\log|w_i-w_j|\right|\le\frac{d(k)(n-d(k)m)}{n^2}\log|1+|\mathcal{I}|+D+\rho(\delta)|\le\ C(k)/n^2$$

forsomefunction $C(k)$.Hencewehave

$$\log|1-2\delta|-\frac{C(k)}{n^2}\le\left|\frac{1}{n^2}\sum_{w_i,w_j\ \text{notinsame}\Lambda_{\ l}}\log|w_i-w_j|-\ \Sigma_{\rm a}(\mu_k)\right|\le\log|1+2\,\delta|+\frac{C(k)}{n^2}. \tag{33}$$

Thisistrueforevery $\nu\in\mathcal{N}_n(\epsilon,\delta)$,andthereforeforevery $\nu\in\widetilde{\mathcal{N}}_n(\epsilon,\delta)$.Bydefiniton of $\widetilde{\mathcal{N}}_n$,wehave$\Sigma_{\ \rm a}(\nu(i))\ge 2\Sigma(U_i)\ge 2B(\delta)$foreach $\nu\in\widetilde{\mathcal{N}}_n$ andeach$0\ \le i\le a(k)$, wherethefunction $B(\delta)=\max(\Sigma(\ U),\Sigma(I))$.Since $m=\lfloor n/d(k)\rfloor$,wehave

$$\sum_{i=1}^{a(k)}\frac{d(k)^2m^2}{a(k)^2n^2}\Sigma_{\rm a}(\nu(i))+\ \frac{(n-d(k)m)^2}{n^2}\Sigma_{\rm a}(\nu(0))\ge\frac{A(\delta)}{a(k)}$$

forsome $A(\delta)\to 0$as $\delta\to 0$.Thiscompletestheproofofpart(ii)oftheproposition.

Part(iii)ofthepropositionisimmediateasthestatementholdsforallmeasures $\nu\in\mathcal{N}_n(\epsilon,\delta)$and $\widetilde{\mathcal{N}}_n(\epsilon,\delta)\subset\mathcal{N}_n(\epsilon,\delta)$. ■

4ProofofProposition3.10

Webeginwithageneralapproximationresult.

Lemma4.1. *Let $\mu\in\mathcal{M}_{\ 1}(\mathbb{C})$ beofcompactsupportandsuchthat μ issymmetric underconjugation, $\Sigma(\mu)<\infty$,and μ doesnotpossessatoms.Thenthereexistsa sequenceofpointconfigurationswithdistinctpointsandempiricalmeasures ν_k such that*

(1) ν_k issymmetricunderconjugation,thesupportof ν_k iscontainedinsidethe $1/k$ thickeningofthesupportof μ,and ν_k doesnotchargetherealline.

(2) $\nu_k\to\mu$ as $k\to\infty$.

(3) $\Sigma_{\rm a}(\nu_k)\to\Sigma(\mu)$ as $k\to\infty$.

Proof. Forandiscretemeasure ν having n distinctatomsandequalmass$1\ /n$ oneach atom,andafunction f,define$\Sigma^{\ f}_{\rm a}(\nu)=\frac{1}{n^2}\sum_{i\neq j}f(x_i,x_{\,j})$where $\{x_i\}_{i=1}^n$ aretheatoms of ν.Thus,for $f(z,w)=\log\ |z-w|$,wehave$\Sigma^{\ f}_{\rm a}(\nu)=\Sigma_{\ \rm a}(\nu)$asdefinedearlier.

Recallthatthemeasure μ iscompactlysupportedandsymmetricunderconjugation. Let μ_R be μ restrictedto $\mathbb{R}$ andlet μ_u be μ restrictedtotheupperhalfplane.Define themeasures $\mu_1 = \frac{1}{2}\mu_R + \mu_u$ and $\mu_2 = \frac{1}{2}\mu_R + \overline{\mu_u}$ where $\overline{\mu_u}$ isthemeasuresupported onthelowerhalfplaneanddefinedby $\overline{\mu_u}(A) = \mu_u(\overline{A})$.Obviously, $\mu = \mu_1 + \mu_2$.

Foreach n,wewillobtainconjugationsymmetricpointsetsofsize2 n whoseempiricalmeasureswillapproximate μ inthefollowingway.First,consider n i.i.d. randomsamples $\{x_1, \cdots, x_n\}$ from μ_1 (toobtainrandomsampleswecontheprobabilitymeasureobtainedbyappropriatelynormalizing μ_1).Considerthepoint set $Y := \{y_1, \cdots, y_n\}$ where $y_j = x_j + \frac{i}{n}$ and i istheimaginaryunit.Considerthe pointset $Z := Y \cup \overline{Y}$,andlet $L_n = \frac{1}{2n}\sum_{z\in Z}\delta_z$ denotethe(random)empiricalmeasureassociatedwith Z;onehasthat $L_n \to \mu$ indistribution(forexample,byan applicationofSanov'stheorem).

Fixapositivenumber M (tobethoughtofaslarge).Let $K > 1$beaboundon thediameterofthesupportof M.Set $f(z,w) = \log|z-w|$.Define $f_M(z,w) = f(z,w) \wedge M \vee (-M)$and $g_M = f - f_M$.Because$\Sigma(\mu) < \infty$ wehavethat

$$\alpha(M) = \iint |g_M(z,w)|d\mu(z)d\mu(w) \to 0$$

as $M \to \infty$.

Wehavethat $\frac{1}{n} \leq |Y_i - \overline{Y}_i| \leq K$.Therefore,for $n > n_0(K)$,

$$\frac{1}{n^2}\left|\sum_{i=1}^{n}\log|Y_i - \overline{Y}_i|\right| \leq \log n/n.$$

Ontheotherhand,for $i \neq j$ wehave $\mathbb{E}[|g_M(Y_i,Y_j)|] = \iint |g_M(z,w)|d\mu_1(z)d\mu_1(w)$and $\mathbb{E}[|g_M(\overline{Y}_i,\overline{Y}_j)|] = \iint |g_M(z,w)|d\mu_2(z)d\mu_2(w)$.

Wenextclaimthat,for $M > \log K$,wehave $\mathbb{E}[|g_M(Y_i,\overline{Y}_j)|] \leq \iint |g_M(z,w)|d\mu_1(z)d\mu_2(w)$. Toseethis,notethat

$$g_M(z,w) = \begin{cases} 0, & \text{if } -M \leq \log|z-w| \leq M, \\ \log|z-w| - M, & \text{iflog } |z-w| \geq M, \\ \log|z-w| + M & \text{iflog } |z-w| \leq -M. \end{cases}$$

Thecaselog $|z-w| \geq M$ doesnotarisewhenweconsider $g_M(Y_j,\overline{Y}_j)$because $M > \log K$.Ontheotherhand,

$$|Y_k - \overline{Y}_j| = |X_k - \overline{X}_j + \frac{2i}{n}| \geq |X_k - \overline{X}_j|$$

becausethe X_j-sbelongtotheupperhalfplane.Thus,log $|Y_k - \overline{Y}_j| \leq -M$ implies $\log|X_k - \overline{X}_j| \leq -M$ andonthiseventwehavelog $|X_k - \overline{X}_j| + M \leq \log|Y_k - \overline{Y}_j| + M$.

Sincethesequantitiesarenegative,thisisequivalentto $|\log|Y_k-\overline{Y}_j|+M|\leq|\log|X_k-\overline{X}_j|+M|$.Thisimpliesthat

$$\mathbb{E}[|g_M(Y_i,\overline{Y}_j)|]\leq\mathbb{E}[|g_M(X_i,\overline{X}_j)|]=\iint|g_M(z,w)|d\mu_1(z)d\mu_2(w),$$

asclaimed.

Wenowhavethat

$$\begin{aligned}\mathbb{E}[|\Sigma_{\mathrm{a}}^{g_M}(L_n)|]&\leq\frac{1}{4n^2}\sum_{i\neq j}\mathbb{E}[|g_M(Z_i,Z_j)|]\\ =\ &\frac{1}{4n^2}\sum_{i\neq j}\mathbb{E}[|g_M(Y_i,Y_j)|]+\frac{1}{4n^2}\sum_{i\neq j}\mathbb{E}[|g_M(Y_i,\overline{Y}_j)|]\\ &+\frac{1}{4n^2}\sum_{i\neq j}\mathbb{E}[|g_M(\overline{Y}_i,\overline{Y}_j)|]+\frac{1}{4n^2}\sum_{i=1}^{n}\mathbb{E}[|g_M(Y_i,\overline{Y}_i)|].\end{aligned}$$

Thus,fromthepreviouscomputations,wecanchoose $n_1=n_1(M)$sothatforall $n>n_1(M)$,

$$\mathbb{E}[|\Sigma_{\mathrm{a}}^{g_M}(L_n)|]\leq 2\alpha(M)\,.$$

Inparticular,forfixed $\delta>0$thereexists $n_2(M)=n_2(M,\delta)$sothatfor $n>n_2(M)$, thereexistsarealization ν_n of L_n suchthat $d(\nu_n,\mu)\leq\delta$ and $|\Sigma_{\mathrm{a}}^{g_M}(\nu_n)|\leq 2\alpha(M)$. Applyingadiagonalizationargument(with $\delta\to 0$while M iskeptfixed),wefinda sequence(withsomeabuseofnotation,denoted ν_k),sothatsuchthat $d(\nu_k,\mu)\leq 1/k$ and $|\Sigma_{\mathrm{a}}^{g_M}(\nu_k)|\leq 2\alpha(M)$.

Now,$\Sigma_{\mathrm{a}}(\nu_k)=\Sigma_{\mathrm{a}}^{f_M}(\nu_k)+\Sigma_{\mathrm{a}}^{g_M}(\nu_k)$.Since f_M isboundedandcontinuous,wehave $\Sigma_{\mathrm{a}}^{f_M}(\nu_k)\to\iint f_M(z,w)d\mu(z)d\mu(w)$as $k\to\infty$.Thisimpliesthat

$$\overline{\lim_{k\to\infty}}\,|\Sigma_{\mathrm{a}}^{f}(\nu_k)-\iint f(z,w)d\mu(z)d\mu(w)|\leq 3\alpha(M).$$

Applyingagainadiagonalizationargument(thistimeover M),oneobtainsthelemma. ■

ProofofProposition3.10. Let ν_k bethesequenceofatomicmeasuresconstructedin Lemma4.1.Notethateachonthe $\nu_k$shas $a(k)$distinct(nonreal)atoms,issupported withinthe $1/a(k)$thickeningofthesupportof μ,andsatisfies$\Sigma_{\mathrm{a}}(\nu_k)\to\Sigma(\nu_k)$.Each ν_k givesrisetoamonicpolynomial Q_k withdistinctzeros(sothatthezerosof Q_k arepreciselytheatomsof ν_k).Since $\mu\in\mathcal{P}$,Step6in[BES13,ProofofTheorem2] showsthatforsome $n(k)$largeenough,thepolynomial $P_k=Q_k^{n(k)}$ hasallcoefficients realandpositive.Notethattheempiricalmeasureofzerosof P_k coincideswiththe empiricalmeasureofzerosof Q_k.Thiscompletestheproof. ■

5 The Bergweiler-Eremenko-Sokal approximation and proof of Proposition 3.8.

We begin with a fixed measure $\mu \in \mathcal{P}$ satisfying $I(\mu) < \infty$. Let

$$\widehat{\mathbb{L}}_\mu(z) = \int_{|z-w|\leq 1} \log(|z-w|) d\mu(w) + \int_{|w|>1} \log(|1-\frac{z}{w}|) d\mu(w)\,,$$

and denote by $u(z) = \mathbb{L}_\mu(z) = \int \log|z-w| d\mu(w)$ its logarithmic potential. Because $I(\mu) < \infty$, the Bergweiler-Eremenko-Sokal condition can be written as $u(z) \leq u(|z|)$ for all $z \in \mathbb{C} \setminus \mathbb{R}_+$.

The Bergweiler-Eremenko-Sokal approximation proceeds in five steps to construct a sequence of approximations $\mu_i = \mu_i^\epsilon$, $i = 1, ..., 5$, with $\mu_5 \in \mathcal{D}$. In the i-th step, one starts with a measure μ_{i-1} (with $\mu_0 = \mu$ and $u_0 = u$) and constructs measures $\mu_i \in \mathcal{P}$ (depending on a small parameter $\epsilon > 0$) such that $\mu_i \to \mu_{i-1}$ weakly as $\epsilon \to 0$. The measures μ_i are defined via subharmonic functions u_i, such that $\mu_i = (2\pi)^{-1}\Delta u_i$ in the sense of distributions. One shows, see Section 4 of [BES13]) that in each of the 5 steps, one has $L_{\mu_i}(z) = u_i(z) + k_i^\epsilon$ where k_i^ϵ is a constant (as a function of z) depending on ϵ, and that $u_i(z) \to u_{i-1}(z)$ for each $z \in \mathbb{R}_+$, while in some of the steps the above convergence will occur pointwise in $\mathbb{C}$. It is a consequence of Proposition 5.1 below that in each of the Steps 1-5, $k_i^\epsilon \to 0$ as $\epsilon \to 0$.

5.1 Preliminaries

We begin with several preliminary proerties concerning the convergence of subharmonic functions.

Proposition 5.1. *Let u_ϵ be a sequence of subharmonic functions converging in the sense of distributions to a subharmonic function u_0 as $\epsilon \to 0$. Assume further that u_ϵ converges pointwise to u_0 on $\mathbb{R}_+$. Let $\mu_\epsilon, \epsilon \geq 0$ denote the Rieszmeasure of u_ϵ and assume that $u_\epsilon(z) = \mathbb{L}_{\mu_\epsilon}(z) + k_\epsilon$ where k_ϵ is independent of z. Then $\mu_\epsilon \to \mu_0$ weakly and $k_\epsilon \to k_0$.*

Proof. By standard results, see [BES13, Appendix A], one gets the weak convergence $\mu_\epsilon \to \mu_0$ and the a.e. (with respect to Lebesguemeasure on $\mathbb{R}_+$) convergence of $\mathbb{L}_{\mu_\epsilon}$ to $\mathbb{L}_{\mu_0}$. In particular, the convergence $L_{\mu_\epsilon}(z) \to L_{\mu_0}(z)$ occurs at a point $z \in \mathbb{R}_+$. This yields the convergence $k_\epsilon \to k_0$. ■

We next show that pointwise monotone convergence of subharmonic functions implies the convergence of the associated logarithmic energies. Results of this nature are known in the literature; here we follow an approach based on the proof of a related result in [D84].

Proposition5.2. *Suppose* $\{u_n\}$ *isasequenceofsubharmonicfunctionsdecreasing pointwisetoasubharmonicfunction* u_0,*as* $n \to \infty$.*Let* μ_n *betheRieszmeasureof* u_n,*andassume* $u_n(z) = \mathbb{L}_{\mu_n}(z) + k_n$,*with* $k_n \to k_0$ *as* $n \to \infty$.*Then* $\Sigma(\mu_n) \to \Sigma(\mu_0)$ *as* $n \to \infty$.

Proof. Bysubtractingoff k_0,wecanassume $u_0 = \mathbb{L}_{\mu_0}$ and $k_\epsilon \to 0$.Introducethe notation$[\,\mu,\nu\,] := \iint \log|z-w|d\mu(z)d\nu(w)$.

Thehypothesesimplyinparticularthat $\mu_n \to \mu_0$ weakly.Thelowersemicontinuity of $-\Sigma(\cdot)$thenimpliesthatlimsup$\Sigma(\mu_n) \leq \Sigma(\mu_0)$.Toseetheotherdirection,note thatifeither $n = 0$or $n > m$ wehave

$$\begin{aligned}\Sigma(\mu_n) &= [\,\mu_n,\mu_n\,] = \int (u_n(z) - k_n)\, d\mu_n(z) = \int u_n(z)d\mu_n(z) - k_n \\ &\leq \int u_m(z)d\mu_n(z) - k_n = [\,\mu_m,\mu_n\,] + k_m - k_n \leq [\mu_m,\mu_m] + 2(k_m - k_n) \\ &= \Sigma(\mu_m) + 2(k_m - k_n),\end{aligned}$$

wherethemonotonicityofthesequence $\{u_n\}$ wasusedintheinequalities.Weconclude thatliminf $_{m\to\infty}\Sigma(\mu_n) \geq \Sigma(\mu_0)$,completingtheproof. ∎

5.2 Approximationsteps

Wedescribeeachoftheapproximationsteps $\mu_i^\epsilon \to \mu_{i-1}$, $i = 1, \ldots, 5$,andshowthat foreach,both

$$\mathbb{L}_{\mu_i^\epsilon}(1) \to \mathbb{L}_{\mu_{i-1}}(1)\,, \quad \Sigma(\mu_i^\epsilon) \to \Sigma(\mu_{i-1}). \tag{34}$$

Inthesequel,weomitthesubscript ϵ whenitisclearfromthecontext.

5.2.1 Step1:

Given $\epsilon > 0$,define

$$u_1(z) = \max\,\{u(ze^{i\alpha}) :\ |\alpha| \leq \epsilon\}.$$

Notethat $u_1 \geq u$ pointwise,andthat u_1 decreasesin ϵ.Itisprovedin[BES13,Section 4]that $u_1 \to_{\epsilon\to 0} u$ weakly;however,duetouppersemicontinuityof u,thisimplies alsothepointwiseconvergence $u_1 \searrow_{\epsilon\to 0} u$.AnapplicationofPropositions5.1and5.2 yields(34)for $i = 1$.

5.2.2 Step 2:

For $\epsilon_1 \in (0, \epsilon)$ where ϵ is as chosen in Step 1, define $D_{\epsilon_1} = \{z : |\arg(z)| \leq \epsilon_1\}$. Let v denote the solution to the Dirichlet problem in D_{ϵ_1} with boundary conditions $u_1(z)$ and $v(z) = O(\log|z|)$ as $z \to \infty$. Define $u_2 = u_2(\epsilon_1)$ to be the function obtained by "balayage" (i.e., sweeping out of the Riesz measure) from the domain D_{ϵ_1}. In other words, $u_2(z) = v(z)$ if $z \in D_{\epsilon_1}$, and $= u_1(z)$ otherwise. In this step too, it follows from [BES13, Section 4] and the upper semicontinuity of u_1 that $u_2 \searrow_{\epsilon_1 \to 0} u_1$ pointwise. An application of Propositions 5.1 and 5.2 yields (34) for $i = 2$.

5.3 Step 3:

For $\epsilon > 0$, define $u_3(z) = u_2(z + \epsilon)$. We have $d\mu_3(z) = d\mu_2(z + \epsilon)$. We have that $u_3 \to u_2$ pointwise on $\mathbb{R}_+$ and hence Proposition 5.1 yields that $\mathbb{L}_{\mu_3}(1) \to \mathbb{L}_{\mu_2}(1)$. Since $\Sigma(\mu_3) = \Sigma(\mu_2)$, we conclude that (34) holds for $i = 3$.

5.4 Step 4:

For $\epsilon > 0$, define $v(z) = u_3(1/z) + \log|z|$ for $z \neq 0$, and extend $v(\cdot)$ to $\mathbb{C}$ by defining $v(0) = \limsup_{z \to 0} v(z)$. This definition preserves the sub-harmonicity of v, and in fact

$$v(0) = \lim_{r \searrow 0} v(r) \quad (35)$$

because $u_3(z) \leq u_3(|z|)$, hence the limsup is attained as $r \searrow 0$, and the convexity of $v(r)$ in $\log r$ then yields the existence of the limit. We claim that in fact, $v(0) > -\infty$ (and therefore, by sub-harmonicity, is finite). Indeed, $\mathbb{L}_{\mu_v}(0) = -\mathbb{L}_{\mu_3}(0)$; by construction, u_3 is harmonic in a neighborhood of 0, see [BES13]. Hence, $\mathbb{L}_{\mu_v}(0) = -\mathbb{L}_{\mu_3}(0)$ is finite, as claimed. We note in passing that $\mathbb{L}_{\mu_v}(1) = \mathbb{L}_{\mu_3}(1)$.

Next, define $w(z) = v(z + \epsilon)$ and, finally, $u_4(z) = w(1/z) + \log|z|$. As in the previous step, we have that $\mathbb{L}_{\mu_4}(1) = \mathbb{L}_{\mu_w}(1) \to_{\epsilon \to 0} \mathbb{L}_{\mu_v}(1)$. So it only remains to check the convergence of the logarithmic energy. To that end, note that under the transformation $v(z) = u(1/z) + \log|z|$, one has

$$\begin{aligned}\Sigma(\mu_v) &= \iint \log|z - w| d\mu_v(z) d\mu_v(w) = \iint \log\left|\frac{1}{z} - \frac{1}{w}\right| d\mu_u(z) d\mu_u(w) \\ &= \iint \log|z - w| d\mu_u(z) d\mu_u(w) - 2\int \log|z| d\mu_u(z) = \Sigma(\mu_u) - 2\mathbb{L}_{\mu_u}(0).\end{aligned}$$

With this computation in hand, we trace the changes in the logarithmic energy Σ in Step 4 as follows:

$$\begin{aligned}\Sigma(\mu_v) &= \Sigma(\mu_{u_3}) - 2\mathbb{L}_{\mu_3}, \Sigma(\mu_w) = \Sigma(\mu_v), \mathbb{L}_{\mu_w}(0) = \mathbb{L}_{\mu_v}(\epsilon) \\ \Sigma(\mu_{u_4}) &= \Sigma(\mu_w) - 2\mathbb{L}_{\mu_w}(0) = \Sigma(\mu_v) - 2\mathbb{L}_{\mu_v}(\epsilon) = \Sigma(\mu_{u_3}) - 2\mathbb{L}_{\mu_{u_3}}(0) - 2\mathbb{L}_{\mu_v}(\epsilon) \\ &= \Sigma(\mu_{u_3}) + 2\mathbb{L}_{\mu_v}(0) - 2\mathbb{L}_{\mu_v}(\epsilon).\end{aligned}$$

But

$$\mathbb{L}_{\mu_v}(\epsilon) - \mathbb{L}_{\mu_v}(0) = v(\epsilon) - v(0) \to 0\,,$$

due to (35). This completes the proof of (34) for Step 4.

5.5 Step 5:

In this step, we slightly differ from the recipe of [BES13]. Let I denote an interval of length θ (which is assumed to be small but fixed) centered at -1. Let α be the normalized Lebesguemeasure supported on the set I. For $\epsilon > 0$, define the measure

$$\mu_5 := (1-\epsilon)\mu_4 + \epsilon\alpha.$$

Define $u_5 = \mathbb{L}_{\mu_5}$. Via a series expansion of $\log|1+z|$ for small and large z, one checks that this has the intended effect of making the coefficients b and c in Step 5 of the [BES13] argument positive, while preserving the inequality $u_5(z) < u_5(|z|)$ for $z \notin \bar{\mathbb{R}}_+$. Thus, $\mu_5 \in \mathcal{D}$.

To see (34) for μ_5, note that $\mathbb{L}_{\mu_5} = (1-\epsilon)\mathbb{L}_{\mu_4} + \epsilon\mathbb{L}_\alpha$ and

$$\Sigma(\mu_5) = (1-\epsilon)^2\Sigma(\mu_4) + \epsilon^2\Sigma(\alpha) + 2\,\epsilon(1-\epsilon)\int \mathbb{L}_\alpha(z)\,d\mu_4(z).$$

Since I is a bounded interval, one has that $\mathbb{L}_\alpha(z)$ is uniformly bounded in z on any compact set, in particular, on the support of μ_4. Letting $\epsilon \to 0$, one gets (34) for μ_5.

This completes the proof of Proposition 3.8. ■

6 Conditioning on all zeros being real

One notes from the expression for the density (5) in case $k = 0$ that $P(L_n \in \mathcal{M}_1(\mathbb{R}_-)) > 0$. One also notes that $\{\mu \in \mathcal{P} : \mathrm{supp}(\mu) = \mathbb{R}_-\} = \mathcal{M}_1(\mathbb{R}_-)$. Thus, one can rerun the proof of the lower bound in Theorem 1.1 replacing throughout $\mathcal{P}$ by $\mathcal{M}_1(\mathbb{R}_-)$ as a particular case of the proof in [BAG97]. One obtains that

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n(L_n \in \mathcal{M}_1(\mathbb{R}_-)) = -I_R\,,$$

and one immediately deduces Theorem 1.2 by noting that the minimizer μ_R is unique due to the strict convexity of I (applied on $\mathcal{M}_1(\mathbb{R}_-)$).

To see Theorem 1.3, we can make the transformation $x \mapsto -x$ to see that we are interested in solving the variational problem

$$\inf_{\mu\in M_1([1,\infty))} \{\int_0^\infty \log(x+1)\,d\mu(x) - \gamma \int_0^\infty \int_0^\infty \log|x-y| d\mu(x) d\mu(y)\,, \qquad (36)$$

with $\gamma = 1/2$.

A standard application of calculus of variation methods (ase.g. in [AGZ10, Lemma 2.6.2]) shows that the minimizer μ of (36) is characterized as the unique solution, for some constant C, of

$$2\gamma \mathbb{L}_{\bar{\mu}}(x) \begin{cases} = \log(x+1) + C, & \bar{\mu} - a.e. \\ > \log|x+1| + C, & x \in \mathbb{R}_+ \setminus \mathrm{supp}(\bar{\mu}). \end{cases} \tag{37}$$

One can proceed by first guessing the form of the minimizer and then verifying that it satisfies indeed (37). For $\gamma > 1/2$, this is can be achieved solving, in a compact interval, the associated Riemann-Hilbert problem, and then taking the limit $\gamma \to 1/2$. We do not detail these computations, instead presenting the *ansatz* that the minimizer in (36) has density with respect to Lebesgue measure on $[0, \infty)$ of the form

$$\psi(x) = \frac{1}{\pi(x+1)\sqrt{x}}. \tag{38}$$

We need to verify that $\psi(x)dx$ satisfies (37). Making the change of variables $w = \sqrt{x}$, we have

$$\mathbb{L}_{\bar{\mu}}(x) = \frac{2}{\pi} \int_0^\infty \frac{\log|x - w^2|}{w^2 + 1} dw.$$

Choosing the contour of integration $\mathcal{C} := \{r\}_{r=-R}^{R} \cup \{Re^{i\theta}\}_{\theta=0}^{\pi}$ for R large, and noting the pole at i, one obtains from a residue computation that $\mathbb{L}_{\bar{\mu}}(x) = \log(x+1)$ for $x \in \mathbb{R}_+$, i.e. that (37) holds with density ψ. This completes the proof of Theorem 1.3.

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