Large deviations for integer partitions

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Abstract We consider deviations from limit shape induced by uniformly distributed partitions (and strict partitions) of an integer n on the associated Young diagrams. We prove a full large deviation principle, of speed $\sqrt{n}$. The proof, based on projective limits, uses the representation of the uniform measure on partitions by means of suitably conditioned independent variables.

1 Introduction and statement of results

For any integer n, a *partition* λ of n is a collection of integers $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ such that $\sum_{i=1}^{k} n_i = n$. A partition is called *strict* if all inequalities between different n_i-s above are strict, i.e. if $n_1 > n_2 > \cdots > n_k \geq 1$. The set of all partitions of an integer n is denoted $\mathcal{P}_n$, while the set of all strict partitions of n is denoted $\mathcal{P}_n^s \subset \mathcal{P}_n$. The set of all partitions (strict partitions) is denoted $\mathcal{P}$ ($\mathcal{P}^s$), i.e. $\mathcal{P} = \cup_n \mathcal{P}_n$ and $\mathcal{P}^s = \cup_n \mathcal{P}_n^s$.

An alternative description of any partition $\lambda \in \mathcal{P}$ is obtained by defining the sequence of integers $\{r_k\}_{k=1}^{\infty}$ such that $r_k = \ell$ if exactly ℓ elements of the partition λ equal k. Note that if λ is a strict partition then the only possible values of r_k are 0 or 1. Note also that if $\lambda \in \mathcal{P}_n$ then at most n of the r_k-s are non-zero, and $n = \sum_{k=1}^{\infty} k r_k$. It is clear that the map from $\mathcal{P}_n$ to the sequences $\{r_k\}_{k=1}^{\infty}$ satisfying the above constraints

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is a bijection. This map also allows one to obtain a graphical description of a partition, simply by defining

$$\varphi_\lambda(t) := \sum_{k=\lceil t \rceil}^{\infty} r_k\,, \quad t > 0.$$

Note that by definition, $\varphi_\lambda(\cdot)$ is a monotone decreasing, piecewise constant function of t, and that if $\lambda \in \mathcal{P}_n$ then $n = \int_0^\infty \varphi_\lambda(t)dt$. We refer to such graphical description of a partition as a *Young diagram* (it is also sometimes called a Ferrers diagram).

Throughout this paper, we denote by Q_n the uniform law on the (finite) set $\mathcal{P}_n$, and by Q_n^s the uniform law on $\mathcal{P}_n^s$. Our goal is to investigate certain asymptotic properties of the measures Q_n and Q_n^s, and specifically to provide large deviation statements and limit shape theorems, suggested in [17], for these measures and the associated (rescaled versions of the) functions $\varphi_\lambda(\cdot)$. That is, define $\tilde{\varphi}_n(t) = \frac{1}{\sqrt{n}}\varphi_\lambda(\lceil t\sqrt{n}\rceil)$, and with some abuse of notations, continue to denote by Q_n and Q_n^s the uniform law induced by partitions (strict partitions) on $\tilde{\varphi}_n(\cdot)$.

Certain asymptotic properties of Q_n and Q_n^s have already been studied in the literature. Erdös and Lehner [7], using generating functions techniques and detailed analysis, studied the distribution of n_1 under Q_n. More recent advances on other individual random variables connected with Q_n, including the results of [16], are nicely reviewed in [14], and we refer to the latter and to Section 2 of [9] for a description and references to these asymptotics.

Fristedt [9] introduced a construction of Q_n and Q_n^s based on the conditioning of an appropriate sequence of i.i.d. random variables (similar constructions appear also in [12]). This idea, related to classical constructions in statistical mechanics (see e.g. [11, 18]), where it is refered to as "equivalence of micro-canonical and macro-canonical ensembles", allowed him to discuss limit theorems, under Q_n, for the sequences $\{n_i/\sqrt{n} - c\log n\}_{i=1}^{o(n^{1/4})}$ with an appropriate constant c, as well as for the sequence $\{n_{k-i}/\sqrt{n}\}_{i=1}^{o(n^{1/4})}$, i.e. for small and large components of a random partition of n. These results were complemented by the analysis of Pittel [14], who considered the intermediate range too, and allowed him to confirm a conjecture made by Arratia and Tavaré [2]. The goal of the study mentioned above is to obtain vanishing bounds, in variation distance, between the law of the random vectors associated with Q_n and some limiting law. We refer to [1] for a general description of these type of results in a variety of random combinatorial structures.

Vershik [17] (see also the statement in [19, Pg. 30]), in an attempt to capture various limiting results concerning particular functionals in a unified framework, posed the question of evaluating limit shapes for $\tilde{\varphi}_n(\cdot)$, under both Q_n and Q_n^s, as well as under a variety of different random combinatorial models. These models all share the property that the measure on the combinatorial structure can be represented as being generated by independent random variables under an appropriate conditioning. In the context of partitions, Vershik notes that under Q_n, $\tilde{\varphi}_n(\cdot) \to_{n\to\infty} \Psi(\cdot)$ in the sense of uniform convergence on compacts, whereas under Q_n^s, $\tilde{\varphi}_n(\cdot) \to_{n\to\infty} \Psi^s(\cdot)$. Here,

$$\Psi(t) = -\frac{1}{\alpha}\log(1 - e^{-\alpha t}) = \int_t^\infty \frac{du}{e^{\alpha u} - 1}\,, \tag{1}$$

for $\alpha = \pi/\sqrt{6}$, while

$$\Psi^s(t) = \frac{1}{\beta}\log(1 + e^{-\beta t}) = \int_t^\infty \frac{du}{e^{\beta u} + 1}\,, \tag{2}$$

for $\beta = \pi/\sqrt{12}$. (An alternative description is that $(x, \Psi(x))$ is the curve satisfying $e^{-\alpha x} + e^{-\alpha y} = 1$, while $(x, \Psi^s(x))$ is the curve satisfying $e^{\beta y} - e^{-\beta x} = 1$.)

Our goal in this paper is to study the *Large Deviations* from the limit shape predicted by Vershik. This is a different regime from the one studied by [9] and [14]. We recall that a sequence of measures $\{\mu_n\}$ on a completely regular Hausdorff topological space $\mathcal{X}$ is said to satisfy the LDP with speed b_n and a rate function I if $I : \mathcal{X} \to [0,\infty]$ is lower semicontinuous, and for any measurable set $X \subset \mathcal{X}$,

$$-\inf_{x\in X^o} I(x) \le \liminf_{n\to\infty} b_n^{-1}\log\mu_n(X) \le \limsup_{n\to\infty} b_n^{-1}\log\mu_n(X) \le -\inf_{x\in\overline{X}} I(x)\,.$$

Here and throughout, X^o, $\overline{X}$ and X^c denote, respectively, the interior, the closure, and the complement of a set $X \subset \mathcal{X}$. The rate function I is called *good* if the sets $I^{-1}[0,b]$ are compact for all $b<\infty$. Our general reference for the existence of an LDP and its consequences is [6]. We also recall from [6] that a sequence of measures $\{\mu_n\}$ is called *exponentially tight* (with speed b_n) if there exist compact sets K_L such that

$$\lim_{L\to\infty}\limsup_{n\to\infty} b_n^{-1}\log\mu_n(K_L^c) = -\infty\,.$$

For $I\subset[0,\infty)$ an interval, let $D(I)$ denote the space of all functions $f : I\to \mathbb{R}$ that are left-continuous and of right limits. Let DF denote the subset of $D(0,\infty)$ consisting of non-increasing functions with $\lim_{t\to\infty} f(t) = 0$ and $f(t)<\infty$ for all $t>0$, and let $\widehat{DF}$ denote the collection of all $f : (0,\infty)\to[0,\infty)$ non-increasing, such that $f = g$ almost everywhere for some $g\in DF$. Using the representation $f(t) = \mu([t,\infty)) < \infty$ for some positive measure μ on $(0,\infty)$, one has for every $f \in DF$ the Lebesgue decomposition $f(t) = f_{ac}(t) + f_s(t)$, where $f_{ac}(\cdot)$ denotes the absolutely continuous part of f and $f_s(\cdot)$ denotes its singular component. The decomposition $f(t) = f_{ac}(t) + f_s(t)$ for $f\in\widehat{DF}$ then corresponds to the Lebesgue decomposition of the element of DF to which f equals almost everywhere. Let $\mathcal{AC}_\infty$ denote the subset of $D([0,\infty))$ consisting of non-increasing absolutely continuous functions $f(\cdot)$ satisfying $\lim_{t\to\infty} f(t) = 0$ (and hence $f(t) = \int_t^\infty(-\dot f(u))du$). For any $L>0$, let $\mathcal{AC}_\infty^{[-L,0]}$ denote the subset of $\mathcal{AC}_\infty$ consisting of functions with derivative belonging Lebesgue-a.e. to the interval $[-L,0]$.

Our main result concerning strict partitions is the following large deviations principle:

Theorem 1 *Under the laws Q_n^s, the random variables $\tilde\varphi_n(\cdot)$ satisfy the LDP in $D[0,\infty)$ (equipped with the topology of uniform convergence), with speed $\sqrt{n}$ and good rate function*

$$I^s(f) = \begin{cases} \beta(1-\int_0^\infty t(-\dot f(t))dt) + \int_0^\infty H(-\dot f(t)| - \dot\Psi^s(t))dt, & f\in\mathcal{AC}_\infty^{[-1,0]}, \int_0^\infty t(-\dot f(t))dt\le 1\,,\\ \infty, & \text{otherwise}\,.\end{cases} \tag{3}$$

(Recall that $\beta = \pi/\sqrt{12}$, and see (4) below for an alternative expression for $I^s(f)$).

Here and throughout,

$$H(x|p) = \begin{cases} x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}\,, & x\in[0,1],\ p\in(0,1)\,,\\ \infty\,, & \text{otherwise}\,,\end{cases}$$

denotes the relative entropy of x with respect to p.

Remark 1 *An alternative expression for $I^s(f)$ when $f\in\mathcal{AC}_\infty^{[-1,0]}$ and $\int_0^\infty(-t)df(t)\le 1$, is*

$$I^s(f) = 2\beta - \int_0^\infty h(-\dot f_{ac}(t))dt\,, \tag{4}$$

where $h(x) = -x\log x - (1-x)\log(1-x)\ge 0$ for $x\in[0,1]$ is the binary entropy of x.

Indeed,

$$H(x|p) = -h(x) + x\log(p^{-1}-1) - \log(1-p),$$

while $\log((-\dot\Psi^s(t))^{-1}-1) = \beta t$, $-\log(1+\dot\Psi^s(t)) = \beta\Psi^s(t)$, and

$$\int_0^\infty \Psi^s(t)dt = \int_0^\infty t(-\dot\Psi^s(t))dt = \beta^{-2}\int_0^\infty \frac{udu}{e^u+1} = 1\,. \tag{5}$$

Note that $I^s(f)=0$ only if $f \in \mathcal{AC}_\infty^{[-1,0]}$ and $\dot f(t) = \dot\Psi^s(t)$ for a.e. $t \in (0,\infty)$, implying that $f(\cdot) = \Psi^s(\cdot)$. By (5), $\int_0^\infty t(-\dot\Psi^s(t))dt = 1$, hence $\Psi^s(\cdot)$ of (2) is the unique function f for which $I^s(f) = 0$. Thus, the concentration described in [17] is a consequence of Theorem 1.

Analogous results, albeit in a weaker topology, hold also in the non-strict case.

Theorem 2 *Under the laws Q_n, the random variables $\tilde\varphi_n(\cdot)$ satisfy the LDP in $\widehat{DF}$ (equipped with the topology of pointwise convergence), with speed $\sqrt{n}$ and good rate function*

$$I(f) = \begin{cases} \alpha(1-\int_0^\infty t(-\dot f_{ac})(t)dt) + \int_0^\infty \widehat{H}(-\dot f_{ac}(t)|-\dot\Psi(t))dt\,, & \int_0^\infty(-t)df(t) \le 1, \\ \infty, & \text{otherwise.}\end{cases} \tag{6}$$

(Recall that $\alpha = \pi/\sqrt{6}$, and see (7) for an alternative expression for $I(f)$).

Here and throughout,

$$\widehat{H}(x|p) = \begin{cases} x\log\frac{x}{p} - (1+x)\log\frac{1+x}{1+p}\,, & x \ge 0,\ p>0\,, \\ \infty\,, & \text{otherwise}\,.\end{cases}$$

Remark 2 *An alternative expression for $I(f)$ when $f \in \widehat{DF}$ and $\int_0^\infty(-t)df(t) \le 1$, is*

$$I(f) = 2\alpha - \int_0^\infty \left(1-\dot f_{ac}(t)\right) h\left(\frac{-\dot f_{ac}(t)}{1-\dot f_{ac}(t)}\right) dt\,. \tag{7}$$

Indeed,

$$\widehat{H}(x|p) = -(1+x)h(x/(1+x)) + x\log(p^{-1}+1) + \log(1+p)\,,$$

while $\log((-\dot\Psi(t))^{-1}+1) = \alpha t$, $\log(1-\dot\Psi(t)) = \alpha\Psi(t)$, and

$$\int_0^\infty \Psi(t)dt = \int_0^\infty t(-\dot\Psi(t))dt = \alpha^{-2}\int_0^\infty \frac{udu}{e^u-1} = 1\,. \tag{8}$$

Note that $I(f) = 0$ only if $f \in \widehat{DF}$ is such that $\int_0^\infty(-t)df(t) \le 1$ and $\dot f_{ac}(t) = \dot\Psi(t)$ for a.e. $t \in (0,\infty)$, implying that $f_{ac}(\cdot) = \Psi(\cdot)$. By (8), $\int_0^\infty(-t)\dot\Psi(t)dt = 1$, hence $\Psi(\cdot)$ of (1) is the unique function f for which $I(f)=0$, providing here too an alternative proof of the concentration phenomenon described in [17].

Remark 3 *The symmetry in the problem becomes even more apparent if one writes the rate function in terms of parametrized curves: indeed, let $f \in \widehat{DF}$ with $\int_0^\infty(-t)df(t) \le 1$ be parametrized by $-\infty < s < \infty$ as $(x(s),y(s))$, with $x(\cdot)$ and $-y(\cdot)$ increasing. Then,*

$$I(f) = 2\alpha - \left(\int h(-y'/(x'-y'))dy + \int h(x'/(x'-y'))dx\right)\,.$$

While working on this paper, we saw an announcement of related LDP for the non-strict case due to Blinovsky [5], with the rate function defined as in (7).

In Section 2, we recall the construction of Q_n and Q_n^s based on conditioning a sequence of independent variables, and in Propositions 1 and 2 state the LDP results for the *unconditional* independent variables. Propositions 1 and 2 are proved in Sections 3 and 4 respectively. Section 5 presents the *area transformation* and applies it to deduce Theorems 1 and 2 out of the corresponding Propositions.

The techniques developed in this paper apply to many of the different problems mentioned e.g. in [17] under the name *multiplicative statistics*. While we do not develop all the details for all such possible applications here, we provide in Section 6 a few representative examples and a discussion placing these result in a more general context.

2 Independent Variables Representation

A key in obtaining our LDP's is the following alternative description of the measures Q_n and Q_n^s on $\mathcal{P}_n$ and $\mathcal{P}_n^s$, due to Fristedt [9], whose usefulness was already pointed out in [17], c.f. also [1],[14]. Similar constructions for other combinatorial structures are known, see e.g. [15], [12] and [1].

Let the probability spaces $(\Omega, \mathcal{F}, \mathbb{P}_x)$ and $(\Omega, \mathcal{F}, \mathbb{P}_x^s)$ be composed of independent, real valued, random variables R_k, $k = 1, 2, \ldots$, with k-dependent laws parametrized by a parameter $x \in (0, 1)$ as follows:

ST: *Strict ensembles*: $R_k \in \{0, 1\}$, and $\mathbb{P}_x^s(R_k = 1) = x^k/(1 + x^k)$ for $k \geq 1$.

NS: *General ensembles*: $R_k \in \mathbb{Z}_+$, with $\mathbb{P}_x(R_k = \ell) = (1 - x^k)x^{\ell k}, \ell \in \mathbb{Z}_+, k \geq 1$.

We fix throughout $R_0 = 0$. Note that by the Borel-Cantelli lemma,

$$\mathbb{P}_x(R_k = 1 \text{ infinitely often}) = 0 \qquad \text{since} \qquad \sum_{k=1}^{\infty} \mathbb{P}_x(R_k \neq 0) \leq \sum_{k=1}^{\infty} x^k < \infty,$$

implying that $\mathbb{P}_x$-a.s., $K_{\max} := \max\{k : R_k \neq 0\} < \infty$. Analogous statements hold also under $\mathbb{P}_x^s$.

We shall define the following random variables:

$$N = \sum_{k=1}^{\infty} kR_k, \quad \varphi(i) = \sum_{k=i}^{\infty} R_k, \quad i = 1, 2, \ldots .$$

Note that $N < \infty$ $\mathbb{P}_x$-a.s. (and also $\mathbb{P}_x^s$-a.s), and $\varphi(i)$ is a monotone non-increasing sequence with $\varphi(1) < \infty$ a.s. and $\varphi(i) = 0, \forall i > K_{\max}$. Thus, each sequence $\{R_k\}_{k=1}^{\infty}$ defines a (random) partition $\lambda \in \mathcal{P}_N$, which further belongs to $\mathcal{P}_N^s$ in case ST.

We shall call N the area of λ, $K_{\max}$ the length of λ and $\varphi(0) = \varphi(1)$ the height of λ. We fix throughout $\Delta_n = n^{-1/2}$. We observe that $\mathbb{P}_x^s$ induces on any particular strict partition λ_o a (positive) probability

$$\mathbb{P}_x^s(\lambda = \lambda_o) = \prod_{k=1}^{\infty} \frac{x^{kR_k}}{(1 + x^k)} = x^{N(\lambda_o)}/Z^s(x), \tag{9}$$

where $Z^s(x) = \prod_{k=1}^{\infty}(1 + x^k) \in (1, \infty)$ for any $x \in (0, 1)$. Moreover, conditional upon $N(\lambda_o) = n$, all such λ_o have the same probability. Thus, with $\tilde{C}_n$ denoting the number of strict partitions of n, we have that the partition function $Z^s(x) = \sum_{n=0}^{\infty} \tilde{C}_n x^n$ and $\mathbb{P}_x^s(\lambda = \lambda_o | N(\lambda) = n) = \tilde{C}_n^{-1}$. Therefore, $\mathbb{P}_x^s$ induces on $\mathcal{P}_n^s$ the law Q_n^s, regardless of the values of $x \in (0,1)$ and $n \in \mathbb{Z}_+$. This random combinatorial structure is thus a particular case of a *selection*, see [10] for more on this topic.

Similar statements hold also for $\mathbb{P}_x$ where

$$\mathbb{P}_x(\lambda = \lambda_o) = x^{N(\lambda_o)}/Z(x) \tag{10}$$

with the partition function $Z(x) = (\prod_{k=1}^{\infty}(1-x^k))^{-1} \in (1,\infty)$ for any $x \in (0,1)$ (this is a particular case of a *multiset*, c.f. [10]). Thus, $\mathbb{P}_x$ induces on $\mathcal{P}_n$ the law Q_n by conditioning on the event $\{N(\lambda_o) = n\}$, that is, $\mathbb{P}_x(\lambda = \lambda_o | N(\lambda) = n) = C_n^{-1}$ with C_n denoting the number of partitions of n and $Z(x) = \sum_{n=0}^{\infty} C_n x^n$.

By an appropriate (n-dependent) choice of x, the event $\{N(\lambda_o) = n\}$ can be made into a "typical" event for $\mathbb{P}_x$ ($\mathbb{P}_x^s$), hence information about random variables under $\mathbb{P}_x$ ($\mathbb{P}_x^s$), can yield the corresponding consequences under Q_n (Q_n^s, respectively).

This idea is related to classical constructions in statistical mechanics and the mathematical literature concerning the Bose-Einstein and Fermi-Dirac models of ideal gas (see e.g. [11]): the distributions $\mathcal{P}_x$ and $\mathcal{P}_x^s$ are "macro-canonical ensembles", the conditional laws Q_n and Q_n^s are "micro-canonical ensembles", and assertions of equivalence of different asymptotics as $n \to \infty$ and $x_n \to 1$ are refered to as "equivalence of micro-canonical and macro-canonical ensembles". For more on this topic, see [18].

We proceed next to produce some easy consequences of the independent variables representation, followed by the statement of the corresponding LDP.

Set $x_n = (1 - \beta/\sqrt{n})$, $n > \beta^2$ for $\beta = \pi/\sqrt{12}$ as in (2). Then, for $a_n = x_n^{-\sqrt{n}} \downarrow e^\beta$,

$$\mathbb{E}_{x_n}^s(n^{-1}N) = \sum_{k=1}^{\infty} \Delta_n \frac{k\Delta_n}{1 + x_n^{-k}} := f_n^s(a_n)$$

where for $\eta \in (1,\infty)$, the function $f_n^s(\eta)$ is the Riemannian sum of $f_\infty^s(\eta) := \int_0^\infty t(1+\eta^t)^{-1}dt$ at the points $k\Delta_n$, $k = 1, 2, \ldots$. By (5), $f_\infty^s(e^\beta) = 1$, implying by monotonicity and continuity of $f_\infty^s(\cdot)$ that

$$\lim_{n\to\infty} \mathbb{E}_{x_n}^s(n^{-1}N) = \lim_{n\to\infty} f_n^s(a_n) = 1 \, .$$

Similarly, with $\mathrm{Var}_{x_n}^s$ denoting the variance under the measure $\mathbb{P}_{x_n}^s$, as $n \to \infty$,

$$\sqrt{n}\ \mathrm{Var}_{x_n}^s(n^{-1}N) = \sum_{k=1}^{\infty} \Delta_n \frac{(k\Delta_n)^2 x_n^k}{(1+x_n^k)^2} \ \to \ \int_0^\infty t^2 e^{\beta t}(1+e^{\beta t})^{-2}dt \ \in (0,\infty),$$

implying by Chebychev's inequality that for any $\delta > 0$,

$$\limsup_{n\to\infty}\ \mathbb{P}_{x_n}^s(|n^{-1}N - 1| > \delta) \ \le \frac{1}{\delta^2}\ \limsup_{n\to\infty}\ \{\mathrm{Var}_{x_n}^s(n^{-1}N) + (\mathbb{E}_{x_n}^s(n^{-1}N) - 1)^2\} = 0$$

(c.f. [9] for better bounds, which are not needed for this work). By similar arguments, as $n \to \infty$ also

$$\mathbb{E}_{x_n}^s(\frac{\varphi(0)}{\sqrt{n}}) = \sum_{k=1}^{\infty}\ \Delta_n\ \frac{x_n^k}{1+x_n^k} \to \int_0^\infty\ \frac{dt}{1+e^{\beta t}} := \mu \in (0,\infty)$$

and

$$\sqrt{n}\ \mathrm{Var}_{x_n}^s(\frac{\varphi(0)}{\sqrt{n}}) = \sum_{k=1}^{\infty}\ \Delta_n\ \frac{x_n^k}{(1+x_n^k)^2}\ \to\ \int_0^\infty\ \frac{dte^{\beta t}}{(1+e^{\beta t})^2} \in (0,\infty),$$

so that

$$\limsup_{n\to\infty} \mathbb{P}^s_{x_n}\big(\,|\frac{\varphi(0)}{\sqrt{n}} - \mu\,| > \delta\big) = 0.$$

In conclusion, for $x_n = 1 - \beta/\sqrt{n}$ and any $\delta > 0$,

$$\lim_{n\to\infty} \mathbb{P}^s_{x_n}\big(\,|\frac{\varphi(0)}{\sqrt{n}} - \mu| \le \delta, \quad |n^{-1}N - 1| \le \delta\big) \;= 1\,.$$

The statements for $\mathbb{P}_x$ are similar, taking in this case $x_n = (1-\alpha/\sqrt{n})$, $n > \alpha^2$ for $\alpha = \pi/\sqrt{6}$ as in (1), so that $a_n = x_n^{-\sqrt{n}} \downarrow e^{\alpha}$. Since $\mathbb{E}_x(R_k) = \frac{x^k}{1-x^k}$ for all $k \ge 1, x \in (0,\infty)$, we now get

$$\mathbb{E}_{x_n}(n^{-1}N) = \sum_{k=1}^{\infty} \Delta_n \frac{k\Delta_n x_n^k}{1-x_n^k} := f_n(a_n)$$

where $f_n(\eta)$ is the Riemannian sum of $f_\infty(\eta) := \int_0^\infty t(\eta^t - 1)^{-1}dt$. By (8), $f_\infty(e^\alpha) = 1$, implying by monotonicity and continuity of $f_\infty(\cdot)$ that also

$$\lim_{n\to\infty} \mathbb{E}_{x_n}(n^{-1}N) = \lim_{n\to\infty} f_n(a_n) = 1\,.$$

Similarly, with $\mathrm{Var}_x(R_k) = \frac{x^k}{(1-x^k)^2}$, for all $k \ge 1, x \in (0,1)$, it follows that as $n \to \infty$

$$\sqrt{n}\mathrm{Var}_{x_n}(n^{-1}N) = \sum_{k=1}^{\infty} \Delta_n \frac{(k\Delta_n)^2 x_n^k}{(1-x_n^k)^2} \quad\to\quad \int_0^\infty t^2 e^{\alpha t}(e^{\alpha t}-1)^{-2}dt \;\in (0,\infty)\,.$$

Consequently, by Chebychev's inequality, we conclude that for all $\delta > 0$,

$$\limsup_{n\to\infty} \mathbb{P}_{x_n}(|n^{-1}N - 1| < \delta) = 0.$$

However, note that

$$\mathbb{E}_{x_n}(\frac{\varphi(0)}{\sqrt{n}}) = \sum_{k=1}^{\infty} \Delta_n \frac{x_n^k}{1-x_n^k} \to \int_0^\infty \frac{dt}{e^{\alpha t}-1} = \infty\,,$$

in contrast with the case of strict partitions considered earlier. In fact, it is not hard to check that

$$\lim_{n\to\infty} \mathbb{E}_{x_n}(\frac{\varphi(0)}{\sqrt{n}\log n}) = \frac{1}{2\alpha} \in (0,\infty)\,,$$

and one can use this computation to rederive and extend the results of [7] (c.f. [9] for details).

Turning to state the large deviations results, let $\dot\varphi_n(t) = -R_k$ for $t \in (\Delta_n(k-0.5), \Delta_n(k+0.5))\,,\ k \ge 0$. Consider the piecewise linear approximation to the points $(k\Delta_n, \varphi(k)\Delta_n)$ defined via

$$\varphi_n(t) = \int_t^\infty (-\dot\varphi_n(u))du.$$

Note that $\tilde\varphi_n(t) = \varphi_n(t)$ for $t = \Delta_n(k-0.5)$, $k \ge 1$. Under $\mathbb{P}^s_{x_n}$ conditioned on $N = n$, the law on $\tilde\varphi_n(\cdot)$ is just Q^s_n. Thus, the next proposition is key to the proof of Theorem 1.

Proposition 1 *For $x_n = 1-\beta/\sqrt{n}$, under the law $\mathbb{P}^s_{x_n}$, the sequence $(n^{-1}N, \varphi_n(\cdot))$ satisfies the LDP in $\mathbb{R}\times C[0,\infty)$ with the topology of uniform convergence, speed $\sqrt{n}$, and good rate function*

$$\widehat{I}^s(\nu,f) = \begin{cases} \beta(\nu - \int_0^\infty t(-\dot f(t))dt) + \int_0^\infty H(-\dot f(t)| - \dot\Psi^s(t))dt\,, & f\in \mathcal{AC}_\infty^{[-1,0]}, \int_0^\infty t(-\dot f(t))dt \le \nu, \\ \infty\,, & \textit{otherwise}. \end{cases} \tag{11}$$

Moreover, the sequence $(n^{-1}N, \tilde\varphi_n(\cdot))$ satisfies the same LDP in $\mathbb{R}\times D[0,\infty)$.

The occurence of the entropy $H(\cdot|\cdot)$ is due to the underlying Bernoulli random variables in the representation, common to random combinatorial structures which are selections.

Under $\mathbb{P}_{x_n}$ conditioned on $N=n$, the law on $\tilde\varphi_n(\cdot)$ is just Q_n, hence the next proposition is key to the proof of Theorem 2.

Proposition 2 *Equipp $\widehat{DF}$ with the topology of pointwise convergence on $(0,\infty)$. Then, for $x_n = 1-\alpha/\sqrt{n}$, under the law $\mathbb{P}_{x_n}$, the sequence $(n^{-1}N, \tilde\varphi_n(\cdot))$ satisfies the LDP in $\mathbb{R}\times\widehat{DF}$ with speed $\sqrt{n}$ and good rate function*

$$\widehat{I}(\nu,f) = \begin{cases} \alpha(\nu - \int_0^\infty t(-\dot f_{ac})(t)dt) + \int_0^\infty \widehat{H}(-\dot f_{ac}(t)| - \dot\Psi(t))dt\,, & \int_0^\infty (-t)df(t) \le \nu, \\ \infty, & \text{otherwise.} \end{cases} \tag{12}$$

The occurence of the entropy $\widehat{H}(\cdot|\cdot)$ is due to the underlying independent Geometric random variables in the representation, as is the case for any multiset.

3 Proof of Proposition 1

Taking $x_n = 1-\beta/\sqrt{n}$ and $n>\beta^2$ throughout this section, the first step of the proof is to show that the sequence $(\tilde\varphi_n(t_1),\ldots,\tilde\varphi_n(t_m), n^{-1}N)$ satisfies the LDP of speed $\sqrt{n}$ in $\mathbb{R}^{m+1}$ under the law $\mathbb{P}^s_{x_n}(\cdot)$, for any fixed $m<\infty$ and $0\le t_1<t_2\ldots<t_m<\infty$.

Lemma 1 *The sequence of random vectors $(\tilde\varphi_n(t_1),\ldots,\tilde\varphi_n(t_m), n^{-1}N)$ satisfies the LDP in $\mathbb{R}^{m+1}$ with speed $\sqrt{n}$ and good rate function $\Lambda^*_{\underline{t}}(y_1-y_2,\ldots,y_{m-1}-y_m,y_m,\nu)$. Here,*

$$\Lambda^*_{\underline{t}}(\eta_1,\ldots,\eta_m,\eta_{m+1}) = \sup_{\lambda_i,\theta\in\mathbb{R}}\left(\sum_{i=1}^m \lambda_i\eta_i + \theta\eta_{m+1} - \Lambda_{\underline{t}}(\lambda_1,\ldots,\lambda_m,\theta)\right) \tag{13}$$

with

$$\Lambda_{\underline{t}}(\lambda_1,\ldots,\lambda_m,\theta) = \begin{cases} \int_0^\infty \log[\frac{1+e^{-(\beta-\theta)t}e^{\lambda(t)}}{1+e^{-\beta t}}]dt\,, & \theta<\beta \\ \infty\,, & \theta\ge\beta\,, \end{cases}$$

where $\lambda(t) := \sum_{i=1}^m \lambda_i 1_{t\in[t_i,t_{i+1})}$, $t_{m+1}=\infty$.

Proof of Lemma 1: Let $t_0=0$, $\lambda_0=0$ and $a_n = x_n^{\sqrt{n}}$. Fixing $\lambda_i,\theta\in\mathbb{R}$, define

$$\Lambda_n(\lambda_1,\ldots,\lambda_m,\theta) = \frac{1}{\sqrt{n}}\log\mathbb{E}^s_{x_n}[\exp\sqrt{n}\{\sum_{i=1}^{m-1}\lambda_i(\tilde\varphi_n(t_i)-\tilde\varphi_n(t_{i+1})) + \lambda_m\tilde\varphi_n(t_m) + \theta n^{-1}N\}]\,. \tag{14}$$

By the independence of R_k under $\mathbb{P}^s_{x_n}$ we have (recalling that $\Delta_n = 1/\sqrt{n}$),

$$\begin{aligned}
\Lambda_n(\lambda_1,\dots,\lambda_m,\theta) &= \sum_{i=0}^{m} \Delta_n \sum_{k=\lceil t_i/\Delta_n\rceil}^{\lceil t_{i+1}/\Delta_n\rceil -1} \log \mathbb{E}^s_{x_n}[\exp((\lambda_i+\theta k\Delta_n)R_k)] \\
&= \sum_{k=1}^{\infty} \Delta_n \sum_{i=0}^{m} \log(\frac{1+x_n^k e^{\theta k\Delta_n} e^{\lambda_i}}{1+x_n^k}) 1_{\{t_{i+1}>k\Delta_n\geq t_i\}} \\
&= \sum_{k=1}^{\infty} \Delta_n \log(1+(a_n e^{\theta})^{k\Delta_n} e^{\lambda(k\Delta_n)}) - \sum_{k=1}^{\infty} \Delta_n \log(1+a_n^{k\Delta_n}) \\
&:= h_n(a_n e^{\theta},\lambda) - h_n(a_n,0) \;. \qquad (15)
\end{aligned}$$

For $\eta \in [0,\infty)$, the function $h_n(\eta,\lambda)$ is the Riemannian sum of

$$h_\infty(\eta,\lambda) := \int_0^\infty \log(1+\eta^t e^{\lambda(t)})dt$$

at the points $k\Delta_n$, $k=1,2,\dots$, with $\eta \mapsto h_\infty(\eta,\lambda)$ monotone increasing, and $h_\infty(\eta,\lambda) < \infty$ if and only if $\eta < 1$. Assume first that $\theta < \beta$, in which case $a_n e^{\theta} \nearrow e^{-(\beta-\theta)} < 1$, implying that

$$\lim_{n\to\infty} [h_n(a_n e^{\theta},\lambda) - h_n(a_n,0)] = h_\infty(e^{-(\beta-\theta)},\lambda) - h_\infty(e^{-\beta},0) = \Lambda_{\underline{t}}(\lambda_1,\dots,\lambda_m,\theta) \;.$$

In case $\theta = \beta > 0$, we have $a_n e^{\theta} \nearrow 1$, so by monotonicity,

$$\liminf_{n\to\infty} h_n(a_n e^{\theta},\lambda) \geq \lim_{\eta\uparrow 1} \liminf_{n\to\infty} h_n(\eta,\lambda) = \lim_{\eta\uparrow 1} h_\infty(\eta,\lambda) = \infty \;,$$

whereas $h_n(a_n,0) \to h_\infty(e^{-\beta},0) < \infty$. Consequently, by (15), $\Lambda_n(\lambda_1,\dots,\lambda_m,\beta) \to \infty$ as $n\to\infty$. Since $\theta \mapsto \Lambda_n(\lambda_1,\dots,\lambda_m,\theta)$ is monotone increasing, we conclude that for any $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} \Lambda_n(\lambda_1,\dots,\lambda_m,\theta) = \Lambda_{\underline{t}}(\lambda_1,\dots,\lambda_m,\theta) \;. \qquad (16)$$

Note that $\Lambda_{\underline{t}}(\cdot)$ is finite and differentiable throughout the set $\mathcal{D}^o_\Lambda = \{(\lambda_1,\dots,\lambda_m,\theta) : \theta < \beta\}$ which contains an open neighborhood of the origin. Moreover, it is easy to check that $\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\dots,\lambda_m^{(\ell)},\theta^{(\ell)}) \to \infty$ as $\ell \to \infty$, whenever $\lambda_i^{(\ell)} \to \lambda_i \in \mathbb{R}$, $i=1,\dots,m$ and $\theta^{(\ell)} \uparrow \beta$. In particular, due to the convexity of $\Lambda_{\underline{t}}(\cdot)$, $|\nabla\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\dots,\lambda_m^{(\ell)},\theta^{(\ell)})| \to \infty$ for $(\lambda_1^{(\ell)},\dots,\lambda_m^{(\ell)},\theta^{(\ell)}) \in \mathcal{D}^o_\Lambda$, hence $\Lambda_{\underline{t}}(\cdot)$ is a lower semi-continuous, steep function. Thus, in view of (16), by the Gärtner-Ellis theorem (see [6, Theorem 2.3.6]), the sequence $(\tilde{\varphi}_n(t_1)-\tilde{\varphi}_n(t_2),\dots,\tilde{\varphi}_n(t_m),n^{-1}N)$ satisfies the LDP of speed $\sqrt{n}$ and good rate function $\Lambda^*_{\underline{t}}(\cdot)$. The stated LDP then follows by applying the contraction principle (see [6, Theorem 4.2.1]) for the continuous bijection $(z_1,\dots,z_m,\nu) \mapsto (\sum_{i=1}^m z_i,\dots,z_{m-1}+z_m,z_m,\nu)$ on $\mathbb{R}^{m+1}$. □

Observe that

$$\| \varphi_n - \tilde{\varphi}_n \|_\infty := \sup_{t\in[0,\infty)} |\varphi_n(t)-\tilde{\varphi}_n(t)| \leq \frac{1}{2\sqrt{n}} \sup_k |R_k| \leq \frac{1}{2\sqrt{n}}. \qquad (17)$$

In particular, by Lemma 1 and the exponential equivalence (17), the vectors $(\varphi_n(t_1),\dots,\varphi_n(t_m),n^{-1}N)$ also satisfy the LDP in $\mathbb{R}^{m+1}$ with speed $\sqrt{n}$ and good rate function $\Lambda^*_{\underline{t}}(y_1-y_2,\dots,y_{m-1}-y_m,y_m,\nu)$ (c.f. [6, Theorem 4.2.13]).

Let $\mathcal{Y} = \mathbb{R} \times \mathcal{X}$, where $\mathcal{X}$ denotes the space of maps from $[0, \infty)$ to $\mathbb{R}$ equipped with the topology of pointwise convergence. Then, applying the Dawson-Gärtner theorem (see [6, Theorem 4.6.1]), we see that under the laws $\mathbb{P}^s_{x_n}(\cdot)$, the sequence $(n^{-1}N, \varphi_n(\cdot))$ satisfies the LDP of speed $\sqrt{n}$ in $\mathcal{Y}$ with the good rate function

$$\begin{aligned} I_{\mathbb{R}\times\mathcal{X}}(\nu, f) &= \sup_{\substack{0\le t_1\dots<t_m<\infty\\ m<\infty}} \sup_{\substack{\lambda_1\dots\lambda_m\\ \theta<\beta}} \sum_{i=1}^{m-1} \lambda_i(f(t_i) - f(t_{i+1})) + \lambda_m f(t_m) + \theta\nu - \Lambda_{\underline{t}}(\lambda_1 \dots \lambda_m, \theta) \\ &= \sup_{\theta<\beta}\{\theta\nu - \int_0^\infty \log(\frac{1+e^{-(\beta-\theta)t}}{1+e^{-\beta t}})dt + I_{\mathcal{X},(\beta-\theta)}(f)\}, \end{aligned} \tag{18}$$

where for $r > 0$ (with the convention $f(\infty) = 0$),

$$I_{\mathcal{X},r}(f) = \sup_{\substack{t_0=0\le t_1\dots<t_m<t_{m+1}=\infty\\ \lambda_0=0,\lambda_1\dots\lambda_m, m<\infty}} \sum_{i=0}^{m} \left\{\lambda_i(f(t_i) - f(t_{i+1})) - \int_{t_i}^{t_{i+1}} \log(\frac{e^{rt}+e^{\lambda_i}}{1+e^{rt}})dt\right\}. \tag{19}$$

Aiming to identify the rate function $I_{\mathbb{R}\times\mathcal{X}}(\cdot)$ we next provide alternative expressions for the functions $I_{\mathcal{X},r}(\cdot)$.

Lemma 2 *Fix $r > 0$ and let*

$$I_r(f) := \begin{cases} \int_0^\infty H(-\dot{f}(t)|\frac{1}{1+e^{rt}})dt, & f \in \mathcal{AC}_\infty^{[-1,0]}, \\ \infty, & \textit{otherwise}. \end{cases}$$

Then, $I_{\mathcal{X},r}(f) = I_r(f)$ for any $f \in \mathcal{X}$.

Proof of Lemma 2: Since $H(x|p) \ge \lambda x - \log(1 - p + pe^\lambda)$ for any $\lambda \in \mathbb{R}, x \in \mathbb{R}, p \in (0,1)$ it follows that for any $f \in \mathcal{AC}_\infty^{[-1,0]}$, $\lambda_i \in \mathbb{R}$ and $0 \le t_i < t_{i+1} < \infty$, we have

$$\int_{t_i}^{t_{i+1}} H(-\dot{f}(t)|\frac{1}{1+e^{rt}})dt \ge \lambda_i(f(t_i) - f(t_{i+1})) - \int_{t_i}^{t_{i+1}} \log(\frac{e^{rt}+e^{\lambda_i}}{1+e^{rt}})dt. \tag{20}$$

Since $\lim_{t\to\infty} f(t) = 0$ for $f \in \mathcal{AC}_\infty^{[-1,0]}$, the above inequality holds also for $t_{i+1} \uparrow \infty$. Hence, comparing (20) and (19), we see that $I_r(f) \ge I_{\mathcal{X},r}(f)$ for any $f \in \mathcal{X}$.

To prove the converse inequality, suppose first that $f(\cdot)$ is not absolutely continuous. Note that $\log(\frac{e^{rt}+e^{\lambda_i}}{1+e^{rt}}) \le |\lambda_i|$ for all t, λ_i and $\log(\frac{e^{rt}+e^{\lambda_i}}{1+e^{rt}}) = 0$ for $\lambda_i = 0$. Fix $\rho > 0$, $\mathcal{I} \subseteq \{1, \dots, m-1\}$, and set $\lambda_i = \rho\,\mathrm{sign}[f(t_i) - f(t_{i+1})]$ for $i \in \mathcal{I}$ and $\lambda_i = 0$ for $i \notin \mathcal{I}$. Then, (19) implies that

$$I_{\mathcal{X},r}(f) \ge \sup_{\substack{0\le t_1\dots<t_m<\infty\\ \mathcal{I}\subseteq\{1,\dots,m-1\},\rho>0}} \rho\{\sum_{i\in\mathcal{I}} |f(t_i) - f(t_{i+1})| - \sum_{i\in\mathcal{I}}(t_{i+1} - t_i)\} \tag{21}$$

Since f is not absolutely continuous, there exist $\delta > 0$, $t_i^{(n)}, \mathcal{I}_n$ such that $\sum_{i\in\mathcal{I}_n} |f(t_i^{(n)}) - f(t_{i+1}^{(n)})| \ge \delta$ while $\sum_{i\in\mathcal{I}_n}(t_{i+1}^{(n)} - t_i^{(n)}) \to 0$, implying that $I_{\mathcal{X},r}(f) \ge \rho\delta$. Taking $\rho \to \infty$ shows that $I_{\mathcal{X},r}(f) = \infty$ for any such f.

Next suppose that $f(T_n) \ge \delta$ for some $\delta > 0$ and $T_n \uparrow \infty$. Then, taking in (19) $m = 1$, $t_1 = T_n$ and arbitrary $\lambda_1 > 0$ results with

$$I_{\mathcal{X},r}(f) \ge \lambda_1\delta - \int_{T_n}^\infty \log(\frac{e^{rt}+e^{\lambda_1}}{1+e^{rt}})dt.$$

Since $\int_T^\infty \log(\frac{e^{rt}+e^\lambda}{1+e^{rt}})dt \to 0$ as $T \to \infty$, considering $n \to \infty$ we have that $I_{\mathcal{X},r}(f) \geq \lambda_1 \delta$. Taking $\lambda_1 \to \infty$ results with $I_{\mathcal{X},r}(f) = \infty$ for any such $f(\cdot)$. The same argument applies when $f(T_n) \leq \delta$ for some $\delta < 0$ and $T_n \uparrow \infty$. Thus, we see that $I_{\mathcal{X},r}(f) < \infty$ only when $\lim_{t\to\infty} f(t) = 0$.

Considering in (19) $m = 2$, $\lambda_2 = 0$ and arbitrary $\lambda_1 < 0$, $0 \leq t_1 < t_2 < \infty$, results with $I_{\mathcal{X},r}(f) \geq \lambda_1(f(t_1) - f(t_2))$. Taking $\lambda_1 \to -\infty$ we see that $I_{\mathcal{X},r}(f) = \infty$ as soon as $f(t_1) < f(t_2)$ for some $0 \leq t_1 < t_2 < \infty$. Considering (21) for $m = 2$ and $\rho \to \infty$ we have that $I_{\mathcal{X},r}(f) = \infty$ as soon as $f(t_1) - f(t_2) > (t_2 - t_1)$ for some $0 \leq t_1 < t_2 < \infty$.

In particular, by the above we have that $I_{\mathcal{X},r}(f) < \infty$ only when $f \in \mathcal{AC}_\infty^{[-1,0]}$, that is, $f(t) = \int_t^\infty g(u)du$ for some measurable $g \in [0,1]$. Fix such $f(\cdot)$, large integers k, T and let $\psi(t) = (1+e^{rt})^{-1}$. Then, considering in (19) $m = kT + 1$, with $\lambda_m = 0$ and $t_i = (i-1)/k$, $i = 1, \ldots, m$, results with

$$\begin{aligned} I_{\mathcal{X},r}(f) &\geq \sup_{\substack{\lambda_i \in \mathbb{R} \\ i=1,\ldots,kT}} \{\frac{1}{k} \sum_{i=1}^{kT} \lambda_i k \int_{\frac{i-1}{k}}^{\frac{i}{k}} g(u)du - k \int_{\frac{i-1}{k}}^{\frac{i}{k}} \log(1 - \psi(t) + \psi(t)e^{\lambda_i})dt\} \\ &\geq \sum_{i=1}^{kT} \frac{1}{k} \sup_{\lambda_i \in \mathbb{R}} \{\lambda_i k \int_{\frac{i-1}{k}}^{\frac{i}{k}} g(u)du - \log(1 - p_i + p_i e^{\lambda_i})\}, \end{aligned}$$

where $p_i = k \int_{\frac{i-1}{k}}^{\frac{i}{k}} \psi(t)dt$, and the second inequality follows by the convexity of the map $p \mapsto -\log(1-p+pe^\lambda)$. Thus, solving for the optimal values of λ_i we get

$$I_{\mathcal{X},r}(f) \geq \sum_{i=1}^{kT} \frac{1}{k} H(k \int_{\frac{i-1}{k}}^{\frac{i}{k}} g(u)du \,|\, k \int_{\frac{i-1}{k}}^{\frac{i}{k}} \psi(u)du) = \int_0^T H(g_k(t) \,|\, \psi_k(t)) \;,$$

where $g_k(t) := k \int_{\frac{[tk]}{k}}^{\frac{([tk]+1)}{k}} g(u)du$ and $\psi_k(t) := k \int_{\frac{[tk]}{k}}^{\frac{([tk]+1)}{k}} \psi(u)du$. By Lebesgue's theorem, as $k \to \infty$, both $g_k(t) \to g(t)$ and $\psi_k(t) \to \psi(t)$ for almost every $t \in [0,T]$. Hence, by Fatou's lemma and the lower semicontinuity of $H(x|p)$ with respect to $(x,p) \in [0,1]^2$,

$$I_{\mathcal{X},r}(f) \geq \liminf_{k\to\infty} \int_0^T H(g_k(t) \,|\, \psi_k(t))dt \geq \int_0^T \liminf_{k\to\infty} H(g_k(t) \,|\, \psi_k(t))dt \geq \int_0^T H(g(t) \,|\, \psi(t))dt.$$

Taking $T \uparrow \infty$, by the non-negativity of $H(\cdot|\cdot)$ we conclude that

$$I_{\mathcal{X},r}(f) \geq \int_0^\infty H(g(t)|\psi(t))dt = \int_0^\infty H(-\dot{f}(t)|\frac{1}{1+e^{rt}})dt = I_r(f)$$

for any $f \in \mathcal{AC}_\infty^{[-1,0]}$. □

By (18) and Lemma 2, we see that $I_{\mathbb{R}\times\mathcal{X}}(\nu, f) = \infty$ unless $f \in \mathcal{AC}_\infty^{[-1,0]}$. Fix $f \in \mathcal{AC}_\infty^{[-1,0]}$. Since $H(x|p) - H(x|q) = x\log(\frac{q}{1-q})(\frac{1-p}{p}) - \log(\frac{1-p}{1-q})$, we have for all r, s positive and almost every $t \in [0,\infty)$,

$$H(-\dot{f}(t)|\frac{1}{1+e^{st}}) = H(-\dot{f}(t)|\frac{1}{1+e^{rt}}) - \log(\frac{1+e^{-rt}}{1+e^{-st}}) + (-\dot{f}(t))t(s-r) \;.$$

Hence, for all $\theta < \beta$

$$\int_0^\infty H(-\dot{f}(t)|\frac{1}{1+e^{\beta t}})dt = \int_0^\infty H(-\dot{f}(t)|\frac{1}{1+e^{(\beta-\theta)t}})dt - \int_0^\infty \log(\frac{1+e^{-(\beta-\theta)t}}{1+e^{-\beta t}})dt + \theta \int_0^\infty t(-\dot{f}(t))dt$$

implying that

$$I_{\mathbb{R}\times\mathcal{X}}(\nu,f)=\sup_{\theta<\beta}\theta[\nu-\int_0^\infty t(-\dot f(t))dt]+\int_0^\infty H(-\dot f(t)|\frac{1}{1+e^{\beta t}})dt\,.$$

It thus follows that $I_{\mathbb{R}\times\mathcal{X}}(\nu,f)=\widehat{I}^s(\nu,f)$ of (11) for any $(\nu,f)\in\mathbb{R}\times\mathcal{X}$.

In particular, $(n^{-1}N,\varphi_n(\cdot))$ satisfies the LDP in $\mathbb{R}\times\mathcal{X}$ with the topology of pointwise convergence and good rate function $\widehat{I}^s(\nu,f)$ such that $\mathcal{D}_I:=\{y\in\mathcal{Y}:\widehat{I}^s(y)<\infty\}\subseteq\mathbb{R}\times C[0,\infty)$. Since $\mathbb{P}^s_{x_n}(\varphi_n(\cdot)\in C[0,\infty))=1$, we can, by [6, Lemma 4.1.5], restrict this LDP to the space $\mathbb{R}\times C[0,\infty)$. Our next goal is to strengthen the topology on $\mathbb{R}\times C[0,\infty)$ from that of pointwise convergence to that of uniform convergence. Using the inverse contraction principle [6, Corollary 4.2.6], it suffices to prove that $(n^{-1}N,\varphi_n(\cdot))$ is exponentially tight in the space $\mathbb{R}\times C[0,\infty)$ equipped with the supremum norm.

The exponential tightness of $n^{-1}N$ follows from (16) and the fact that $\Lambda_{\underline{t}}(0,\dots,0,\theta)<\infty$ for all $|\theta|<\beta$ (using the same argument as in the proof of [6, Theorem 2.4.6, part (a)]). Since $\varphi_n(\cdot)$ are Lipschitz functions of Lipschitz constant 1, it follows that $\{\varphi_n(\cdot):n\geq 1\}$ is in an equicontinuous set of functions in $C[0,\infty)$. Moreover, $\varphi_n(\cdot)\geq 0$ are monotonically non-increasing, hence $\sup_{t\geq 0}|\varphi_n(t)|=\varphi_n(0)=\tilde\varphi_n(0)$ and exponential tightness of $\{\varphi_n(\cdot)\}_n$ in $C[0,\infty)$ follows by the Arzela-Ascolli theorem from the exponential tightness of $\{\tilde\varphi_n(0)\}$ in $\mathbb{R}$. The latter is a consequence of having $\Lambda_{\underline{t}}(\lambda_1,\theta)<\infty$ for $m=1$, $t_1=0$, $\theta=0$ and any value of λ_1 (again adapting the proof of [6, Theorem 2.4.6, part (a)]).

4 Prooof of Proposition 2

Taking $x_n=1-\alpha/\sqrt{n}$ and $n>\alpha^2$ throughout this section, the first step in proving Proposition 2 is to show that the sequence $(\tilde\varphi_n(t_1),\dots,\tilde\varphi_n(t_m),n^{-1}N)$ satisfies the LDP of speed $\sqrt{n}$ in $\mathbb{R}^{m+1}$ under the law $\mathbb{P}_{x_n}(\cdot)$, for any fixed $m<\infty$ and $0<t_1<t_2\dots<t_m<\infty$.

Lemma 3 *The sequence of random vectors $(\tilde\varphi_n(t_1),\dots,\tilde\varphi_n(t_m),n^{-1}N)$ satisfies the LDP in $\mathbb{R}^{m+1}$ with speed $\sqrt{n}$ and good rate function $\Lambda^*_{\underline{t}}(y_1-y_2,\dots,y_{m-1}-y_m,y_m,\nu)$ for $\Lambda^*_{\underline{t}}(\cdot)$ of (13), now with*

$$\Lambda_{\underline{t}}(\lambda_1,\dots,\lambda_m,\theta)=\begin{cases}\int_0^\infty\log[\frac{1-e^{-\alpha t}}{1-e^{-(\alpha-\theta)t}e^{\lambda(t)}}]dt\,, & \theta<\alpha\,,\lambda_i\leq(\alpha-\theta)t_i, i=1,\dots,m\\ \infty\,, & \text{otherwise,}\end{cases}\qquad(22)$$

where $\lambda(t):=\sum_{i=1}^m\lambda_i 1_{t\in[t_i,t_{i+1})}$, $t_{m+1}=\infty$.

Proof of Lemma 3: Let $a_n=x_n^{\sqrt{n}}\uparrow e^{-\alpha}$ and fixing $\lambda_i,\theta\in\mathbb{R}$, define $\Lambda_n(\lambda_1,\dots,\lambda_m,\theta)$ as in (14), but now with $\mathbb{E}_{x_n}$ instead of $\mathbb{E}^s_{x_n}$. Let $\lambda^\theta(t)=\lambda(t)+\theta t$, and hereafter use the convention that $\log z=-\infty$ whenever $z\leq 0$. In parallel with the derivation of (15), the independence of R_k under $\mathbb{P}_{x_n}$ leads to

$$\begin{aligned}\Lambda_n(\lambda_1,\dots,\lambda_m,\theta)&=-\sum_{k=1}^\infty\Delta_n\log(1-a_n^{k\Delta_n}e^{\lambda^\theta(k\Delta_n)})+\sum_{k=1}^\infty\Delta_n\log(1-a_n^{k\Delta_n})\\&:=h_n(a_n,\lambda^\theta)-h_n(a_n,0)\,.\end{aligned}\qquad(23)$$

Note that $h_n(a_n,0)$ is the Riemannian sum of

$$h_\infty(\eta,0):=-\int_0^\infty\log(1-\eta^t)dt$$

at the points $k\Delta_n$, $k = 1, 2, \ldots$, evaluated for $\eta = a_n \uparrow e^{-\alpha}$. Since $\eta \mapsto h_\infty(\eta, 0)$ is monotone increasing, continuous and finite for $\eta \in [0, 1)$ it follows that

$$\lim_{n\to\infty} h_n(a_n, 0) = h_\infty(e^{-\alpha}, 0) = -\int_0^\infty \log(1 - e^{-\alpha t})dt \,. \tag{24}$$

If $\lambda_i > (\alpha - \theta)t_i$ for some $i = 1, \ldots, m$ then $\lambda^\theta(t_i) > \alpha t_i$ and

$$a_n^{k\Delta_n} e^{\lambda^\theta(k\Delta_n)} \geq 1$$

for $k = \lceil t_i/\Delta_n \rceil$ and all n large enough, resulting with $h_n(a_n, \lambda^\theta) = \infty$. If $\theta = \alpha$, then for $\delta > 0$, n large enough such that $a_n e^\theta \geq (1 - \delta)$ and $k < t_1/\Delta_n$,

$$-\log(1 - a_n^{k\Delta_n} e^{\lambda^\theta(k\Delta_n)}) \geq -\log(1 - (1 - \delta)^{t_1}) \,,$$

implying that

$$\liminf_{n\to\infty} h_n(a_n, \lambda^\theta) \geq \lim_{\delta\downarrow 0}\{-t_1 \log(1 - (1 - \delta)^{t_1})\} = \infty \,.$$

The monotonicity of $\theta \mapsto h_n(a_n, \lambda^\theta)$ implies that $h_n(a_n, \lambda^\theta) \to \infty$ for any $\theta \geq \alpha$. Let

$$\mathcal{D}_\Lambda := \{(\lambda_1, \ldots, \lambda_m, \theta) : \ \theta < \alpha, \ \lambda_i \leq (\alpha - \theta)t_i, \, i = 1, \ldots, m\} \,,$$

concluding in view of (23) and (24) that $\Lambda_n(\lambda_1, \ldots, \lambda_m, \theta) \to \infty$ whenever $(\lambda_1, \ldots, \lambda_m, \theta) \notin \mathcal{D}_\Lambda$.

Suppose next that $(\lambda_1, \ldots, \lambda_m, \theta) \in \mathcal{D}_\Lambda$ in which case $g(t) := (\lambda^\theta(t) - \alpha t)$ is non-positive and decreasing within each $[t_i, t_{i+1})$. Then, for all k, n,

$$a_n^{k\Delta_n} e^{\lambda^\theta(k\Delta_n)} < e^{g(k\Delta_n)} \leq \sup_{t\geq 0} e^{g(t)} \ \leq 1$$

implying that $h_n(a_n, \lambda^\theta) < \infty$ for all n and is the Riemannian sum of

$$h_\infty(\eta, \lambda^\theta) := -\int_0^\infty \log[1 - \eta^t e^{\lambda^\theta(t)}]dt$$

at the points $k\Delta_n$, $k = 1, 2, \ldots$, evaluated for $\eta = a_n \uparrow e^{-\alpha}$. Note that $\eta \mapsto h_\infty(\eta, \lambda^\theta)$ is now monotone increasing, continuous and finite for $\eta \in [0, e^{-\alpha}]$, hence

$$\limsup_{n\to\infty} h_n(e^{-\alpha}, \lambda^\theta) \geq \limsup_{n\to\infty} h_n(a_n, \lambda^\theta) \geq \liminf_{n\to\infty} h_n(a_n, \lambda^\theta) \geq \lim_{\eta\uparrow e^{-\alpha}} h_\infty(\eta, \lambda^\theta) = h_\infty(e^{-\alpha}, \lambda^\theta).$$

Since $(\lambda_1, \ldots, \lambda_m, \theta) \in \mathcal{D}_\Lambda$, it follows that $t \mapsto -\log(1 - e^{g(t)})$ is monotone decreasing within each interval $[t_i, t_{i+1})$, so for all n,

$$h_n(e^{-\alpha}, \lambda^\theta) = -\int_0^\infty \log[1 - e^{g(\lceil t/\Delta_n \rceil \Delta_n)}]dt \leq -\int_0^\infty \log[1 - e^{g(t)}]dt = h_\infty(e^{-\alpha}, \lambda^\theta) \,.$$

Consequently, in view of (23) and (24), $\Lambda_n(\lambda_1, \ldots, \lambda_m, \theta) \to \Lambda_{\underline{t}}(\lambda_1, \ldots, \lambda_m, \theta)$ of (22).

Note that $\Lambda_{\underline{t}}(\cdot)$ of (22) is finite and differentiable throughout the set $\mathcal{D}_\Lambda^o$ which contains an open neighborhood of the origin. Also, $\Lambda_{\underline{t}}(\cdot)$ is continuous throughout the set $\mathcal{D}_\Lambda$ and infinite outside this set. Fix an arbitrary sequence $(\lambda_1^{(\ell)}, \ldots, \lambda_m^{(\ell)}, \theta^{(\ell)}) \in \mathcal{D}_\Lambda$ such that $\theta^{(\ell)} \uparrow \alpha$ and $\lambda_i^{(\ell)} \to \lambda_i$ for $i = 1, \ldots, m$. Then,

$$-\int_0^{t_1} \log(1 - e^{-(\alpha - \theta^{(\ell)})t})dt \to \infty$$

implying that $\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) \to \infty$. Consequently, $\Lambda_{\underline{t}}(\cdot)$ is a lower semi-continuous function. Moreover, $|\frac{\partial}{\partial\theta}\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)})| \uparrow \infty$ whenever $(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) \in \mathcal{D}_\Lambda^o$ is such that $\theta^{(\ell)} \uparrow \alpha$ and $\lambda_i^{(\ell)} \to \lambda_i$ for $i = 1,\ldots,m$. Consider next $(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) \in \mathcal{D}_\Lambda^o$ such that $\theta^{(\ell)} \to \theta < \alpha$, $\lambda_i^{(\ell)} \to \lambda_i$, $i = 1,\ldots,m$ and $\lambda_j = (\alpha-\theta)t_j$ for some $j \in \{1,\ldots,m\}$. In this case, it does not hold true that $\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) \to \infty$. However, for any value of ℓ,

$$\frac{\partial}{\partial\lambda_j}\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) = \int_{t_j}^{t_{j+1}} \frac{dt}{e^{(\alpha-\theta^{(\ell)})t-\lambda_j^{(\ell)}}-1},$$

implying that for any $\tau \in (t_j, t_{j+1})$,

$$\liminf_{\ell\to\infty}\frac{\partial}{\partial\lambda_j}\Lambda_{\underline{t}}(\lambda_1^{(\ell)},\ldots,\lambda_m^{(\ell)},\theta^{(\ell)}) \geq \lim_{\gamma\downarrow 1}\int_{t_j}^{\tau}\frac{dt}{\gamma e^{(\alpha-\theta)(t-t_j)}-1} = \infty .$$

It thus follows that $|\nabla\Lambda_{\underline{t}}(\cdot)| \to \infty$ for any sequence of points in $\mathcal{D}_\Lambda^o$ approaching a point on the boundary of this set, that is, $\Lambda_{\underline{t}}(\cdot)$ is a steep function. The proof of Lemma 3 is completed by applying the Gärtner-Ellis theorem (see [6, Theorem 2.3.6]), as done in the proof of Lemma 1. □

Fix $\tau > 0$ and let $\mathcal{Y} = \mathbb{R}\times\mathcal{X}_\tau$, where $\mathcal{X}_\tau$ denotes the space of maps from $[\tau,\infty)$ to $\mathbb{R}$ equipped with the topology of pointwise convergence. Then, applying the Dawson-Gärtner theorem (see [6, Theorem 4.6.1]), in analogy with (18) and (19), we see that under the laws $\mathbb{P}_{x_n}(\cdot)$, the sequence $(n^{-1}N, \tilde{\varphi}_n(\cdot)\,|_{[\tau,\infty)})$ satisfies the LDP of speed $\sqrt{n}$ in $\mathcal{Y}$ with the good rate function

$$I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f) = \sup_{\theta<\alpha}\{\theta\nu + \int_\tau^\infty \log(\frac{1-e^{-(\alpha-\theta)t}}{1-e^{-\alpha t}})dt + I_{\mathcal{X}_\tau,(\alpha-\theta)}(f)\}, \tag{25}$$

where for $\tau \in (0,\infty)$ and $\gamma > 0$

$$I_{\mathcal{X}_\tau,\gamma}(f) = \sup_{\substack{\tau\leq t_1<\ldots<t_m<t_{m+1}=\infty \\ \lambda_i\leq\gamma t_i, m<\infty}} \sum_{i=1}^m\{\lambda_i(f(t_i)-f(t_{i+1})) - \int_{t_i}^{t_{i+1}}\log(\frac{1-e^{-\gamma t}}{1-e^{-\gamma t}e^{\lambda_i}})dt\} \tag{26}$$

(with the convention $f(\infty) = 0$).

Let DF_τ denote the subset of $D[\tau,\infty)$ consisting of non-increasing functions with $\lim_{t\to\infty} f(t) = 0$ and $\widehat{DF}_\tau$ the collection of all $f : [\tau,\infty) \to [0,\infty)$ non-increasing, such that $f = g$ almost everywhere for some $g \in DF_\tau$. The representation $f(t) = \mu([t,\infty))$ for $f \in DF_\tau$ and some finite positive measure μ on $[\tau,\infty)$ provides a unique decomposition of each $f \in \widehat{DF}_\tau$ to its absolutely continuous and singular components, denoted $f_{ac}(\cdot)$ and $f_s(\cdot)$, respectively.

In analogy with Lemma 2, we next provide an explicit simple expression for $I_{\mathcal{X}_\tau,\gamma}(\cdot)$.

Lemma 4 *Fix $\tau \in (0,\infty)$, $\gamma > 0$ and let*

$$I_\gamma(f) = \begin{cases} \int_\tau^\infty \widehat{H}(-\dot{f}_{ac}(t)|\frac{1}{e^{\gamma t}-1})dt + \gamma\int_\tau^\infty(-t)df_s(t), & f \in \widehat{DF}_\tau, \\ \infty, & \text{otherwise.}\end{cases} \tag{27}$$

Then, $I_{\mathcal{X}_\tau,\gamma}(f) = I_\gamma(f)$ for any $f \in \mathcal{X}_\tau$.

Proof of Lemma 4: Considering in (26) $m = 2$, $\lambda_2 = 0$ and arbitrary $\lambda_1 < 0$, $t_1 < t_2$ results with $I_{\mathcal{X}_\tau,\gamma}(f) \geq \lambda_1(f(t_1)-f(t_2))$. Taking $\lambda_1 \to -\infty$ we see that $I_{\mathcal{X},\gamma}(f) = \infty$ as soon as $f(t_1) < f(t_2)$ for some

$\tau \leq t_1 < t_2 < \infty$. Similarly, considering in (26) $m = 1$ and arbitrary $\lambda_1 < 0$, $t_1 \in [\tau, \infty)$ results with $I_{\mathcal{X}_\tau,\gamma}(f) \geq \lambda_1 f(t_1)$, implying that $I_{\mathcal{X}_\tau,\gamma}(f)$ is finite only if f is nonnegative and non-increasing on $[\tau, \infty)$. Considering hereafter only such functions, by (26) for $m = 1$ and $\lambda_1 = \gamma t_1$, we get in the limit $t_1 \to \infty$ that

$$I_{\mathcal{X}_\tau,\gamma}(f) \geq \gamma \limsup_{t\to\infty}\{tf(t)\} + \int_0^\infty \log(1 - e^{-\gamma u})du \,.$$

Since $\int_0^\infty \log(1 - e^{-\gamma u})du \in (-\infty, 0]$ we conclude that $I_{\mathcal{X}_\tau,\gamma}(f) = \infty$ unless $f \in \widehat{DF}_\tau$. Any such f has left and right limits everywhere on $[\tau, \infty)$ with at most countably many jump discontinuity points. Suppose $\sigma \in (\tau, \infty)$ is a jump discontinuity point of f with $f(\sigma^-) > f(\sigma^+)$ denoting the left and right limits of $f(\cdot)$ at σ and $f(\sigma) \in [f(\sigma^+), f(\sigma^-)]$. Fix $m < \infty$, $j \in \{1, \ldots, m\}$, $\lambda \leq \gamma\sigma$, $\tau = t_0 \leq t_1 < \ldots < t_{j-1} < \sigma < t_{j+1} < t_m < t_{m+1} = \infty$ and $\lambda_i \leq \gamma t_i$, $i = 1, \ldots, m$, $i \neq j$, $\lambda_0 = 0$. For $\delta := \min\{t_{j+1} - \sigma, \sigma - t_{j-1}\} > 0$ and any $x \in (-\delta, \delta)$ let $t_j = \sigma + x$, $\lambda_j = \lambda - \gamma|x|$ and define

$$V(x) := \sum_{i=1}^m \{\lambda_i(f(t_i) - f(t_{i+1})) - \int_{t_i}^{t_{i+1}} \log(\frac{1 - e^{-\gamma t}}{1 - e^{-\gamma t}e^{\lambda_i}})dt\} \,. \tag{28}$$

It is not hard to check that

$$\max\{V(0^+), V(0^-)\} - V(0) = \max\{(\lambda - \lambda_{j-1})(f(\sigma^+) - f(\sigma)), (\lambda - \lambda_{j-1})(f(\sigma^-) - f(\sigma))\} \geq 0 \,. \tag{29}$$

Note that for any $x \neq 0$ the value of $V(x)$ is independent of $f(\sigma)$, hence so are $V(0^+)$ and $V(0^-)$. Thus, by (28) and (29) the supremum in (26) over all t_i and $\lambda_i \leq \gamma t_i$ results with $I_{\mathcal{X}_\tau,\gamma}(f)$ that does not depend on the specific value of $f(\sigma)$ within the interval $[f(\sigma^+), f(\sigma^-)]$. Suppose a monotone function f on $[\tau, \infty)$ equals almost everywhere to $f_o \in DF_\tau$, so in particular the left and right limits of f are the same as the corresponding limits for f_o (everywhere on $[\tau, \infty)$). Since the above argument applies to any jump discontinuity point $\sigma \in (\tau, \infty)$ of f (and f_o), it follows that $I_{\mathcal{X}_\tau,\gamma}(f) = I_{\mathcal{X}_\tau,\gamma}(f_o)$.

Thus, without loss of generality, assume that $f \in DF_\tau$, with the decomposition to singular and absolutely continuous components

$$f(t) = f_s(t) + f_{ac}(t) = -\int_{\Omega\,\cap[t,\infty)} df_s(u) + \int_t^\infty g(u)du \,,$$

for some nonnegative $g \in L_1([\tau, \infty))$ such that $g = -\dot{f}_{ac}$ almost everywhere and $f_s \in DF_\tau$ is supported on the set $\Omega \subset [\tau, \infty)$ of zero Lebesgue measure.

Fixing $p \in (0, \infty)$ and $\lambda < \log(1 + p^{-1})$, the function $\xi(x) := \widehat{H}(x|p) - \lambda x$ is convex, differentiable on $(0, \infty)$ with global minimum obtained for $x^* = 1/((1 + p^{-1})e^{-\lambda} - 1)$, leading to $\xi(x^*) = \log(1 + p - pe^\lambda)$. Consequently, for any $p, x \in (0, \infty)$ and $\lambda < \log(1 + p^{-1})$,

$$\widehat{H}(x|p) \geq \lambda x + \log(1 + p - pe^\lambda) \tag{30}$$

By continuity, the inequality (30) extends to $x = 0$ and $\lambda = \log(1 + p^{-1})$. Thus, for any $\lambda_i \leq \gamma t_i$ and $\tau \leq t_i < t_{i+1} < \infty$ we have

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \widehat{H}(g(t)\,|\,\frac{1}{e^{\gamma t} - 1})dt &\geq \int_{t_i}^{t_{i+1}} [\lambda_i g(t) + \log(\frac{e^{\gamma t} - e^{\lambda_i}}{e^{\gamma t} - 1})]dt \\ &= \lambda_i(f_{ac}(t_i) - f_{ac}(t_{i+1})) - \int_{t_i}^{t_{i+1}} \log(\frac{1 - e^{-\gamma t}}{1 - e^{-\gamma t}e^{\lambda_i}})dt \,. \end{aligned} \tag{31}$$

Since $f_s(\cdot)$ is non-increasing, also

$$\gamma \int_{t_i}^{t_{i+1}} (-t) df_s(t) \geq -\gamma t_i \int_{t_i}^{t_{i+1}} df_s(t) = \gamma t_i (f_s(t_i) - f_s(t_{i+1})) \geq \lambda_i (f_s(t_i) - f_s(t_{i+1})) \,. \tag{32}$$

Combining (31) and (32), we have that

$$\int_{t_i}^{t_{i+1}} \widehat{H}(g(t)|\frac{1}{e^{\gamma t}-1})dt + \gamma \int_{t_i}^{t_{i+1}} (-t) df_s(t) \geq \lambda_i (f(t_i) - f(t_{i+1})) - \int_{t_i}^{t_{i+1}} \log(\frac{1-e^{-\gamma t}}{1-e^{-\gamma t}e^{\lambda_i}})dt \,. \tag{33}$$

Since $\lim_{t\to\infty} f(t) = 0$, the inequality (33) holds also for $t_{i+1} = \infty$. Thus, comparing (26) and (27) we see that $I_\gamma(f) \geq I_{\mathcal{X}_\tau,\gamma}(f)$.

Turning to prove the converse inequality, fix $\sigma > \tau$ and let $f_\sigma(t) = -\int_{\Omega\,\cap[t,\sigma]} df_s(u) + \int_t^\infty g_\sigma(u)du$, where $g_\sigma(u) = g1_{u\leq\sigma} + (e^{\gamma u}-1)^{-1}1_{u>\sigma}$. Since $\widehat{H}(p|p) = 0$, it follows that

$$I_\gamma(f_\sigma) = \int_\tau^\sigma \widehat{H}(g(t)\,|\,\frac{1}{e^{\gamma t}-1})dt + \gamma \int_\tau^\sigma (-t) df_s(t) \,, \tag{34}$$

and it is not hard to verify that

$$I_\gamma(f_\sigma) + \int_\tau^\sigma \log(1-e^{-\gamma t})dt = h(\lambda^*) \tag{35}$$

for

$$\lambda^*(t) = \gamma t + \log\left(\frac{g(t)}{1+g(t)}\right) 1_{t\in\Omega^c} \in [-\infty, \gamma t] \,,$$

and

$$h(\lambda) := \int_\tau^\sigma (-\lambda(t)) df(t) + \int_\tau^\sigma \log(1-e^{-\gamma t}e^{\lambda(t)})dt \,.$$

Setting $t_m = \sigma$, $\lambda_m = 0$ in (26), it follows that

$$I_{\mathcal{X}_\tau,\gamma}(f) + \int_\tau^\infty \log(1-e^{-\gamma t})dt \geq \sup_{\substack{\lambda(t)\in(-\infty,\gamma t],\, t\in[\tau,\sigma] \quad \text{piecewise} \\ \text{constant of finitely many pieces}}} h(\lambda) \,. \tag{36}$$

Comparing (27) and (34) we see that $I_\gamma(f_\sigma) \uparrow I_\gamma(f)$ for $\sigma \uparrow \infty$. Hence, in view of (35) and (36), suffices to construct for any fixed $\sigma < \infty$, a sequence $\{\lambda_n\}$ of piecewise constant functions each having finitely many pieces, such that $\liminf_{n\to\infty} h(\lambda_n) \geq h(\lambda^*)$. To this end, fix $\delta > 0$ and define $\lambda^\delta(t) = \lambda^*(t) \vee (-1/\delta) \wedge (\gamma t - \delta)$. Since

$$\int_\tau^\sigma -(\lambda^\delta(t) - \lambda^*(t)) df(t) \geq -\delta(f(\tau) - f(\sigma))$$

and

$$\int_\tau^\sigma \log\left(\frac{1-e^{-\gamma t}e^{\lambda^\delta(t)}}{1-e^{-\gamma t}e^{\lambda^*(t)}}\right) dt \geq (\sigma-\tau)\log(1-e^{-1/\delta}) \,,$$

it follows that $\liminf_{\delta\to 0} h(\lambda^\delta) \geq h(\lambda^*)$. By Egorov's theorem, for each $\epsilon > 0$ there exists a continuous function $\lambda^{\delta,\epsilon}(t)$ with values in $[-1/\delta, \gamma t - \delta]$ such that

$$\int_\tau^\sigma 1_{\lambda^\delta(t)\neq\lambda^{\delta,\epsilon}(t)} (df(t) + dt) < \epsilon \,.$$

It is easy to check that then,

$$h(\lambda^{\delta,\epsilon})-h(\lambda^{\delta})\geq -2\epsilon(\delta^{-1}\vee\gamma\sigma+|\log(1-e^{-\delta})|)\ ,$$

so that $\liminf_{\epsilon\to 0}h(\lambda^{\delta,\epsilon})\geq h(\lambda^{\delta})$. Now, with $\lambda^{\delta,\epsilon}\in C_b[\tau,\sigma]$, there exist piecewise constant functions of finitely many pieces $\lambda_n(\cdot)$ with $\lambda_n(t)\in[-\delta^{-1},\gamma t-\delta]$ such that

$$\sup_{t\in[\tau,\sigma]}|\lambda^{\delta,\epsilon}(t)-\lambda_n(t)|\leq n^{-1}\,.$$

Noting that

$$h(\lambda_n)-h(\lambda^{\delta,\epsilon})\geq -n^{-1}(\sigma-\tau)(1+(e^{\delta}-1)^{-1})\ ,$$

we complete the proof by taking $n\to\infty$ followed by $\epsilon\downarrow 0$ and then $\delta\downarrow 0$. □

By (25) and Lemma 4, we see that $I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f)=\infty$ unless $f\in\widehat{DF}_\tau$. Moreover, if $f\in\widehat{DF}_\tau$ equals almost everywhere to $f_o\in DF_\tau$, then $I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f)=I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f_o)$. Hence, without loss of generality, fix $f\in DF_\tau$, with $g(t)=-\dot f_{ac}(t)$. Since $\widehat H(x|p)-\widehat H(x|q)=x\log(\frac{q}{1+q})(\frac{1+p}{p})-\log(\frac{1+q}{1+p})$, we have for any r,s positive and almost every $t\geq\tau$,

$$\widehat H(g(t)\,|\,\frac{1}{e^{st}-1})=\widehat H(g|\frac{1}{e^{rt}-1})+\log(\frac{1-e^{-rt}}{1-e^{-st}})+(s-r)tg(t)$$

Hence, for all $\theta<\alpha$

$$\int_\tau^\infty \widehat H(g(t)|\frac{1}{e^{\alpha t}-1})dt=\int_\tau^\infty \widehat H(g(t)|\frac{1}{e^{(\alpha-\theta)t}-1})dt+\int_\tau^\infty \log(\frac{1-e^{-(\alpha-\theta)t}}{1-e^{-\alpha t}})dt+\theta\int_\tau^\infty tg(t)dt$$

implying that

$$I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f)=\sup_{\theta<\alpha}\theta[\nu-\int_\tau^\infty(-t)df(t)]+\int_\tau^\infty \widehat H(-\dot f_{ac}(t)|\frac{1}{e^{\alpha t}-1})dt+\alpha\int_\tau^\infty(-t)df_s(t)\ .$$

Consequently,

$$I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f)=\begin{cases}\int_\tau^\infty \widehat H(-\dot f_{ac}(t)|\frac{1}{e^{\alpha t}-1})dt+\alpha[\nu-\int_\tau^\infty(-\dot f_{ac}(t))tdt], & f\in\widehat{DF}_\tau,\nu\geq\int_\tau^\infty(-t)df(t)\\ \infty & \text{otherwise}\end{cases}\tag{37}$$

In particular, $(n^{-1}N,\widetilde\varphi_n(\cdot)|_{[\tau,\infty)})$ satisfies the LDP in $\mathbb{R}\times\mathcal{X}_\tau$ with the topology of pointwise convergence and good rate function $I_{\mathbb{R}\times\mathcal{X}_\tau}(\cdot)$ such that $\mathcal{D}_I:=\{y\in\mathbb{R}\times\mathcal{X}_\tau\,:\,I_{\mathbb{R}\times\mathcal{X}_\tau}(y)<\infty\}\subseteq\mathbb{R}\times\widehat{DF}_\tau$. Since $\mathbb{P}_{x_n}(\tilde\varphi_n(\cdot)\in\widehat{DF}_\tau)=1$, we can, by [6, Lemma 4.1.5], restrict this LDP to the space $\mathbb{R}\times\widehat{DF}_\tau$ equipped with the topology of pointwise convergence. Since the projective limit of the spaces $\{(\mathbb{R}\times\widehat{DF}_\tau):\tau>0\}$ is precisely the set $\mathbb{R}\times\widehat{DF}$ equipped with the topology of pointwise convergence, by the Dawson-Gärtner theorem (see [6, Theorem 4.6.1]), the sequence $(n^{-1}N,\widetilde\varphi_n(\cdot)|_{(0,\infty)})$ satisfies the LDP with speed $\sqrt{n}$ in the latter space and the good rate function,

$$\sup_{\tau>0}I_{\mathbb{R}\times\mathcal{X}_\tau}(\nu,f|_{[\tau,\infty)})=\widehat I(\nu,f)$$

(compare (12) with (37)).

5 Proof of Theorems 1 and 2

In order to analyze the effect of the conditioning $\{N(\lambda)=n\}$ on the LDP, we introduce the *area transformation* $F_n(\cdot)$, which for any Young diagram λ of finite, nonnegative, area $N(\lambda)\le n$ yields a Young diagram $F_n(\lambda)$ of area n, such that $F_n(\lambda)$ is a strict diagram whenever λ is. This transformation is defined as follows:

1. If $N(\lambda)=n$ then $F_n(\lambda)=\lambda$.
2. If $0\le N(\lambda)<n$ then extend the last row of λ (of index $K_{\max}(\lambda)$) by $[n-N(\lambda)]$.

Note the following properties of the transformation $\lambda\mapsto F_n(\lambda)$ (in addition to the fact that $N(F_n(\lambda))=n$):

1. $0\le\varphi_{F_n(\lambda)}(i)-\varphi_\lambda(i)\le 1$ for all i.
2. For any λ with $N(\lambda)=n$, the cardinality of the set $F_n^{-1}(\lambda)$ is bounded by n.
3. Recall (9) implying that $\mathbb{P}^s_x(\lambda=\tilde\lambda)=x^{N(\tilde\lambda)-N(\lambda_o)}\mathbb{P}^s_x(\lambda=\lambda_o)$. Hence, for $x_n=1-\beta/\sqrt{n}$ and any λ_o such that $N(\lambda_o)=n$,

$$\mathbb{P}^s_{x_n}(\lambda\in F_n^{-1}(\lambda_o),n^{-1}N(\lambda)>(1-\delta))=\sum_{k>(1-\delta)n}^{n}x_n^{k-n}\mathbb{P}^s_{x_n}(\lambda=\lambda_o)\le ne^{2\beta\delta\sqrt{n}}\,\mathbb{P}^s_{x_n}(\lambda=\lambda_o)\,. \tag{38}$$

Similarly, for $\mathbb{P}_{x_n}(\cdot)$ and $x_n=1-\alpha/\sqrt{n}$,

$$\mathbb{P}_{x_n}(\lambda\in F_n^{-1}(\lambda_o),n^{-1}N(\lambda)>(1-\delta))\le ne^{2\alpha\delta\sqrt{n}}\,\mathbb{P}_{x_n}(\lambda=\lambda_o)\,. \tag{39}$$

Proof of Theorem 1: Since $\widehat{I}^s(\nu,f)$ of (11) is a good rate function on $\mathbb{R}\times D[0,\infty)$, where $D[0,\infty)$ is equipped with the supremum-norm topology, it follows that $I^s(f)=\widehat{I}^s(1,f)$, given by (3), is a good rate function on $D[0,\infty)$ with the same topology.

Turning to prove the LDP lower bound, it suffices to show that for any $\phi\in D[0,\infty)$ such that $I^s(\phi)<\infty$,

$$\lim_{\delta\to0}\liminf_{n\to\infty}\frac{1}{\sqrt{n}}\log Q^s_n(B_{\phi,2\delta})\ge -I^s(\phi)=-\widehat{I}^s(1,\phi)\ , \tag{40}$$

where $B_{\phi,\delta}=\{f\in D[0,\infty):\|f-\phi\|_\infty<\delta\}$. To prove (40), fix $\delta\in(0,1)$ and $\phi\in\mathcal{AC}^{[-1,0]}_\infty$ with $\int_0^\infty t(-\dot\phi(t))dt\le 1$. Let $\widehat{B}_{\phi,\delta}:=\{(\nu,f):\|f-\phi\|_\infty<\delta,\,1-\delta<\nu<1\}$, an open subset of $\mathbb{R}\times D[0,\infty)$. Identifying $\lambda\in\mathcal{P}^s$ with the corresponding $\tilde\varphi_n\in D[0,\infty)$, for all $n>\delta^{-2}$,

$$\{(n^{-1}N(\lambda),\lambda)\in\widehat{B}_{\phi,\delta}\}\implies\{F_n(\lambda)\in B_{\phi,2\delta}\}\,.$$

Thus, for such n, by (38),

$$\begin{aligned}\mathbb{P}^s_{x_n}((n^{-1}N,\tilde\varphi_n)\in\widehat{B}_{\phi,\delta}) &= \sum_{\lambda_o\in B_{\phi,2\delta},N(\lambda_o)=n}\mathbb{P}^s_{x_n}(F_n(\lambda)=\lambda_o,\ (n^{-1}N(\lambda),\lambda)\in\widehat{B}_{\phi,\delta})\\ &\le \sum_{\lambda_o\in B_{\phi,2\delta},N(\lambda_o)=n}\mathbb{P}^s_{x_n}(\lambda\in F_n^{-1}(\lambda_o),\ n^{-1}N(\lambda)>(1-\delta))\\ &\le ne^{2\beta\delta\sqrt{n}}\ \mathbb{P}^s_{x_n}(n^{-1}N=1,\ \tilde\varphi_n\in B_{\phi,2\delta})\,.\end{aligned} \tag{41}$$

Applying the lower bound of the LDP of Proposition 1 for the random variables $(n^{-1}N, \tilde{\varphi}_n)$ and the open subset $\widehat{B}_{\phi,\delta}$ of $\mathbb{R} \times D[0,\infty)$, results with

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in B_{\phi,2\delta}) \geq - \inf_{(\nu,\psi)\in\widehat{B}_{\phi,\delta}} \widehat{I}^s(\nu,\psi) - 2\beta\delta \tag{42}$$

Let $\phi_\delta(t) = (\phi(t) - \delta/2)_+$. Then, $\int_0^\infty t(-\dot{\phi}_\delta(t))dt < 1$, implying that $\widehat{I}^s(1-\epsilon,\phi_\delta) \to I^s(\phi_\delta)$ as $\epsilon \downarrow 0$. Moreover, $(1-\epsilon,\phi_\delta) \in \widehat{B}_{\phi,\delta}$ for all $\epsilon \in (0,\delta)$, so by (42)

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in B_{\phi,2\delta}) \geq - \limsup_{\delta\to 0} I^s(\phi_\delta) = -I^s(\phi)\,. \tag{43}$$

This proves (40) since

$$Q^s_n(B_{\phi,2\delta}) = \frac{\mathbb{P}^s_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in B_{\phi,2\delta})}{\mathbb{P}^s_{x_n}(n^{-1}N = 1)}\,.$$

Considering (43) for $\phi = \Psi^s$ of (2), we see that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1) \geq -I^s(\Psi^s) = 0\,. \tag{44}$$

(In fact, better bounds are available, c.f. [9], but we will not need them here).

Turning to prove the LDP upper bound, fix a closed set $F \subseteq D[0,\infty)$. Clearly, $\{1\}\times F$ is a closed subset of $\mathbb{R}\times D[0,\infty)$, so the LDP of Proposition 1 for $(n^{-1}N, \tilde{\varphi}_n(\cdot))$ results with

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log Q^s_n(F) &\leq \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1, \tilde{\varphi}_n \in F) - \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1) \\
&\leq - \inf_{\nu=1, f\in F} \widehat{I}^s(\nu, f) - \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}^s_{x_n}(n^{-1}N = 1)
\end{aligned}$$

The upper bound of the LDP of Theorem 1 thus follows from the lower bound (44), completing the proof of the theorem. □

Proof of Theorem 2: The proof parallels that of Theorem 1, with few modifications due to the fact that the topology of pointwise convergence is not a metric topology on $\widehat{DF}$. Obviously, $I(f) = \widehat{I}(1,f)$, given by (6), is a good rate function on $\widehat{DF}$. To prove the LDP lower bound, suffices to show that for any $\phi \in \widehat{DF}$ such that $I(\phi) < \infty$, $m < \infty$ and $0 < t_1 < \cdots < t_m < \infty$,

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log Q_n(U_{\underline{t},\phi,2\delta}) \geq -I(\phi)\,, \tag{45}$$

where the sets $U_{\underline{t},\phi,\delta} = \{f : \max_{i=1}^m |f(t_i) - \phi(t_i)| < \delta\}$ form a base of the topology of pointwise convergence on $\widehat{DF}$. To this end, fix $\delta > 0$, $\phi \in \widehat{DF}$ such that $\int_0^\infty (-t)d\phi(t) \leq 1$, and an open set $U_{\underline{t},\phi,2\delta}$ as above. Let $\widehat{U}_{\underline{t},\phi,\delta} = \{(\nu, f) : f \in U_{\underline{t},\phi,\delta},\, 1-\delta < \nu < 1\}$ an open subset of $\mathbb{R}\times\widehat{DF}$. Identifying $\lambda \in \mathcal{P}$ with the corresponding $\tilde{\varphi}_n \in \widehat{DF}$, for all $n > \delta^{-2}$,

$$\{(n^{-1}N(\lambda), \lambda) \in \widehat{U}_{\underline{t},\phi,\delta}\} \implies \{F_n(\lambda) \in U_{\underline{t},\phi,2\delta}\}\,.$$

For such n and $x_n = 1 - \alpha/\sqrt{n}$, in analogy with the derivation of (41), it follows from (39) that

$$\mathbb{P}_{x_n}((n^{-1}N, \tilde{\varphi}_n) \in \widehat{U}_{\underline{t},\phi,\delta}) \leq n e^{2\alpha\delta\sqrt{n}}\, \mathbb{P}_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in U_{\underline{t},\phi,2\delta})\,.$$

Consequently, applying the lower bound of the LDP of Proposition 2 for the random variables $(n^{-1}N, \tilde{\varphi}_n)$ and the open set $\widehat{U}_{\underline{t},\phi,\delta}$ we see that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in U_{\underline{t},\phi,2\delta}) \geq - \inf_{(\nu,\psi)\in\widehat{U}_{\underline{t},\phi,\delta}} I(\nu,\psi) - 2\alpha\delta \tag{46}$$

Since $\int_0^\infty (-t) d\phi_\delta(t) < 1$ (for $\phi_\delta(t) := (\phi(t) - \delta/2)_+$), it follows that $\widehat{I}(1-\epsilon, \phi_\delta) \to I(\phi_\delta)$ as $\epsilon \downarrow 0$. Moreover, $(1-\epsilon, \phi_\delta) \in \widehat{U}_{\underline{t},\phi,\delta}$ for all $\epsilon \in (0,\delta)$ and hence

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in U_{\underline{t},\phi,2\delta}) \geq - \limsup_{\delta\to 0} I(\phi_\delta) = -I(\phi)\ . \tag{47}$$

This proves (45) since

$$Q_n(U_{\underline{t},\phi,2\delta}) = \frac{\mathbb{P}_{x_n}(n^{-1}N = 1,\, \tilde{\varphi}_n \in U_{\underline{t},\phi,2\delta})}{\mathbb{P}_{x_n}(n^{-1}N = 1)}\ .$$

Considering (47) with $\phi = \Psi$ of (1) for which $I(\Psi) = 0$ (see Remark 2), we see that in analogy with (44),

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}_{x_n}(n^{-1}N = 1) = 0\ . \tag{48}$$

The lower bound (48) and the LDP of Proposition 2 yield the upper bound of the LDP of Theorem 2 by the same argument we used when deducing the upper bound for Theorem 1 out of (44) and Proposition 1. □

6 Extensions and discussion

We present briefly in this section a (non-exhaustive) list of possible extensions and related results. As is aparent from our proof, our techniques are suitable in situations where a measure μ_n on $\mathcal{P}_n$ can be obtained by restricting a measure on $\mathcal{P}$ involving independent components (with law parametrized by a finite number of parameters) to $\mathcal{P}_n$ in such a way that the restriction does not depend on the parameters. Such measures are called *multiplicative statistics* in [17]. We note that examples involving an i.i.d representation by means of Bernoulli random variables are related to the combinatorial structures called selections, representations by means of Geometric random variables are related to multisets, and representations by means of Poisson random variables are related to assemblies, c.f. [10] and [8]. While the techniques of proof are similar, the details and the scalings may vary. Our choice of examples was motivated by our desire to exhibit an example of an assembly, c.f. Section 6.1, and an example of some current physical interest, c.f. Section 6.2. We distinguish between theorems where we have checked the details, and other situations where we have not.

6.1 Trees and partitions

Consider a *set* partition of $\{1, 2, \ldots, n\}$ to non-empty disjoint sets $S_1, S_2, \ldots, S_k$, of cardinality $n_1 \geq n_2 \geq \cdots \geq n_k$. To each S_i associate a vertebrate (that is, a bipointed tree, with "head" and "tail" marked, see [4, 10]) such that each element in S_i corresponds to a specific vertex of the vertebrate (there are $n_i^{n_i}$ possibilities for such trees). We have thus associated to each set partition a *forest of vertebrates*, and we denote by $\mathcal{F}_n$ the collection of all such forests. To each element of $\mathcal{F}_n$ we associate an integer partition λ represented by $n_1, \ldots, n_k$ and the corresponding rescaled shape $\tilde{\varphi}_n(t) = n^{-1/3}\varphi_\lambda(\lceil tn^{2/3}\rceil)$. Let $\widetilde{Q}_n$ denote the measure induced on $\mathcal{P}_n$ by the uniform measure on $\mathcal{F}_n$, and note that for some constant K_n and all $(n_1, \ldots, n_j) \in \mathcal{P}_n$,

$$\widetilde{Q}_n(n_1, \ldots, n_j) = K_n \prod_{i=1}^{j} \frac{n_i^{n_i}}{n_i!}.$$

See [3] for a discussion of some other possible measures on random forests of trees.

Define next $\gamma = 1/2$ and

$$\widetilde{\Psi}(t) = \int_t^\infty \frac{e^{-\gamma u}}{\sqrt{2\pi u}} du.$$

Theorem 3 *Under the laws $\widetilde{Q}_n$, the random variables $\tilde{\varphi}_n(\cdot)$ satisfy the LDP in $D[0,\infty)$ (equipped with the topology of uniform convergence), with speed $n^{1/3}$ and good rate function*

$$\widetilde{I}(f) = \begin{cases} \gamma(1 - \int_0^\infty t(-\dot{f}(t))dt) + \int_0^\infty \widetilde{H}(-\dot{f}(t)| - \dot{\widetilde{\Psi}}(t))dt, & f \in \mathcal{AC}_\infty, \int_0^\infty t(-\dot{f}(t))dt \leq 1, \\ \infty, & \textit{otherwise}. \end{cases} \quad (49)$$

Here, $\widetilde{H}(x|p) = x\log(x/p) - x + p$. In particular, $\tilde{\varphi}_n$ concentrates around the curve $\widetilde{\Psi}$.

Sketch of proof of Theorem 3 For any $x \in (0,1)$, let $\widetilde{\mathbb{P}}_x$ denote the law of a sequence of independent Poisson random variables R_k with parameters $\lambda_k(x) = k^k x^k e^{-k}/k!$. By the Borel-Cantelli lemma, $N < \infty$, $\widetilde{\mathbb{P}}_x$ a.s. for any $x \in (0,1)$. Setting $\Delta_n = n^{-2/3}$ and $x_n = (1 - \gamma\Delta_n)$, one checks that

$$\widetilde{\mathbb{E}}_{x_n}(n^{-1}N) = \sum_{k=1}^\infty \Delta_n^{3/2} k\lambda_k(x_n) \to_{n\to\infty} (2\pi)^{-1/2} \int_0^\infty \sqrt{t} e^{-\gamma t} dt = 1.$$

Similarly,

$$n^{1/3}\widetilde{\mathrm{Var}}_{x_n}(n^{-1}N) = \sum_{k=1}^\infty \Delta_n^{5/2} k^2 \lambda_k(x_n) \to_{n\to\infty} (2\pi)^{-1/2} \int_0^\infty t^{3/2} e^{-\gamma t} dt < \infty,$$

implying the concentration of $n^{-1}N$ under $\widetilde{\mathbb{P}}_{x_n}$. Concentration of $\tilde{\varphi}_n(\cdot)$ to the limit shape $\widetilde{\Psi}(\cdot)$ follows by the same technique. In proving the large deviations principle, one now proceeds as in the proof of Theorem 1 and Proposition 1: first (compare with (14)), letting

$$\Lambda_n(\eta_1, \ldots, \eta_m, \theta) = n^{-1/3} \log \widetilde{\mathbb{E}}_{x_n}[\exp n^{1/3}\{\sum_{i=1}^{m-1} \eta_i(\tilde{\varphi}_n(t_i) - \tilde{\varphi}_n(t_{i+1})) + \eta_m \tilde{\varphi}_n(t_m) + \theta n^{-1}N\}],$$

one checks that

$$\lim_{n\to\infty} \Lambda_n(\eta_1, \ldots, \eta_m, \theta) = \begin{cases} \int_0^\infty (e^{\eta(t)+\theta t} - 1)(-\dot{\widetilde{\Psi}}(t))dt, & \theta < \gamma, \\ \infty & \theta \geq \gamma, \end{cases}$$

where $\eta(t) = \sum_{i=1}^m \eta_i 1_{t\in[t_i,t_{i+1})}$, $t_{m+1} = \infty$.

One may now repeat the computations in Lemma 1 in order to deduce a finite-dimensional LDP for $(\{\tilde{\varphi}_n(t_i)\}_{i=1}^m, n^{-1}N)$. Further, note that $\widetilde{\mathbb{P}}_{x_n}(K_{\max} > n^2) \leq \exp(-Cn^{4/3})$ for some $C > 0$ whereas $\lambda_k(x_n) \leq C_1$ for some finite C_1 and all k, n. The exponential equivalence of $\tilde{\varphi}_n$ and the corresponding piecewise linear approximation φ_n is obtained by replacing (17) with the bound, valid for any $\eta > 0$,

$$\begin{aligned} \widetilde{\mathbb{P}}_{x_n}(\| \varphi_n - \tilde{\varphi}_n \|_\infty > \delta) &\leq n^2 \sup_{k\leq n^2} \widetilde{\mathbb{P}}_{x_n}(R_k > \delta n^{1/3}) + \widetilde{\mathbb{P}}_{x_n}(K_{\max} > n^2) \\ &\leq n^2 e^{-\eta\delta n^{1/3}} \sup_{k\leq n^2} e^{\lambda_k(x_n)(e^\eta - 1)} + e^{-Cn^{4/3}} \leq n^2 e^{-\eta\delta n^{1/3}} e^{C_1(e^\eta-1)} + e^{-Cn^{4/3}}, \end{aligned}$$

and taking first $n \to \infty$, then $\eta \to \infty$. Similarly, the exponential tightness of $(n^{-1}N, \varphi_n(\cdot))$ follows by using the compact sets $K_L = \mathcal{AC}_\infty^{[-L,0]} \cap \{\phi : \phi(0) \le L\}$ and the bound

$$\widetilde{\mathbb{P}}_{x_n}(\varphi_n \notin \mathcal{AC}_\infty^{[-L,0]}) \le n^2 \sup_{k \le n^2} \widetilde{\mathbb{P}}_{x_n}(R_k > Ln^{1/3}) + \widetilde{\mathbb{P}}_{x_n}(K_{\max} > n^2)$$

By the same proof as in Lemma 2, one deduces that $(n^{-1}N, \tilde{\varphi}_n(\cdot))$ satisfy the LDP in $\mathbb{R} \times D[0,\infty)$, equipped with the topology of uniform convergence, speed $n^{1/3}$, and rate function

$$\widetilde{I}(\nu, f) = \begin{cases} \gamma(\nu - \int_0^\infty t(-\dot{f})dt) + \int_0^\infty \widetilde{H}(-\dot{f}(t)| - \dot{\widetilde{\Psi}}(t))dt, & f \in \mathcal{AC}_\infty, \int_0^\infty t(-\dot{f}(t))dt \le \nu\,, \\ \infty, & \text{otherwise.} \end{cases}$$

Finally, Theorem 3 follows from the above in the same manner that Theorem 1 followed from Proposition 1, by noting that any $n \ge n_0(\delta)$, $\lambda_o = F_n(\widetilde{\lambda})$ with $N(\lambda_o) = n$ and $N(\widetilde{\lambda}) \ge n(1-\delta)$ satisfy the inequality

$$e^{3\gamma\delta n^{1/3}} \widetilde{\mathbb{P}}_{x_n}(\lambda = \lambda_o) \ge \widetilde{\mathbb{P}}_{x_n}(\lambda = \widetilde{\lambda})$$

(compare with (38)). □

6.2 Partitions with multiplicities and constrained partitions

We next describe a class of measures over partitions obtained from having parts of the partition corresponding to possibly different “types”. Let $\{c_k\}_{k=1}^\infty$ denote a deterministic sequence of non-negative integers. We say that $\{c_k\}$ is of *type* $(q, b) \in (0,\infty) \times (0,\infty)$ if

$$\lim_{\epsilon \to 0} \lim_{L \to \infty} \epsilon^{-1} L^{-q} \sum_{k=L}^{(1+\epsilon)L} c_k = b\,.$$

Let $R_k^1, \dots, R_k^{c_k}$ denote independent copies of the random variable R_k of Section 2 (under either $\mathbb{P}_x$ or $\mathbb{P}_x^s$), and denote by $\mathbb{P}_{x,c}$ and $\mathbb{P}_{x,c}^s$ the induced law on $\{R_k^\ell\}$, that is, for any sequence of integers $\{r_k^\ell\}_{k=1,\ell=1}^{k=\infty,\ell=c_k}$ with only finitely many non-zero elements,

$$\mathbb{P}_{x,c}(\{R_k^\ell = r_k^\ell\}) = \prod_{k=1}^\infty \prod_{\ell=1}^{c_k} \mathbb{P}_x(R_k^\ell = r_k^\ell)\,, \quad \mathbb{P}_{x,c}^s(\{R_k^\ell = r_k^\ell\}) = \prod_{k=1}^\infty \prod_{\ell=1}^{c_k} \mathbb{P}_x^s(R_k^\ell = r_k^\ell)\,.$$

Define next the shape

$$\varphi_\lambda(i) = \sum_{k=i}^\infty \sum_{\ell=1}^{c_k} R_k^\ell\,,$$

and area $N(\lambda) = \sum_{k=1}^\infty \sum_{\ell=1}^{c_k} kR_k^\ell$, with

$$Q_{n,c}(\lambda = \lambda_0) = \mathbb{P}_{x,c}(\lambda = \lambda_o | N(\lambda) = n)\,, \quad Q_{n,c}^s(\lambda = \lambda_0) = \mathbb{P}_{x,c}^s(\lambda = \lambda_o | N(\lambda) = n)\,.$$

Note that Theorems 1 and 2 correspond to the sequence $c_k = 1$ for all k (and $b = q = 1$). Another interesting example is the sequence $c_k = 1_{k^{1/\ell} \in \boldsymbol{N}}$, i.e. $c_k = 1$ if k is a perfect ℓ square of an integer and $c_k = 0$ otherwise. The latter sequence is of type $(1/\ell, 1/\ell)$, leading to $Q_{n,c}$ being the uniform measure over all partitions of n to perfect ℓ roots, $\ell = 2, 3, \dots$. Related sequences are $c_k = c_{k,d}$ corresponding to the number of different ways of expressing k as a sum of d perfect squares, which is of type $(d/2, d2^{-(d+1)}\mathrm{Vol}(\{x \in \mathbb{R}^d : |x|_2 \le 1\}))$, or $c_k = [k^{q-1}]$, $q \geq 1$, which is of type $(q,1)$, see [17] for the last two examples. Note that in these examples, one may write explicitely (c.f. [17]) the generating function associated with $\mathbb{P}_{x,c}$, all of the form

$$\prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{c_k}} .$$

Let $m_{q,b}$ denote the positive, σ-finite measure on $[0,\infty)$ with density $dm_{q,b}/dt = bt^{q-1}$, and let

$$\beta_{q,b} = (\int_0^\infty u(1+e^u)^{-1} dm_{q,b}(u))^{1/(q+1)} , \quad \alpha_{q,b} = (\int_0^\infty u(e^u-1)^{-1} dm_{q,b}(u))^{1/(q+1)} .$$

It is easy to check that then, with $\Delta_n = n^{-1/(q+1)}$ and $x_n = 1 - \alpha_{q,b}\Delta_n$, it holds that $n^{-1}N \to 1$ under $\mathbb{P}_{x_n,c}$, with similar conclusion under $\mathbb{P}^s_{x_n,c}$ if $x_n = 1 - \beta_{q,b}\Delta_n$. Define $\tilde{\varphi}_n(t) = n^{-q/(1+q)}\varphi_\lambda(\lceil t/\Delta_n \rceil)$. With

$$\Psi_{q,b}(t) = \int_t^\infty \frac{dm_{q,b}(u)}{e^{\alpha_{q,b}u}-1} , \quad \Psi^s_{q,b}(t) = \int_t^\infty \frac{dm_{q,b}(u)}{1+e^{\beta_{q,b}u}} ,$$

one now has concentration of $\tilde{\varphi}_n$ around the curves $\Psi_{q,b}(t)$ and $\Psi^s_{q,b}(t)$ under $\mathbb{P}_{x_n,c}$ and $\mathbb{P}^s_{x_n,c}$, respectively, see e.g. [17] for the particular case $c_k = [k^{q-1}]$.

The techniques of this paper lead us to believe that the LDP holds under the measure $Q_{n,c}$ (respectively, $Q^s_{n,c}$) in the space $D[0,\infty)$ equipped with the topology of pointwise convergence (respectively, topology of uniform convergence), speed $n^{q/(q+1)}$ and good rate functions

$$\hat{I}_{q,b}(f) = \begin{cases} \alpha_{q,b}(1 - \int_0^\infty t(-\dot{f}_{ac}(t))dt) + \int_0^\infty \widehat{H}(-\frac{df_{ac}}{dm_{q,b}} | - \frac{d\Psi_{q,b}}{dm_{q,b}}) dm_{q,b} , & f \in \widehat{DF}, 1 \geq \int_0^\infty (-t) df(t) , \\ \infty & \text{otherwise} , \end{cases}$$

and, respectively in the strict case,

$$\hat{I}^s_{q,b}(f) = \begin{cases} \beta_{q,b}(1 - \int_0^\infty t(-\dot{f}(t))dt) + \int_0^\infty H(-\frac{df}{dm_{q,b}} | - \frac{d\Psi^s_{q,b}}{dm_{q,b}}) dm_{q,b} , & f \in \mathcal{AC}_\infty, 1 \geq \int_0^\infty (-t) df(t) , \\ \infty & \text{otherwise} . \end{cases}$$

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