A Large Deviations Analysis of Range Tracking Loops

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Abstract Large deviations theory is applied to the analysis of a discrete time range tracking loop. It is shown that the resulting asymptotics differ from those of the continuous time diffusion limit.

1 Introduction and main results

In the design of tracking loops in the presence of noise, two performance measures are of outmost importance: the steady state error ("accuracy of the loop") and the time till lock is lost ("stability" of the loop). While the analysis of the loop accuracy may usually be performed by considering a linearized version of the loop, it is well known that this is not a good approach for studying stability. At least when the tracking loop may be modeled as a Markov process, analytic solutions for the latter question exist. Those solutions are hardly ever explicitly computable. Since often one is interested in systems where some natural parameter ϵ is small (for example, the "Noise to Signal"

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ratio, or the bandwidth), an asymptotic study of the stability question, which hopefully yields explicit expressions, is of interest.

In recent years, large deviations methods have been applied extensively to the latter problem. Beginning with the pioneering work of Friedlin and Wentzell [11], it became clear that in many cases the question of loss of lock ("problem of exit from a domain"), which involves longer and longer (in ϵ) time intervals, may be reduced to the analysis of fixed intervals large deviations estimates. Such analysis has been carried out for many Markov processes, and in particular, for diffusion processes (see [7] and, in the context of tracking systems, [6, 8, 12]). It seems that the discrete time version of this problem has not received much attention in the literature. An often used approach, namely the use of the diffusion limit of the discrete time chain as an intermediate step in the exit problem analysis, may lead to completely wrong estimates if the process noise is not Gaussian (see remark (c) below).

In this article we focus on a discrete time model for a range tracking loop. As will be clear from our exposition, the approach presented is quite general, but we chose to present it in the simplest possible situation which still captures the main features of the problem. A related discussion and some other examples will appear in the book [4].

There exists a vast literature on tracking systems and algorithms. For a guide to the literature, we refer the reader to [1]. The model we discuss here is as follows. By transmitting a pulse $s(t)$ and analyzing its return from a target $s(t-\tau)$, a radar receiver may estimate τ, the time it took the pulse to travel to the target and return. Dividing by twice the speed of light, an estimate of the distance to the target is obtained.

A range tracker keeps track of changes in τ. Since the range of the target is unknown to the tracker, and fluctuates, it is common to model it, or actually its represention by τ, as a random process. In order to keep the analysis simple, and yet to provide a meaningful model, we describe the range of the target as a first order AR process. That is, we take

$$\tau_{k+1} = \tau_k - \epsilon T \beta \tau_k + \epsilon T^{1/2} v_k \tag{1}$$

where τ_k denotes the value of τ at the k-th pulse transmission instant, $\{v_k\}$ denotes a sequence of zero mean i.i.d. random variables, T is a deterministic constant which denotes the time interval between successive pulses (so $1/T$ is the pulse repetition frequency), and β is a deterministic constant related to the target's motion bandwidth and to the speed in which the target is approaching the radar while making random maneuvers. The changes in the dynamics of the target are slow in comparison with the pulse repetition frequency, i.e. small bandwidth of its motion. This is indicated by the $\epsilon \ll 1$ factor in both terms in the right side of (1).

If $\{v_k\}$ are standard Normal, then (1) may be obtained by discretizing at T-intervals the solution of the stochastic differential equation $d\tau = -\epsilon\tilde{\beta}\tau dt + \epsilon\tilde{\alpha}dv_t$, with v_t a standard Brownian motion, $\beta = (1 - e^{-\epsilon\tilde{\beta}T})/\epsilon T \simeq \tilde{\beta}$ and $\tilde{\alpha}^2 = 2\epsilon\tilde{\beta}T/(1 - e^{-2\epsilon\tilde{\beta}T}) \simeq 1$. We return to this remark below.

The radar transmits pulses with shape $s(\cdot)$ at times kT , $k = 0, 1, \ldots$ where $s(t) = 0$ for $|t| \geq \frac{\delta}{2}$ and $\delta \ll T$. The k-th pulse appears at the receiver as $s(t - kT - \tau_k)$, and to this additive noise is added. Hence, the receiver input is

$$dy_t = \sum_{k=0}^{\infty} s(t - kT - \tau_k)\, dt + N_0^{1/2}\, dw_t$$

where w_t is a standard Brownian motion independent of $\{v_k\}$ and τ_0, while N_0 is a deterministic fixed constant reflecting the noise power level. Usually, T is chosen such that no ambiguity occurs between adjacent pulses, i.e. T is much larger then the dynamic range of the increments $(\tau_{k+1} - \tau_k)$. A typical radar receiver is depicted in figure 1. It contains a filter $h(\cdot)$ (the "range gate") which is

Figure 1: Block diagram of a range tracker

normalized such that $\int_{-\delta/2}^{\delta/2} |h(t)|^2\, dt = 1/\delta$ and is designed so that the function

$$g(x) = \int_{-\delta/2}^{\delta/2} s(t + x) h(t)\, dt$$

is bounded, uniformly Lipschitz continuous, with $g(0) = 0$, $g'(0) < 0$ and $xg(x) < 0$ for $0 < |x| < \delta$.

A typical example is $s(t) = 1_{[-\frac{\delta}{2},\frac{\delta}{2}]}(t)$ and $h(t) = \frac{1}{\delta}\text{sign}(t)1_{[-\frac{\delta}{2},\frac{\delta}{2}]}(t)$ for which

$$g(x) = -\text{sign}(x)\left(\frac{|x|}{\delta}1_{[0,\frac{\delta}{2}]}(|x|) + \left(1 - \frac{|x|}{\delta}\right)1_{(\frac{\delta}{2},\delta]}(|x|)\right) .$$

The receiver forms the estimates

Figure 2: Typical range gate characteristics

$$\hat{\tau}_{k+1} = \hat{\tau}_k - \epsilon T\beta\,\hat{\tau}_k + \epsilon KT\int_{\hat{\tau}_k+kT-\delta/2}^{\hat{\tau}_k+kT+\delta/2} h(t-\hat{\tau}_k-kT)dy_t\,, \tag{2}$$

where $K > 0$ is the receiver gain. The correction term in the above estimates is taken of order ϵ in order to reduce the effective measurement noise to the same level as the random maneuvers of the target. We assume that ϵ is known, but this is not a severe limitation because it may (and in practice, is) estimated from the target's dynamics, and the effect of a mismatch in the value of ϵ (say, by a constant factor) is equivalent to an effective change in β and in the shape of the range gate function.

Since $T \gg \delta$, adjacent pulses do not overlap in (2), and the update formula for $\hat{\tau}_k$ may be rewritten as

$$\hat{\tau}_{k+1} = \hat{\tau}_k - \epsilon T\beta\hat{\tau}_k + \epsilon TKg(\hat{\tau}_k - \tau_k) + \epsilon\frac{N_0^{1/2}KT}{\delta^{1/2}}w_k$$

where $w_k \sim N(0,\,1)$ is a sequence of i.i.d. Normal random variables. Assume throughout that $\hat{\tau}_0 = \tau_0$, i.e. the tracker starts with perfect lock. Let $Z_k = \hat{\tau}_k - \tau_k$ denote the range error process. Note that Z_k satisfies the equation

$$Z_{k+1} = Z_k + \epsilon b(Z_k) + \epsilon \nu_k,\; Z_0 = 0 \tag{3}$$

where $b(z) \triangleq -T\beta z + TKg(z)$ and $\nu_k \triangleq \frac{N_0^{1/2}KT}{\delta^{1/2}} w_k - T^{1/2} v_k$.

A loss of track is the event $\{|Z_k| \geq \delta\}$. We wish to analyze the asymptotic probability of such an event, as it will determine the performance of the range tracker.

In the absence of the noise sequence $\{\nu_k\}$ and when $\beta \geq 0$, the dynamical system (3) has 0 as its unique stable point in the interval $[-\delta, \delta]$ due to the design condition $zg(z) < 0$ (this stability extends to $\beta < 0$ if K is large enough). Therefore, as $\epsilon \to 0$, the probability of loss of lock in any finite interval is small and it is reasonable to rescale time. Define the continuous time process $Z_\epsilon(t),\; t \in [0,1]$ via

$$Z_\epsilon(t) = Z_{[\frac{t}{\epsilon}]} + \frac{1}{\epsilon}\left(t - \epsilon\left[\frac{t}{\epsilon}\right]\right)\left(Z_{[\frac{t}{\epsilon}]+1} - Z_{[\frac{t}{\epsilon}]}\right),$$

where $[t/\epsilon]$ denotes the integer part of t/ϵ.

In order to state our results, we need some basic definitions from the theory of large deviations. We follow the notations of [4]. For other relevant expositions of the large deviations theory, see [3, 5, 7, 10].

Let $C_0[0,1]$ denote the space of continuous functions $\phi(\cdot)$ on $[0,1]$ satisfying $\phi(0) = 0$, equipped with the supremum norm. Let μ_ϵ be a sequence of probability measures on $C_0[0,1]$. We say that $\{\mu_\epsilon\}$ satisfies the Large Deviations Principle (LDP) with the good rate function $I : C_0[0,1] \to \mathbb{R}$ if the latter possesses compact level sets and

$$-\inf_{\Gamma^o} I(x) \leq \liminf_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq \limsup_{\epsilon \to 0} \epsilon \log \mu_\epsilon(\Gamma) \leq -\inf_{\bar{\Gamma}} I(x)$$

where Γ is an arbitrary Borel measurable set in $C_0[0,1]$, and $\Gamma^0(\bar{\Gamma})$ denote, respectively, the interior of Γ and its closure in $C_0[0,1]$. As mentioned above, the existence of a LDP for μ_ϵ is the main tool required for the analysis of the loss of lock asymptotics.

The main result required for infering information on the asymptotics of the time to lose track is summarized in the following theorem, whose proof is delayed to Section 2. Let $\Lambda(\cdot)$ denote the logarithmic moment generating function associated with the random variables ν_k, i.e. $\Lambda(\lambda) = \log E(\exp \lambda \nu_k)$. Define $\Lambda^*(\cdot)$ to be the Legendre transform of Λ, i.e.

$$\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda(\lambda)\} .$$

Theorem 1 *Assume that $\Lambda(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$. Then, $\{Z_\epsilon(\cdot)\}$ satisfies the LDP in $C_0([0,1])$ with the good rate function*

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}(t) - b(\phi(t)))\, dt & \phi \in \mathcal{AC}, \phi(0) = 0 \\ \infty & \text{otherwise ,} \end{cases} \tag{4}$$

where $\mathcal{AC}$ denotes the space of absolutely continuous functions, i.e.

$$\mathcal{AC} \stackrel{\triangle}{=} \left\{ \phi \in C([0,1]) : \sum_{\ell=1}^{k} |t_\ell - t_{\ell-1}| \to 0 \Longrightarrow \sum_{\ell=1}^{k} |\phi(t_\ell) - \phi(t_{\ell-1})| \to 0 \right\} . \tag{5}$$

Remarks:

(a) As with any asymptotic study, it is hard to predict for which values of ϵ one gets close to the asymptotic limit. The importance of the asymptotic study lies in the fact that it allows for a comparative study of different trackers, and it yields intuition on the influence of various systems parameters on the performance of the system.

(b) Theorem 1 applies to any random process which evolves according to (3), provided that $b(\cdot)$ is bounded and uniformly Lipschitz continuous and ν_k are i.i.d. random variables whose logarithmic moment generating function $\Lambda(\cdot)$ is finite everywhere. The proof below does not use any other special property of the range tracking problem. Likewise, this theorem extends to any finite interval $t \in [0, T]$ and to $\mathbb{R}^d$-valued processes, with the appropriate obvious modifications.

(c) Our reason for assuming that w_k is a Gaussian sequence while v_k is not stems from the fact that w_k typically represents the discretized version of a continuous time white noise process, which may be assumed Gaussian in many applications, whereas v_k models random maneuvers of the target, which may be of discrete nature (in particular, the important case of binary disturbance may be analyzed). Thus, there is no reason to assume that the process noise sequence is Gaussian. From a mathematical point of view, w_k could also be taken with an arbitrary law, as long as the exponential tails condition on $\Lambda(\cdot)$ is satisfied.

(d) It is interesting to note that a common model in the literature is to approximate first τ_k by a continuous diffusion model (obtained after a time rescaling) and then compute the asymptotics of rare events. Since the time rescaled continuous time model involves Brownian motion, this computation results with the rate function

$$I_N(\phi) = \begin{cases} \frac{c}{2}\int_0^1 (\dot{\phi}_t - b(\phi_t))^2\, dt\,, & \phi \in H_1 \\ \infty & \text{otherwise} \end{cases} \tag{6}$$

for some $c > 0$. As is obvious from the analysis, this approach is justified only if either the random maneuvers v_k are modeled by i.i.d. Normal random variables or if v_k are multiplied by a power of ϵ larger than 1.

(e) Focusing back on the range tracking application, since in that case v_k has zero mean and $b(0) = 0$, it follows from Jensen's inequality and the strict convexity of Λ^* that $I(\phi) = 0$ only when $\phi_t \equiv 0$. Moreover, by choosing K large enough such that $Kg'(0) < \beta$, the point 0 becomes a stable point of the deterministic ordinary differential equation

$$\dot{x}_s = b(x_s) = -T\beta x_s + TKg(x_s)\ ,$$

and it is the unique such point in some neighborhood of 0 which may be strictly included in the domain $G = (-\delta, \delta)$. However for $\beta \geq 0$ indeed $x = 0$ is the unique stable point in G and any trajectory with initial condition $x_0 \in G$ converges to 0 as $s \to \infty$. Then, the analysis of the mean time till loss of track follows the approach presented in [7] for diffusion processes. In particular, denote the number of time steps till exit occurs by $\tau^* = \inf\{k : |Z_k| \geq \delta\}$, and assume that the process equation is stable (i.e., $\beta \geq 0$). Repeating the arguments in [7], it follows that

$$\lim_{\epsilon\to 0} \epsilon \log E(\tau^*) = \inf_{T>0} \inf_{\substack{\phi:\, \phi(0)=0 \\ |\phi(T)|=\delta}} I_T(\phi) \stackrel{\triangle}{=} \bar{\tau}\ , \tag{7}$$

where

$$I_T(\phi) = \begin{cases} \int_0^T \Lambda^*(\dot{\phi}(t) - b(\phi(t)))\, dt & \phi \in \mathcal{AC}, \phi(0) = 0 \\ \infty & \text{otherwise}\ , \end{cases}$$

and, moreover, for any $\delta > 0$,

$$\lim_{\epsilon\to 0} P\left(\left|\frac{\epsilon \log \tau^*}{\bar{\tau}} - 1\right| > \delta\right) = 0\,.$$

In the particular case that v_k are Gaussian, the above is a direct consequence of Corollary 5.7.16 in [4].

(f) Since ν_k possesses a Gaussian component, it follows that $\Lambda(\cdot)$ is strictly convex. Let now $V(\cdot)$ be a solution to the equation

$$V(x)b(x) = \Lambda(V(x)), -\delta \leq x \leq \delta .$$

Then (see [7], Theorem 4.3 and exploit the symmetry of $b(\cdot)$), a time shifted version of the minimizing path in (7) satisfies the equation

$$\dot{\phi}(t) = b(\phi(t)) - \Lambda'(V(\phi(t))), \phi(-\infty) = 0, \phi(0) = \delta.$$

In particular, if v_k is also standard Normal $N(0,1)$, then $\Lambda(\lambda) = \lambda^2(N_0K^2T^2 + \delta T)/2\delta$, $\Lambda^*(x) = x^2\delta/2(N_0K^2T^2 + \delta T)$, and the minimizing path satisfies the equation $\dot{\phi}(t) = -b(\phi(t))$. In the particular example of loop nonlinearity $g(\cdot)$ described above, this yields the expression

$$\lim_{\epsilon \to 0} \epsilon \log E(\tau^*) = \frac{\delta^2(\beta\delta + K/2)}{\delta + N_0K^2T}. \tag{8}$$

As expected, to maximize the time to lose lock, it is best to have δ large, T small and N_0 small. Moreover, all other parameters being fixed, there is an optimal gain

$$K^* = \sqrt{(2\beta\delta)^2 + \delta/N_0T} - 2\beta\delta ,$$

and as $N_0 \to 0$ one has $K^*N_0^{1/2} \to \sqrt{\delta/T}$, as one would guess from the analysis of [12]. Note that this last limit relates the optimal gain to the energy of the signal, i.e. the optimal gain is asymptotically related to the signal to noise ratio $\sqrt{\delta/TN_0}$.

2 Proofs

Proof of Theorem 1: We will need below a large deviations estimate for random walk. Let $\{\nu_k\}$ be a sequence of i.i.d. random variables, and let $\Lambda(\lambda)$ be the associated logarithmic moment generating function. Let $Y_\epsilon(t) = \epsilon\sum_{k=1}^{[t/\epsilon]} \nu_k$ denote the rescaled random walk generated by $\{\nu_k\}$, and let $\tilde{Y}_\epsilon(t) = Y_\epsilon(t) + (t - \epsilon[t/\epsilon])(Y_\epsilon(t+\epsilon) - Y_\epsilon(t))$ denote the (continuous) linearly interpolated version of $Y_\epsilon(\cdot)$. Note that $\tilde{Y}_\epsilon(\cdot) \in C_0[0,1]$. The following theorem is due to Mogulskii and to Varadhan [9], and in the present form may be found in [4], Theorem 5.1.2.

Theorem 2 *Assume that for all* $\lambda \in \mathbb{R}$, $\Lambda(\lambda) < \infty$. *Then, the law of* $\tilde{Y}_\epsilon(\cdot)$ *satisfies in* $C_0[0,1]$ *the LDP with the good rate function*

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}(t))\,dt, & \text{if } \phi \in \mathcal{AC}, \phi(0) = 0 \\ \\ \infty & \text{otherwise} \end{cases} \tag{9}$$

Remark: Theorem 2 holds also for $\mathbb{R}^d$ valued random walk with properly modified $\Lambda(\cdot), \Lambda^*(\cdot)$. It also holds for the random walk $Y_\epsilon(\cdot)$ itself, if $C_0[0,1]$ is replaced throughout by $D_0[0,1]$, the space of functions satisfying $\phi(0) = 0$ which are continuous from the right and possess left limits, and the latter space is equipped with the uniform topology, see [4, 9]).

Since $b(\cdot)$ is Lipschitz continuous, the map $\psi = F(\phi)$ defined by

$$\psi(t) = \int_0^t b(\psi(s))ds + \phi(t)$$

is a continuous map of $C_0([0,1])$ onto itself. Therefore, by the contraction principle of large deviations theory (see [10], page 5, remark 1, or [4], Theorem 4.2.1), $\tilde{Z}_\epsilon \triangleq F(\tilde{Y}_\epsilon)$ satisfies the LDP in $C_0([0,1])$ with the good rate function $I(\cdot)$ of (4). Thus, the proof is completed provided that $Z_\epsilon(\cdot)$ and $\tilde{Z}_\epsilon(\cdot)$ are exponentially equivalent, namely that for any $\eta' > 0$

$$\lim_{\epsilon \to 0} \epsilon \log \mathrm{P}(\sup_{t \in [0,1]} |\tilde{Z}_\epsilon(t) - Z_\epsilon(t)| \geq \eta') = -\infty\,. \tag{10}$$

To this end, recall that

$$Z_\epsilon(t) - Z_\epsilon\left(\epsilon\left[\frac{t}{\epsilon}\right]\right) = \left(t - \epsilon\left[\frac{t}{\epsilon}\right]\right)\left(b(Z_{[\frac{t}{\epsilon}]}) + \nu_{[\frac{t}{\epsilon}]}\right)\,,$$

and therefore,

$$\int_0^1 \left| Z_\epsilon(t) - Z_\epsilon\left(\epsilon\left[\frac{t}{\epsilon}\right]\right)\right| dt \leq \frac{\epsilon^2}{2}\left(\max_{k=0}^{[\frac{1}{\epsilon}]} |\nu_k| + \sup_{z \in \mathbb{R}} |b(z)|\right)\,.$$

Since $b(\cdot)$ is bounded, it now follows that for any $\eta > 0$ and any $\lambda > 0$,

$$\begin{aligned} \lim_{\epsilon \to 0} \epsilon \log \mathrm{P}\left(\int_0^1 \left| Z_\epsilon(t) - Z_\epsilon\left(\epsilon\left[\frac{t}{\epsilon}\right]\right)\right| dt \geq \eta\right) &\leq \lim_{\epsilon \to 0} \epsilon \log \frac{1}{\epsilon}\mathrm{P}\left(|\nu_0| \geq \frac{\eta}{2\epsilon}\right) \\ &\leq \lim_{\epsilon \to 0} \epsilon \log\left(e^{-\lambda\eta/2\epsilon} E(e^{\lambda|\nu_0|})\right) = -\lambda\eta/2, \end{aligned} \tag{11}$$

where the assumption that $\Lambda(\lambda)$ is finite was used. Taking $\lambda \to \infty$, it follows that

$$\lim_{\epsilon \to 0} \epsilon \log \mathrm{P}\left(\int_0^1 \left| Z_\epsilon(t) - Z_\epsilon\left(\epsilon\left[\frac{t}{\epsilon}\right]\right)\right| dt \geq \eta\right) = -\infty$$

Observe that by iterating (3),

$$Z_\epsilon(t) = \int_0^t b\left(Z_\epsilon\left(\epsilon\left[\frac{s}{\epsilon}\right]\right)\right) ds + \tilde{Y}_\epsilon(t) \ ,$$

while

$$\tilde{Z}_\epsilon(t) = \int_0^t b(\tilde{Z}_\epsilon(s))ds + \tilde{Y}_\epsilon(t) \ .$$

Let $e(t) = |\tilde{Z}_\epsilon(t) - Z_\epsilon(t)|$. Then,

$$\begin{aligned} e(t) &\leq \int_0^t \left| b(\tilde{Z}_\epsilon(s)) - b\left(Z_\epsilon\left(\epsilon\left[\frac{s}{\epsilon}\right]\right)\right)\right| ds \\ &\leq B\int_0^t e(s)ds + B\int_0^t \left|Z_\epsilon(s) - Z_\epsilon\left(\epsilon\left[\frac{s}{\epsilon}\right]\right)\right| ds \ , \end{aligned}$$

where B is the Lipschitz constant of $b(\cdot)$. Hence, Gronwall's inequality yields

$$\sup_{0\leq t\leq 1} e(t) \leq Be^B \int_0^1 \left|Z_\epsilon(s) - Z_\epsilon\left(\epsilon\left[\frac{s}{\epsilon}\right]\right)\right| ds \ ,$$

and the exponential equivalence of $Z_\epsilon(\cdot)$ and $\tilde{Z}_\epsilon(\cdot)$ follows by (11). □

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