Precise large deviation estimates for one-dimensional random walk in random environment

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Abstract

Suppose that the integers are assigned i.i.d. random variables $\{\omega_x\}$ (taking values in the interval [1/2,1)), which serve as an environment. This environment defines a random walk $\{X_k\}$ (called a RWRE) which, when at x, moves one step to the right with probability ω_x, and one step to the left with probability $1-\omega_x$. Solomon (1975) determined the almost-sure asymptotic speed (=rate of escape) of a RWRE, in a more general set-up. Dembo, Peres and Zeitouni (1996), following earlier work by Greven and den Hollander (1994) on the quenched case, have computed rough tail asymptotics for the empirical mean of the annealed RWRE. They conjectured the form of the rate function in a full LDP. We prove in this paper their conjecture. The proof is based on a "coarse graining scheme" together with comparison techniques.

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1 Introduction

In this paper, we continue the study, initiated in [3], [2], and [4], of tail estimates for a nearest-neighbor random walk on $\mathbb{Z}$ with site-dependent transition probabilities. We start with the sample space $\Omega = [1/2,\,1)^{\mathbb{Z}} = \{\omega = (\omega)_{x\in\mathbb{Z}}\,;\, 1/2 \le \omega_x < 1\}$ which serves as a "random environment". For a given probability distribution α on $[1/2,\,1)$, we denote by $\mathbf{P}^\alpha$ the product measure $\otimes^{\mathbb{Z}}\alpha$ on Ω. In order to describe a random walk in our random environment, we first set $W = \{(w_n)_{n\in\mathbb{N}}\,;\, w_n \in \mathbb{Z}\}$ and $X_n(w) = w_n$. For every fixed ω, we consider the Markov chain $(X_n)_{n\ge 0}$ on $\mathbb{Z}$ starting at x with transition probabilities

$$\mathbf{P}_x^\omega[X_{n+1} = y \mid X_n = z] = \begin{cases} \omega_z & \text{if } y = z+1 \\ 1-\omega_z & \text{if } y = z-1 \\ 0 & \text{otherwise}\,, \end{cases} \tag{1}$$

where $\mathbf{P}_x^\omega[\,\cdot\,]$ denotes the measure on the path space W for the given environment ω. We usually write $\mathbf{P}^\omega$ instead of $\mathbf{P}_0^\omega$. Finally we define the "annealed" measure by setting

$$\mathbf{P}_x[\,\cdot\,] = \int \mathbf{P}_x^\omega[\,\cdot\,]\,\mathbf{P}^\alpha(d\omega)\,,$$

and omit the subscript x for $x = 0$. The process X, when governed by $\mathbf{P}_x$, is an example of a *random walk in random environment* (RWRE). Note that $\mathbf{P}_x^\omega$ can be viewed as the conditional measure on the path space given the environment ω, and is often referred to as the "quenched" setting. For a discussion of the different regimes that the RWRE (X_n) exhibits, we refer to the introduction in [2].

Abbreviate $\rho = \rho(x,\omega) = (1-\omega_x)/\omega_x$ and set $\langle\rho\rangle = \int \rho_0(\omega)\mathbf{P}^\alpha(d\omega) = \mathbf{E}^\alpha[\,\rho_0\,]$, where here and throughout $\mathbf{E}^\alpha[\ \ ]$ denotes expectation with respect to the measure $\mathbf{P}^\alpha$. In the situation where $\langle\rho\rangle < 1$, the RWRE is transient (c.f. [7]), and, $\mathbf{P}$-a.s.,

$$\lim_{n\to\infty} n^{-1}X_n = v_\alpha = (1-\langle\rho\rangle)/(1+\langle\rho\rangle)\,.$$

Tail estimates for X_n/n have been derived for the quenched setting in [3]. In particular, it was shown there that, for $\mathbf{P}$-a.e. ω, the random variables X_n/n satisfy with respect to $\mathbf{P}^\omega$ a large deviation principle of speed n and explicit, deterministic, rate function which vanishes on $(0, v_\alpha)$.

Rough tail estimates for the annealed case have been derived in [2]. In the case considered here, the following was shown in Theorem 1.2 of [2].

Theorem 1 (Positive and zero drifts) *Suppose that $\alpha(\{1/2\}) > 0$ and $\langle\rho\rangle < 1$. Then, for any open and nonempty $G \subset (0, v_\alpha)$ such that $\overline{G} \subset [0, v_\alpha)$, we have*

$$-\inf_{v\in G} I(v) \le \liminf_{n\to\infty} n^{-1/3}\log\mathbf{P}(n^{-1}X_n \in G) \le \limsup_{n\to\infty} n^{-1/3}\log\mathbf{P}(n^{-1}X_n \in G) < 0\,, \tag{2}$$

where $\overline{G}$ denotes the closure of G and $I(v)$ is given by

$$I(v) = \inf_{s \geq 0} \left\{ |\log \alpha(\{1/2\})| \, s + \frac{\pi^2}{8s^2}(1 - v/v_\alpha) \right\} = \frac{3}{2} |\frac{\pi}{2} \log \alpha(\{1/2\})|^{2/3} \, (1 - v/v_\alpha)^{1/3}. \tag{3}$$

It was further conjectured in [2] (Remark 2, p. 681) that the lower bound in Theorem 1 is sharp, namely that a full LDP holds with speed $n^{1/3}$ and rate function I. Our goal in this paper is to prove this conjecture. Namely, we will show the following.

Theorem 2 (Positive and zero drifts) *Suppose that $\alpha(\{1/2\}) > 0$ and $\langle \rho \rangle < 1$. Then, for any open and nonempty $G \subset (0, v_\alpha)$ such that $\overline{G} \subset [0, v_\alpha)$,*

$$- \inf_{v \in G} I(v) \leq \liminf_{n \to \infty} n^{-1/3} \log \mathbf{P}(n^{-1} X_n \in G) \leq \limsup_{n \to \infty} n^{-1/3} \log \mathbf{P}(n^{-1} X_n \in \overline{G}) \leq - \inf_{v \in \overline{G}} I(v)$$

We continue with some comments on the strategy of the proof the upper bound in Theorem 2. First, it is worthwile to recall how the lower bound in Theorem 1 was obtained. Fix $v < v_\alpha$. The essence of the derivation (see beginning of Section 4 as well as the bottom of p. 670 in [2]) is that the walk, being in the vicinity of the point nv at time n, has lost time in a single region which is made of fair sites. (We call a site x fair if $\omega_x = 1/2$ and biased otherwise.) It turns out that the optimal length of this fair region (optimal in the sense that it gives the largest lower bound) is of the order $n^{1/3}$. Inside this region, the walk behaves like a simple random walk reflected at the leftmost point of this region and it stays there for approximately $n(1 - v/v_\alpha)$ steps. Outside this region we expect the walk and the environment to be typical.

In order to prove the upper bound, our aim is to show that the event $\{X_n/n \leq v\}$ comes typically from configurations where the walk stays at least $n(1-v/v_\alpha)$ steps in "long" regions of length of order $n^{1/3}$ where biased coins are "sparse," and that such configurations are responsible for the leading asymptotics. We then consider the "cost" that the environment produces such large regions as well as the cost for the walk to spend approximately $n(1 - v/v_\alpha)$ steps in them. This will give us a variational problem leading to the function $I(v)$.

At this point it is noteworthy to mention that the model we study here is to some extent analogous to annealed Brownian motion in a Poissonian potential, see [6] and Sznitman [10]. Indeed, the specification of what we mean by long fair regions and what is sparse, is in the same spirit as in the coarse graining scheme developed in [6].

In our situation, the coarse graining works as follows. For the sake of simplicity, we now assume that α is concentrated on two atoms; $1/2$ and, say, $2/3$. We partition $\mathbb{Z}$ into blocks of length n^θ, where $\theta = 1/3 - \delta$, and $\delta \in (0, 1/24)$. (The choice $\delta < 1/24$ will play a role later in the paper). We pick a parameter $\varepsilon \in (0, 1)$, which eventually will go to zero, and declare a block to be "biased", if the proportion of biased sites in it is larger then ε. If this is not the case, the block will be called fair. A long region where biased sites are sparse corresponds to connected components of fair blocks which have a length at least $rn^{1/3}$ (r will eventually go to zero). We will then show, using bounds in the spirit of Lemma 2.2 in [2], that we can neglect configurations where the walk makes more than Kn^δ left crossings of biased blocks, provided K is chosen large enough. This, together with the fact that we can assume that there are at most Kn^δ fair blocks in the natural range of the walk, plays an important role. It allows us to control the number of visits of fair regions, since, by construction, they have a biased block attached to their left and to their right side. Furthermore, we will see that (provided r is small enough) the probability that the walk spends at least $n\,v/v_\alpha$ steps in the complement of long fair regions, is negligible.

At this point we should mention, that the analogous step in [6] (in the context of Brownian motion in the random potential V) heavily relied on good lower bounds on the principal Dirichlet eigenvalue of the operator $-\frac{1}{2}\frac{d^2}{dx^2}+V$ in an interval I, which could be found in [9]. In our situation of random drift, however, we lack the knowledge of an analogous result.

Before we close the introduction with some remarks, let us explain how the article is organized. In Section 2.1 we introduce our coarse graining scheme and some basic objects which are needed in the sequel. In Section 2.2 we proceed with some preparatory lemmas and with the basic reduction step. In particular, we partition the space $\mathbf{\Omega}$, over which the integration is performed, into three events A_1^c, A_2, and A_3. Here, A_1^c corresponds to the event that there is an atypically long excursion to the left or that there is an atypically large number of fair blocks or that there is an atypical number of crossings of biased blocks from right to left, while A_2 denotes the event that the walk spends a large time in *short* fair blocks and A_3 denotes the event that it spends a large time in *long* fair blocks. We also show in that section that A_1^c is negligible compared to the upper bound in Theorem 2. In Section 2.3 we investigate the event A_3 which gives our desired upper bound. In Section 2.4, which is the most delicate part of the proof of Theorem 2, we show that the event A_2 is negligible, provided our parameters are chosen in an appropriate way. In the Appendix, we give the proof of Lemma 5, which is crucial in the investigation of A_2. The proof of this lemma uses results and techniques from [2] and [4].

Remarks

1. A challenge is to extend the above considerations to the case where the support of α includes points both to the right and to the left of $1/2$. In this case, the annealed rate of decay is polynomial while the quenched rate of decay is exponential in n^β, some $\beta > 0$, and is conjectured to fluctuate on subsequences (c.f. [4]).

2. Observe that due to the right continuity of $I(\cdot)$ at 0, the upper bound in Theorem 2 also holds for $\overline{G} = [-a, 0]$, $a > 0$, where in this case we have to replace $\inf_{v\in\overline{G}} I(v)$ by $I(0)$. It is also easy to see that the strategy which is used to prove the lower bound in Theorem 1 can be extended to the event $\{X_n \leq 0\}$, again with rate $I(0)$.

3. We have concentrated in the region $[0, v_\alpha)$ because this is the region where the RWRE exhibits behaviour significantly different from a simple random walk. Outside the region $[0, v_\alpha]$, the decay rate is exponential

in n, c.f. [2], Theorem 1.3.

2 The proof of Theorem 2.

In this section we give the proof of the upper bound in Theorem 2. Observe that it is enough to establish the upper bound for $\overline{G} = [0, v]$, with $0 < v < v_\alpha$. We suppose that from now on, we are working with a α for which the assumptions of Theorem 2 are valid. In particular, we have that $\alpha(\{1/2\}) \in (0, 1)$.

2.1 Basic definitions.

As previously mentioned in the introduction, the proof of Theorem 2 is based on a description of the environment in terms of fair and biased blocks. Let $\delta \in (0, 1/24)$ be fixed, and set

$$\theta = 1/3 - \delta \tag{4}$$

Let $n > 1$, and divide $\mathbb{Z}$ into blocks B_j of length $\lfloor n^\theta \rfloor$:

$$B_j = [j\lfloor n^\theta \rfloor, (j+1)\lfloor n^\theta \rfloor) \cap \mathbb{Z} \qquad (j \in \mathbb{Z})$$

where $\lfloor n^\theta \rfloor$ denotes the integer part of n^θ. Pick a $\xi \in (0, 1/2)$ such that

$$\alpha\left([1/2, 1/2 + \xi)\right) < 1 \tag{5}$$

Observe that thanks to our assumptions on α, such a ξ always exists. Pick ε with $0 < \varepsilon < 1 - \alpha([1/2, 1/2+\xi))$. We say that the block B_j is *biased* if the proportion of sites x in B_j with $\omega_x \geq 1/2 + \xi$ is strictly larger than ε. Otherwise the block is called *fair* (at the end we let $\varepsilon \to 0$, followed by $\xi \to 0$). The random sets of indices of biased and fair blocks are given by

$$J^b = \{j \in \mathbb{Z}\,;\, B_j \text{ is biased}\} \quad \text{ and } J^f = (J^b)^c.$$

It will be convenient to declare a subset $S \subset \mathbb{Z}$ to be connected, if for any pair of sites $x, y \in S$ there exists a sequence of consecutive pairs of nearest neighbors in S linking x to y. Note that all the sites contained in fair blocks can be decomposed into connected components, which we call *fair regions.* We will divide these regions into two classes according to whether they are "long" or "short". More precisely, for fixed parameter $r \in (0, 1)$ (which will tend to zero finally), and for $s \in \{1, 2\}$, we define

$$\mathcal{F}^{(s)} = \bigcup_{i \in I(s)} \mathcal{F}_i^{(s)} \tag{6}$$

where the $\mathcal{F}_i^{(s)}$-s are the connected components of the set $\bigcup_{j \in J^f} B_j$ with the property

$$|\mathcal{F}_i^{(1)}| \geq rn^{1/3} \quad \text{and} \quad |\mathcal{F}_i^{(2)}| < rn^{1/3} \tag{7}$$

and $I(1)$ and $I(2)$ denote arbitrary index sets. We will specify these sets later, according to our needs. In order to control the time the walk spends in various regions, we enlarge these components by attaching, if present, one biased block to them on the left and one to the right, respectively. Observe that only for $\mathcal{F}_i^{(1)}$ which have an infinite length, possibly both of these biased blocks are not present. The enlargement of $\mathcal{F}_i^{(s)}$ will be denoted by $\overline{\mathcal{F}}_i^{(s)}$, and we set

$$\overline{\mathcal{F}}^{(s)} = \bigcup_{i \in I(s)} \overline{\mathcal{F}}_i^{(s)} \tag{8}$$

with obvious notations if for some $i \in I(1)$: $|\mathcal{F}_i^{(1)}| = \infty$. The next step is to introduce stopping times at which the walk enters $\mathcal{F}_i^{(s)}$ and leaves the enlarged region $\overline{\mathcal{F}}_i^{(s)}$, consecutively. For $s \in \{1, 2\}$, we define

$$R_{i,1}^{(s)} = \inf\{k \geq 0\,;\, X_k \in \mathcal{F}_i^{(s)}\}, \tag{9}$$

the first time the walk visits $\mathcal{F}_i^{(s)}$, and

$$D_{i,0}^{(s)} = 0, \qquad D_{i,1}^{(s)} = \inf\{k > R_{i,1}^{(s)}\,;\, X_k \notin \overline{\mathcal{F}}_i^{(s)}\} \tag{10}$$

the first time after $R_{i,1}^{(s)}$, the process exits $\overline{\mathcal{F}}_i^{(s)}$. For $i \in I(s)$, we set inductively

$$R_{i,k+1}^{(s)} = D_{i,k}^{(s)} + R_{i,1}^{(s)} \circ \vartheta_{D_{i,k}^{(s)}}, \qquad D_{i,k+1}^{(s)} = R_{i,k+1}^{(s)} + D_{i,1}^{(s)} \circ \vartheta_{R_{i,k+1}^{(s)}}, \qquad (k \geq 1) \tag{11}$$

where ϑ denotes the canonical shift. $R_{i,k}^{(s)}$ and $D_{i,k}^{(s)}$ are called the *begin* and the *end of the k-th visit* of the region $\mathcal{F}_i^{(s)}$, respectively. Observe that it is of course possible that $R_{i,k}^{(s)} = D_{i',k'}^{(s')}$. Using these variables we express the *duration of the k-th visit* of the region $\mathcal{F}_i^{(s)}$, as

$$T_{i,k}^{(s)} = D_{i,k}^{(s)} \wedge n - R_{i,k}^{(s)} \wedge n \qquad (k \geq 1) \tag{12}$$

For the region $\mathcal{F}_i^{(s)}$, we denote the total number of visits by $V_i^{(s)} = |\{k \geq 1; T_{i,k}^{(s)} > 0\}|$. It will be important to look at $S_n^{(s)}$, the total amount of time spent visiting the fair regions, up to time n

$$S_n^{(s)} = \sum_{i \in I(s)} \sum_{k=1,\ldots,V_i^{(s)}} T_{i,k}^{(s)} \tag{13}$$

with obvious notations if $V_i^{(s)} = 0$.

Finally, we introduce left crossings of biased blocks (up to time n), as follows. For $x \in \mathbb{Z}$, $k \geq 1$, we set

$$\tau_x^0 = 0, \qquad \tau_x^k = \inf\{t > \tau_x^{k-1}\,;\, X_t = x\} \tag{14}$$

the successive times of hitting the site x. During the time intervals $(\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]$, $(k \geq 1)$, excursions (starting at x) take place either to the left or to the right of x (unless the time interval is empty.) The height of such an excursion is defined by

$$H_x^k = \begin{cases} \max\left\{X_t - x\,;\, t \in (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]\right\} & ; \quad X_{\tau_x^k+1} > x \quad \text{(right excursion)} \\ \min\left\{X_t - x\,;\, t \in (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n]\right\} & ; \quad X_{\tau_x^k+1} < x \quad \text{(left excursion)} \\ 0 & ; \quad (\tau_x^k \wedge n, \tau_x^{k+1} \wedge n] = \emptyset \end{cases} \tag{15}$$

The number of left crossings of the block B_j until time n is given by

$$N_{j,n} = |\{k\,;\, H^k_{(j+1)\lfloor n^\theta \rfloor} \leq -\lfloor n^\theta \rfloor\}| \tag{16}$$

and the total number of left crossings of biased blocks up to time n can be written as

$$N_n = \sum_{j \in J^b} N_{j,n}. \tag{17}$$

2.2 Basic reduction: few fair blocks, not too many or too long left crossings.

In order to prove the upper bound in Theorem 2, we have to estimate the probability of the event $\{X_n/n \leq v\}$ for $v \in (0, v_\alpha)$. In this section we break up this event into three events: A_1^c, A_2 and A_3 and we will show about the first of these events that it is negligible for the purpose of proving Theorem 2. The other two events will be discussed in the following sections. We start with the description of A_1 which is itself the intersection of three further events. Pick $K > 0$ and set

$$G_1(K,n) = \left\{\exists \text{ left crossing of length } Kn^{1/3} \text{ up to time } n\right\} = \bigcup_{0 \leq t < s \leq n} \left\{X_s \leq X_t - Kn^{1/3}\right\} \tag{18}$$

We will drop the arguments of G_1. By Lemma 2.2 in [2], we have that

$$\mathbf{P}[G_1] \leq 2n(n+1)^2 \frac{1}{1 - \langle\rho\rangle} \langle\rho\rangle^{Kn^{1/3}},$$

which immediately implies

$$\limsup_{n\to\infty} n^{-1/3} \log \mathbf{P}[G_1] \leq -K |\log\langle\rho\rangle| \,. \tag{19}$$

Hence, when K is chosen large enough, G_1 is negligible for the purpose of proving Theorem 2. In particular, on the set $\{X_n/n \leq v\} \cap G_1^c$, the walk is confined to the interval

$$U = U(v,K,n) = [-Kn^{1/3}, \lfloor vn + Kn^{1/3} \rfloor] \tag{20}$$

We next want to get an upper bound on the number of fair blocks in the range of the walk up to time n. For $K > 0$ we set

$$G_2(\xi, \varepsilon, \delta, K, n) = \left\{|\{j \in J^f\,;\, B_j \cap [-n,n] \neq \emptyset\}| > Kn^\delta\right\} \tag{21}$$

Lemma 1 *Let $\delta < 1/24$ and $\varepsilon \in (0, 1-\alpha([1/2, 1/2+\xi)))$. We then have*

$$\limsup_{n\to\infty} n^{-1/3} \log \mathbf{P}^\alpha[G_2] \le -K\Lambda^*_{p(\xi)}(\varepsilon)\,,$$

where $\Lambda^*_{p(\xi)}(\cdot)$ denotes the logarithmic moment generating function of a Bernoulli variable with parameter $p(\xi) = 1-\alpha([1/2, 1/2+\xi)) \in (0,1)$. (Note that $\varepsilon < p(\xi)$, thus $\Lambda^*_{p(\xi)}(\varepsilon) > 0$.)

Proof: First, by using Cramer's Theorem we easily obtain the following estimate on the probability that a given block is fair:

$$\mathbf{P}^\alpha[B_1 \text{ is fair }] \le \exp(-\lfloor n^\theta \rfloor \Lambda^*_{p(\xi)}(\varepsilon)) \tag{22}$$

where we recall that

$$\Lambda^*_p(x) = 1_{[0,1]^c}(x)\cdot \infty + 1_{[0,1]}(x)\Big(x\log\frac{x}{p} + (1-x)\log\frac{1-x}{1-p}\Big)$$

Since $\mathbf{P}[G_2] \le (2n+2)^{Kn^\delta+2}\mathbf{P}^\alpha[B_1 \text{is fair}]^{Kn^\delta}$, and recalling that $\theta = 1/3-\delta$, $\delta < 1/24$, Lemma 1 follows. □

The next Lemma plays a key role in what follows. It gives control over N_n, the total number of left crossings of biased blocks until time n, cf. (17). For $K > 0$ we define

$$G_3(\xi, \varepsilon, \delta, K, n) = \{N_n > Kn^\delta\} \tag{23}$$

Lemma 2 *Let $\delta < 1/24$ and $\varepsilon \in (0, 1-\alpha([1/2, 1/2+\xi)))$. Then*

$$\limsup_{n\to\infty} n^{-1/3} \log \mathbf{P}[G_3] \le -\varepsilon K|\log \rho^*(\xi)| \tag{24}$$

where $\rho^*(\xi) = (1/2-\xi)/(1/2+\xi) \in (0,1)$.

Proof: Instead of working directly with the annealed measure $\mathbf{P}$, we derive an appropriate upper bound on $\mathbf{P}^\omega[G_3]$ which is *uniform* in the environment ω. Set $x_j = j\lfloor n^\theta \rfloor$, $(j \in \mathbb{Z})$. The main ingredient is the following estimate: for every $j \in \mathbb{Z}$,

$$\begin{aligned}\sup_{y\in\mathbb{Z}} \mathbf{P}^\omega_y[N_{j-1,n} \ge 1] &= \mathbf{P}^\omega_{x_j}[N_{j-1,n} \ge 1] \le \mathbf{P}^\omega_{x_j}[\tau^1_{x_{j-1}} < \tau^1_{n+x_j},\ \tau^1_{x_{j-1}} < \infty] \\ &\le \sum_{r=x_j}^{n+x_j-1} \prod_{y=x_{j-1}+1}^{r} \rho(y,\omega)\,,\end{aligned} \tag{25}$$

where $N_{j-1,n}$ denotes the number of left crossings of B_{j-1} until time n, and the last inequality follows from [1], eq. preceding (5) on p. 73. Note that if the block B_{j-1} is biased, then there are at least $(\varepsilon\lfloor n^\theta \rfloor - 1)$ ρ-s in the product above with $\rho(y,\omega) \geq \rho^*(\xi)$. Therefore,

$$\sup_{x\in\mathbb{Z}} \mathbf{P}_x^\omega\Big[\bigcup_{\substack{j\in J^b(\omega)\\ -n^{1-\theta}\leq j\leq n^{1-\theta}}} \{N_{j,n}\geq 1\}\Big] \leq 2n^{2-\theta}\exp\big(-(\varepsilon\lfloor n^\theta\rfloor -1)\,|\log\rho^*(\xi)|\big) \tag{26}$$

We now want to apply the strong Markov property. To this end we define L_1 to be the first time a left crossing of a biased block has been completed, i.e.

$$L_1 = \inf\big\{t\geq 0\,;\, X_t\in\{j\lfloor n^\theta\rfloor\,;\, j\in J^b\},\, \exists s\in[0,t) : X_s - X_t = \lfloor n^\theta\rfloor\big\}$$

and inductively ($k\geq 1$): $L_{k+1} = L_k + L_1\circ\vartheta_{L_k}$. Since on G_3 the $\lfloor Kn^\delta\rfloor$-th left crossing of a biased block had been finished before time n, we find by iterating the strong Markov property

$$\begin{aligned}\mathbf{P}^\omega[G_3] &\leq \mathbf{P}^\omega[L_{\lfloor Kn^\delta\rfloor}<n] \leq \mathbf{P}^\omega[L_{\lfloor Kn^\delta\rfloor-1}<n,\, L_1\circ\vartheta_{L_{\lfloor Kn^\delta\rfloor-1}}<n] \leq \sup_{x\in\mathbb{Z}}\mathbf{P}_x^\omega[L_1<n]^{\lfloor Kn^\delta\rfloor}\\ &\leq (2n^{2-\theta})^{\lfloor Kn^\delta\rfloor}\exp\big(-(\varepsilon\lfloor n^\theta\rfloor-1)\,\lfloor Kn^\delta\rfloor|\log\rho^*|\big)\ .\end{aligned}$$

Since $\delta<1/24$ and $\theta+\delta=1/3$, the lemma follows. □

We are now ready to define the (typical) event A_1:

$$A_1(\xi,\varepsilon,\delta,K,n) = \big\{\ \nexists \text{ left crossing of length } Kn^{1/3}\ ,\ |\{j\in J^f\,;\, B_j\subset[-n,n]\}|\leq Kn^\delta\ ,\ N_n\leq Kn^\delta\big\} \tag{27}$$

Since $A_1^c = G_1\cup G_2\cup G_3$, by (19) and Lemmas 1 and 2, we have

$$\limsup_{K\to\infty}\ \limsup_{n\to\infty} n^{-1/3}\log\mathbf{P}[A_1^c] = -\infty \tag{28}$$

Fix $v\in(0,v_\alpha)$. As a last global parameter, we introduce $\zeta>0$ specified in (29) (at the end we will let ζ go to zero). It will be useful to give a list of all our global parameters with their allowed range (and their final limits)

v :	$0<v<v_\alpha$	fixed
δ :	$0<\delta<1/24$	fixed
K :	$1<K<\infty$	($K\to\infty$, makes A_1^c negligible)
r :	$0<r<1$	($r\to 0$, controls the maximal length of short fair blocks)
ε :	$0<\varepsilon<1-\alpha([1/2,1/2+\xi)$	($\varepsilon\to 0$, controls the maximal proportion of biased sites in a fair block)
ζ :	$0<\zeta<1-v/v_\alpha$	($\zeta\to 0$, controls the time spent outside long fair regions)
ξ :	$0<\alpha([1/2,1/2+\xi))<1$	($\xi\to 0$, controls biasdness of sites) (29)

The parameters K and r are used to get rid of various negligible terms, whereas ε, ζ and ξ are used to get the control in the upper bound arbitrarily close to the rate function.

Finally, as promised, we break up the event $\{X_n/n \le v\}$ as follows:

$$\begin{aligned}\{X_n/n \le v\} &= \Big(A_1^c \cap \{X_n/n \le v\}\Big) \cup \Big(A_1 \cap \{X_n/n \le v,\ S_n^{(1)} \le n(1-v/v_\alpha-\zeta)\}\Big) \cup \\ &\qquad \cup \Big(A_1 \cap \{X_n/n \le v, S_n^{(1)} > n(1-v/v_\alpha-\zeta)\}\Big) \\ &:= (A_1^c \cap \{X_n/n \le v\}) \cup A_2 \cup A_3 \,, \end{aligned} \tag{30}$$

where $S_n^{(1)}$ was defined in (13). Note that the events A_2 and A_3 depend on the parameters $(\xi, \varepsilon, \delta, K, v, \zeta, n)$. In view of (28) we know that $A_1 \cap \{X_n/n \le v\}$ is negligible for the purpose of proving Theorem 2. In the next two sections we will show that $P[A_3]$ is the leading term giving the rate $I(v)$ and $P[A_2]$ is negligible.

2.3 The leading term: time spent in long fair regions.

In this section we will show that by suitably adjusting our parameters, the event A_3 is giving the rate $I(v)$, where we recall that $I(v) = \frac{3}{2}\,|\frac{\pi}{2}\log\alpha(\{1/2\})|^{2/3}\,(1-v/v_\alpha)^{1/3}$. Indeed, we have

Proposition 2.1

$$\limsup_{\xi\to 0}\ \limsup_{\zeta\to 0}\ \limsup_{\varepsilon\to 0}\ \limsup_{r\to 0}\ \limsup_{K\to\infty}\ \limsup_{n\to\infty}\ \frac{1}{n^{1/3}}\log \mathbf{P}[\,A_3\,] \le -I(v) \tag{31}$$

Before we begin with the proof of Proposition 2.1, let us highlight some important steps. On the set A_3, we look at all distinct long fair regions $\mathcal{F}_i^{(1)}$ in $[-n, n]$, which is the natural range of the walk up to time n. It turns out that on A_3, there is at least one such region and that their number is bounded above by K/r, a constant which is independent of n and ω. We will then bound the probability of A_3 by the probability that the walk stays at least $n(1-v/v_\alpha-\zeta)$ steps in $\bigcup_i \overline{\mathcal{F}}_i^{(1)} \cap [-n,n]$ multiplied by the probability that the environment produces such regions.
To get hands on the first factor, we will use the following fact. Since our environment produces a drift to the right, the probability that the RWRE doesn't exit an interval J until time $t > 0$ is bounded above by the probability that the RWRE, reflected at the left endpoint of J, doesn't exit this interval until time $t > 0$, and J only consists of fair coins. We are then in a situation where we can proceed by using classical eigenvalue estimates.
Finally, the second factor is easily controlled: Choosing our parameters appropriatly, the probability that $\overline{\mathcal{F}}_i^{(1)} \cap [-n,n] \neq \emptyset$ occurs is, up to correction terms of the order $\exp\{o(n^{1/3})\}$, bounded above by the probability that $\overline{\mathcal{F}}_i^{(1)} \cap [-n,n]$ consists only of fair sites.

Proof: For fixed parameters, as in (29), we will partition the set A_3 into a family $\mathcal{A}$ of cardinality $|\mathcal{A}| = \exp o(n^{1/3})$. Thus, once we have shown

$$\limsup_{\xi\to 0}\ \limsup_{\zeta\to 0}\ \limsup_{\varepsilon\to 0}\ \limsup_{r\to 0}\ \limsup_{K\to\infty}\ \limsup_{n\to\infty}\ \sup_{A\in\mathcal{A}}\ n^{-1/3}\log \mathbf{P}[\,A\,] \le -I(v)\ , \tag{32}$$

(31) will follow. In order to describe the events $A \in \mathcal{A}$, the first step is to introduce the random variable $H(\omega)$ which stands for the number of distinct long fair regions intersecting the interval $[-n,\, n]$. We claim that on the set A_3, the following bounds hold:

$$1 \le H \le (2 + K/r) \tag{33}$$

The lower bound is implied by $S_n^{(1)} > 0$. The upper bound follows from the fact that, on A_3, we have by the definition of A_1 at most Kn^δ fair blocks with total length $l \le Kn^\delta \lfloor n^\theta \rfloor$. On the other hand the total length of all the (at least $H-2$) long fair regions completely contained in $[-n,\, n]$ is at least $(H-2)rn^{1/3}$, and we already know that it can not exceed $Kn^\delta \lfloor n^\theta \rfloor$. Thus we have (33). It will be convenient to use the following labeling of the long fair regions on the set A_3. $\mathcal{F}_1^{(1)}$ is the leftmost long fair stretch intersecting $[-n,\, n]$, $\mathcal{F}_2^{(1)}$ is the next one to the right, etc., and $\mathcal{F}_H^{(1)}$ is the righmost one which intersects $[-n,\, n]$. Observe that for $k \ne j$, $\mathcal{F}_k^{(1)} \cap \mathcal{F}_j^{(1)} = \emptyset$. The times $R_{i,k}^{(1)}$, $T_{i,k}^{(1)}$ etc. will be labeled accordingly. We have to keep track of all the possible configurations (locations) of the regions $\mathcal{F}_i^{(1)}$, $1 \le i \le H$, as well. To this end, we specify the left and right endpoints (of the relevant parts) of these regions as follows: For $i = 1, ..., H$, we define x_i ($\tilde{y}_i$) to be the leftmost (rightmost) point of the leftmost (righmost) block which has a nonempty intersection with $\mathcal{F}_i^{(1)} \cap [-n, n]$. For notational convenience we set $y_i = \tilde{y}_i + 1$. On A_3, the number of all such sequences $(x_i, y_i)_{i=1,\dots,H}$ is certainly bounded by $(2n)^H \le (2n)^{(2+K/r)}$. As next, we define

$$\mathcal{M} = \mathcal{M}(\omega, w) = \left\{(i,k)\,;\, T_{i,k}^{(1)} > 0\right\} \tag{34}$$

the index set of all times associated with visits of long fair regions up to time n. Note that $T_{i,k}^{(1)} > 0$ implies that $i \in \{1, ..., H\}$. We set $M = |\mathcal{M}|$, and estimate it as follows. When the walk starts, it will possibly cross all the H essentially fair regions from the left making thereby only right crossings of the attached biased blocks. Note that after one of these blocks have been (right-) crossed, the next crossing of the same block is necessarily a left crossing. Since on A_3 the number of all left crossings of biased blocks is bounded by Kn^δ we have for all $n > 1$ and $K, 1/r$ large,

$$1 \le M \le H + 2Kn^\delta \le 4n^\delta K/r. \tag{35}$$

In order to use the strong Markov property, we have to describe the *historical* order of the relevant stopping times $R_{i,k}^{(1)}$, $(i,k) \in \mathcal{M}$. Most conveniently we do this by looking at the time evolution of the walk and enumerate the pairs of indices $(i,k) \in \mathcal{M}$ by attaching an additional index l to them, corresponding to the (historical) order they occur. Thus, for example, on the set $A_3 \cap \{H = h,\ M = m\}$, we have

$$0 \le R^{(1)}_{i_1,k_1} < D^{(1)}_{i_1,k_1} \le R^{(1)}_{i_2,k_2} < D^{(1)}_{i_2,k_2} \le \dots \le R^{(1)}_{i_l,k_l} < D^{(1)}_{i_l,k_l} \le \dots \le R^{(1)}_{i_m,k_m} < D^{(1)}_{i_m,k_m} \wedge n \ , \tag{36}$$

with $i_l \in \{1, ..., h\}$ for each $l \le m$. An upper bound on the number $\mathcal{O}$ of all such orderings of $\mathcal{M}$ on the set $A_3 \cap \{H = h,\ M = m\}$ is given by

$$\mathcal{O} \le h 3^{m-1} \le (2 + K/r) 3^{4n^\delta K/r} \tag{37}$$

Indeed, the factor h corresponds to the choice of i_1. Since the walk can not jump, we have

$$i_{l+1} \in \{i_l - 1, i_l, i_l + 1\} \qquad l = 1, ..., m-1 \ , \tag{38}$$

which explains the factor 3 in (37). Note finally that the reason why there is no additional factor in (37) which would correspond to the choice of the k_l-s is that for given $i_1, ..., i_m$, the indices $k_1, ..., k_m$ are *uniquely determined*. In fact they are given by

$$k_l = |\{1 \le r \le l;\ i_r = i_l\}| \tag{39}$$

We say that the indices $(k_l)_{1\le l\le m}$ satisfying (39) are *associated with* $(i_l)_{1\le l \le m}$. Since the right hand side of (37) is independent of h and m, the bound is valid on the entire set A_3. Observe that the ordering itself is uniquely determined for each fixed (ω, w). Hence, i_l and k_l are well defined random variables for $1 \le l \le M$. We now partition the set A_3 according to the values of H, M, $(i_l)_{1\le l\le M}$ and $(x_i, y_i)_{1 \le i \le H}$. We set

$$\begin{aligned} A \ &= \ A_3 \cap \{H = h,\ M = m\} \cap \\ &\qquad \cap \{(i_l)_{1\le l\le m} = (\bar{i}_l)_{1\le l\le m};\ \ (x_i, y_i)_{1\le i\le h} = (\bar{x}_i, \bar{y}_i)_{1\le i\le h}\} \ , \end{aligned} \tag{40}$$

where $(\bar{i}_l)_{1\le l\le m}$ and $(\bar{x}_i, \bar{y}_i)_{1\le i\le h}$ are fixed sequences in the image of the random variables $(i_l)_{1\le l\le m}$ and $(x_i, y_i)_{1\le i\le h}$ on the set $A_3 \cap \{H = h,\ M = m\}$ (with the (i_l)-s satisfying (38)).
We will use the convention that for given $(\bar{i}_l)_{1\le l\le m}$, $(\bar{k}_l)_{1\le l\le m}$ denote the indices associated with the (i_l)-s. The number $|\mathcal{A}|$ of sets into which A_3 has been partitioned can be bounded as follows:

$$|\mathcal{A}| \le (2 + K/r)\, (4n^\delta K/r)\, 3^{4n^\delta K/r} (2n)^{(2+K/r)} \tag{41}$$

Note that this bound is a function $\exp o(n^{1/3})$, for $n \to \infty$, as claimed previously. It remains to control the probabilities of the sets $A \in \mathcal{A}$, uniformly in A. Pick $\lambda > 0$. By using the Chebyshev inequality,

$$\mathbf{P}[\,A\,] \le \exp\Big(- \lambda n (1 - v/v_\alpha - \zeta)\Big)\, \mathbf{E}^\alpha\big[\,\mathbf{E}^\omega[\,1_A \exp(\lambda S^{(1)}_n)\,]\,\big] \ . \tag{42}$$

The following estimate (which can easily be verified) will be useful to control boundary effects in connection with the strong Markov property. For $(i,k) \in \mathcal{M}$:

$$T^{(1)}_{i,k} \le (D^{(1)}_{i,1} \wedge T_{[-n,\,n]}) \circ \vartheta_{R^{(1)}_{i,k}} \ , \tag{43}$$

where $T_{[-n,n]}$ denotes the exit time of the interval $[-n, n]$. Observe also that on A,

$$\big\{(x_i, y_i)_{1\le i\le h} = (\bar{x}_i, \bar{y}_i)_{1\le i\le h}\big\} \subset \bigcap_{i=1,\dots,h} \big\{[\bar{x}_i, \bar{y}_i) \text{ consists of fair blocks}\big\} =: C \tag{44}$$

Set $\widetilde{A} = \Big\{0 \le R^{(1)}_{\bar{i}_1,\bar{k}_1} < D^{(1)}_{\bar{i}_1,\bar{k}_1} \le R^{(1)}_{\bar{i}_2,\bar{k}_2} < D^{(1)}_{\bar{i}_2,\bar{k}_2} \le \dots \le R^{(1)}_{\bar{i}_{m-1},\bar{k}_{m-1}} < D^{(1)}_{\bar{i}_{m-1},\bar{k}_{m-1}} < n\Big\}$.

Using (43) and (44), we find

$$\begin{aligned}
\mathbf{E}^\omega\big[\, 1_A \, \exp(\lambda S_n^{(1)})\,\big] \;&=\; \mathbf{E}^\omega\big[\, 1_A \, \exp(\lambda \sum_{l=1,\dots,m} T^{(1)}_{\bar{i}_l,\bar{k}_l})\,\big] \\
\le 1_C(\omega)\mathbf{E}^\omega\big[\, 1_{\widetilde{A}} &\exp\Big(\lambda \sum_{l=1,\dots,m-1} T^{(1)}_{i,k}\Big) \exp\Big(\lambda(D^{(1)}_{\bar{i}_m,1} \wedge T_{[-n,n]}) \circ \vartheta_{R^{(1)}_{\bar{i}_m,\bar{k}_m}}\Big)\,\big] \\
\le 1_C(\omega)\mathbf{E}^\omega\big[\, 1_{\widetilde{A}} &\exp\Big(\lambda \sum_{l=1,\dots,m-1} T^{(1)}_{i,k}\Big)\,\big] \max_{x\in\{\bar{x}(\bar{i}_m),\, \bar{y}(\bar{i}_m)\}} \mathbf{E}^\omega_x\big[\, \exp\, \lambda(D^{(1)}_{\bar{i}_m,1} \wedge T_{[-n,n]})\big]\,,
\end{aligned} \tag{45}$$

where we applied the strong Markov property to the stopping time $R^{(1)}_{\bar{i}_m,\bar{k}_m}$. Note that we replaced $\bar{x}_{\bar{i}_m}$ by $\bar{x}(\bar{i}_m)$. By successively conditioning on the times $R^{(1)}_{\bar{i}_l,\bar{k}_l}$, we find

$$\mathbf{E}^\omega\big[\, 1_A \, \exp(\lambda S_n^{(1)})\,\big] \le 1_C(\omega) \prod_{l=1,\dots,m} \max_{x\in\{\bar{x}(\bar{i}_l),\, \bar{y}(\bar{i}_l)\}} \mathbf{E}^\omega_x\big[\exp \lambda(D^{(1)}_{\bar{i}_l,1} \wedge T_{[-n,n]})\big] \tag{46}$$

The next step is to give an upper bound on the exponential moment occurring in (46), uniformly in ω. For $i = 1, \dots, h$ we set $J_i = [\bar{x}_i - \lfloor n^\theta \rfloor, \bar{y}_i + \lfloor n^\theta \rfloor) \cap \mathbb{Z}$, and denote the interval obtained by doubling J_i symmetrically about its leftmost point by $2J_i$. (Note that $|2J_i| = 2|J_i| - 1$). We claim that for $1 \le l \le m$,

$$\sup_{\omega\in\Omega} \max_{x\in\{\bar{x}(\bar{i}_l),\, \bar{y}(\bar{i}_l)\}} \mathbf{E}^\omega_x\big[\exp \lambda(D^{(1)}_{\bar{i}_l,1} \wedge T_{[-n,n]})\big] \le \max_{x\in\{\bar{x}(\bar{i}_l),\, \bar{y}(\bar{i}_l)\}} \mathbf{E}^f_x\big[\exp \lambda(T_{2J_{\bar{i}_l}})\big]\,, \tag{47}$$

where T_I denotes the exit time from the interval I and $\mathbf{E}^f_x$ is the path measure of a (simple) symmetric random walk (SRW) starting from x. To show (47), we observe that for all $t \ge 0$, for all (finite) intervals $J \subset \mathbb{Z}$, $x \in J$ and $\omega \in \Omega$,

$$\mathbf{P}^\omega_x[T_J \ge t] \le \mathbf{P}^{\omega,\to}_x[T_J \ge t] \le \mathbf{P}^{f,\to}_x[T_J \ge t]\,, \tag{48}$$

where $\mathbf{P}^{\omega,\to}_x$ and $\mathbf{P}^{f,\to}_x$ denote the path measure of a RWRE and of a SRW starting at x with reflection at the left end point of J, respectively. But $\mathbf{P}^{f,\to}_x[T_J \ge t] = \mathbf{P}^f_x[T_{2J} \ge t]$. Hence, we have (47). Coming back to (46), we find

$$\mathbf{E}^\omega\big[\, 1_A \, \exp(\lambda S_n^{(1)})\,\big] \le 1_C \prod_{l=1,\dots,m} \max_{x\in\{\bar{x}(\bar{i}_l),\, \bar{y}(\bar{i}_l)\}} \mathbf{E}^f_x\big[\exp\big(\lambda T_{2J_{\bar{i}_l}}\big)\big] \tag{49}$$

A good bound on $\mathbf{E}^f_x[\exp(\lambda T_I)]$ is provided by the next (well known) lemma, which is also needed for later purpose. For the reader's convenience, we include a short proof.

Lemma 3 *Let $I=[-a,a]$, $a\geq 1$. For $\rho\in(0,1)$, set $\lambda=(1-\rho)\frac{\pi^2}{8a^2}$. There exists a constant $\chi=\chi(\rho)\in(1,\infty)$ such that for all a large enough*

$$\sup_{x\in I}\mathbf{E}_x^f[\exp(\lambda T_I)]\leq\chi(\rho) \tag{50}$$

Let us emphasize that χ does not depend on a.

Proof: By using symmetry and the strong Markov property it is easy to see that

$$\mathbf{E}_x^f[\exp(\lambda T_I)]\leq\mathbf{E}_0^f[\exp(\lambda T_I)]$$

As next we use

$$\mathbf{E}_0^f[\exp(\lambda T_I)]=1+\lambda\int_0^\infty e^{\lambda u}\mathbf{P}_0^f[T_I>u]\,du=1+\lambda a^2\int_0^\infty e^{\lambda a^2 u}\mathbf{P}_0^f[T_I/a^2>u]\,du \tag{51}$$

It follows from Spitzer [8], p. 243 that for all $\varepsilon'\in(0,1)$, there exists $c'(\varepsilon')\in(0,\infty)$ such that for all a large enough

$$\mathbf{P}_0^f[T_I/a^2>u]\leq c'(\varepsilon')\exp\Big(-u(1-\varepsilon')\pi^2/8\Big)\ ,\forall u\geq 0 \tag{52}$$

By setting $\varepsilon'=\rho/2$ and combining these estimates, (50) follows. □

In order to apply the previous lemma in (49), we pick $\rho\in(0,1)$ and set

$$\lambda=(1-\rho)\frac{\pi^2}{8\Big(\sum_{1\leq i\leq h}|J_i|\Big)^2} \tag{53}$$

Since $\bar{i}_l\in\{1,..,h\}$, we have for all $1\leq l\leq m$ that $\lambda\leq(1-\rho)\pi^2/(8|J_{\bar{i}_l}|^2)$. Hence, in view of Lemma 3 and (49) we have

$$\mathbf{E}^\omega[\,1_A\ \exp(\lambda S_n^{(1)})\,]\leq 1_C(\omega)\ \chi(\rho)^m \tag{54}$$

Using (42), (54) and the fact that the random environment is independent (with respect to $\mathbf{P}^\alpha$) in disjoint regions, we obtain, recalling that $m\leq 4n^\delta K/r$

$$\mathbf{P}[A]\leq\chi(\rho)^{4n^\delta K/r}\exp\Big(-n(1-\frac{v}{v_\alpha}-\zeta)\frac{(1-\rho)\pi^2}{8(\sum\limits_{1\leq i\leq h}|J_i|)^2}\Big)\prod_{1\leq i\leq h}\mathbf{P}^\alpha[\,[\bar{x}_i,\bar{y}_i)\text{ consists of fair blocks}\,] \tag{55}$$

Finally, using the fact that $\mathbf{P}^\alpha[\,B_1\text{ is fair}\,]\leq\exp(-\lfloor n^\theta\rfloor\Lambda^*_{p(\xi)}(\varepsilon))$ cf. (22), we find

$$\begin{aligned}\prod_{1\leq i\leq h}\mathbf{P}^\alpha[\,[\bar{x}_i,\bar{y}_i)\text{ consists of fair blocks}\,]\ &\leq\ \exp\Big(-\Lambda^*_{p(\xi)}(\varepsilon)\sum_{1\leq i\leq h}(\bar{y}_i-\bar{x}_i)\Big)\\ &=\ \exp(2h\lfloor n^\theta\rfloor\Lambda^*_{p(\xi)}(\varepsilon))\exp\Big(-\Lambda^*_{p(\xi)}(\varepsilon)\sum_{1\leq i\leq h}|J_i|\Big)\end{aligned} \tag{56}$$

Set $L(n) = n^{-1/3}\sum_{1\le i\le h}|J_i|$ and recall that $h \le (2+K/r)$. Using (55) and (56), we obtain

$$\begin{aligned}
\mathbf{P}[A] &\le \chi(\rho)^{4n^\delta \frac{K}{r}} \exp\left((4+\frac{2K}{r})\lfloor n^\theta\rfloor \Lambda^*_{p(\xi)}(\varepsilon)\right) \exp(-n^{1/3}\Big(L(n)\Lambda^*_{p(\xi)}(\varepsilon) + \frac{(1-\rho)\pi^2}{8L(n)^2}(1-\frac{v}{v_\alpha}-\zeta)\Big)) \\
&\le \chi(\rho)^{4n^\delta \frac{K}{r}} \exp\Big((4+\frac{2K}{r})\lfloor n^\theta\rfloor \Lambda^*_{p(\xi)}(\varepsilon)\Big) \exp(-n^{1/3}\inf_{s\ge 0}\Big(s\Lambda^*_{p(\xi)}(\varepsilon) + \frac{(1-\rho)\pi^2}{8s^2}(1-\frac{v}{v_\alpha}-\zeta)\Big) \quad (57)
\end{aligned}$$

Since the right hand side of (57) is independent of the choice of $A \in \mathcal{A}$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\sup_{A\in\mathcal{A}} n^{-1/3}\log \mathbf{P}[A] &\le -\inf_{s\ge 0}\Big(s\Lambda^*_{p(\xi)}(\varepsilon) + \frac{(1-\rho)\pi^2}{8s^2}(1-\frac{v}{v_\alpha}-\zeta)\Big) \\
&= -\frac{3}{2}\left((\Lambda^*_{p(\xi)}(\varepsilon))^2(1-\rho)\frac{\pi^2}{4}\Big(1-\frac{v}{v_\alpha}-\zeta\Big)\right)^{1/3} \quad (58)
\end{aligned}$$

Finally, taking the limit $\rho \to 0$, observing that $\lim_{\varepsilon\to 0}\Lambda^*_{p(\xi)}(\varepsilon) = |\log\alpha\Big([1/2, 1/2+\xi)\Big)|$, and performing the remaining limsup operations, we arrive at (31). This finishes the proof of Proposition 2.1. □

Remark: Note that after taking the limit $n\to\infty$, our upper bound in (58) does not depend on K and r. However, for later purpose, we need to know that the limsup operations carried out in the given order do give the correct upper bound $-I(v)$.

2.4 The negligible term: time spent in short fair regions and traversing biased blocks.

We begin with the identification of the time (until n) which has not yet been covered by the times $S_n^{(1)}$ and $S_n^{(2)}$. To this end, we introduce the notion of a *double biased block*, which is the union of two neighboring biased blocks. More precisely, for each $i \in \mathbb{Z}$ such that B_{i-1} and B_i are biased,we set $\mathcal{B}_i = B_{i-1}\cup B_i$. The (random) set of all such i-s is denoted by $I(3)$. The *center* $\mathcal{C}_i$ of the double biased block $\mathcal{B}_i$ is the set $\{i\lfloor n^\theta\rfloor - 1,\ i\lfloor n^\theta\rfloor\}$. Note that n consecutive biased blocks give rise to $n-1$ double biased blocks. The analogue of $T^{(1)}_{i,k}$ (the time spent in long fair regions) will be described by differences of appropriate stopping times. For $i\in I(3)$ we set

$$R^{(3)}_{i,1} = \inf\{t\ge 1\,;\ X_t\in\mathcal{C}_i,\ X_{t-1}\in\mathcal{C}_i\}, \quad (59)$$

the first time the walk *traverses* the center of $\mathcal{B}_i$, and

$$D^{(3)}_{i,0} = 0, \qquad D^{(3)}_{i,1} = \inf\{k > R^{(3)}_{i,1}\,;\ X_k\notin\mathcal{B}_i\} \quad (60)$$

the first time after $R^{(3)}_{i,1}$, the process exits $\mathcal{B}_i$. For $i\in I(3)$, we set inductively

$$R^{(3)}_{i,k+1} = D^{(3)}_{i,k} + R^{(3)}_{i,1}\circ\vartheta_{D^{(3)}_{i,k}}, \qquad D^{(3)}_{i,k+1} = R^{(3)}_{i,k+1} + D^{(3)}_{i,1}\circ\vartheta_{R^{(3)}_{i,k+1}}, \qquad (k\ge 1) \quad (61)$$

where ϑ denotes the canonical shift. Using these variables we express the duration of the k-th visit of $\mathcal{B}_i$ up to time n, as follows:

$$T_{i,k}^{(3)} = D_{i,k}^{(3)} \wedge n - R_{i,k}^{(3)} \wedge n \qquad (k \geq 1) \tag{62}$$

The total number of visits in $\mathcal{B}_i$ will be denoted by $V_i^{(3)}$. The total time of visits of double biased blocks up to time n is given by

$$S_n^{(3)} = \sum_{i \in I(3)} \sum_{k=1,\ldots,V_i^{(3)}} T_{i,k}^{(3)} \tag{63}$$

Finally we set

$$T_n^{(0)} = \begin{cases} \inf\{t \geq 0\,;\, X_t \notin B_0\} \wedge n & ; \quad \text{on } \{B_0 \text{ is biased}\} \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{64}$$

A moment thought, using the fact that the time spent visiting biased blocks which are not double biased blocks is absorbed completely in $S_n^{(1)}$ and $S_n^{(2)}$, shows that the time until n can be broken up as follows:

$$n = T_n^{(0)} + S_n^{(1)} + S_n^{(2)} + S_n^{(3)} \tag{65}$$

In the rest of the paper we will use constants c_1, c_2, ... etc., whose values do not depend on any of the parameters $v, n, K, \varepsilon, \xi, \delta, r, \zeta$ (but may depend on the distribution α.)

Proposition 2.2 *For the parameters as in (29), we have*

$$\limsup_{n\to\infty} n^{-1/3} \log \mathbf{P}[A_2] \leq -c_1 \frac{\zeta}{r^2} \tag{66}$$

for an appropriate constant $c_1 > 0$. *Therefore,*

$$\limsup_{\xi\to 0}\ \limsup_{\zeta\to 0}\ \limsup_{\varepsilon\to 0}\ \limsup_{r\to 0}\ \limsup_{K\to\infty}\ \limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbf{P}[\,A_2\,] = -\infty \tag{67}$$

Proof: By (65), the probability of the event A_2 can be estimated as

$$\begin{aligned} \mathbf{P}[A_2] &= \mathbf{P}\big[\,A_1, X_n/n \leq v,\ T_n^{(0)} + S_n^{(2)} + S_n^{(3)} > n(v/v_\alpha + \zeta)\}\,\big] \\ &\leq \mathbf{P}\big[\,A_1, X_n/n \leq v,\ \sum_{i\in I(2)} \sum_{k \leq V_i^{(2)}} T_{i,k}^{(2)} \geq n\zeta/2\,\big] + \\ &\quad + \mathbf{P}\big[\,A_1, X_n/n \leq v,\ T_n^{(0)} + \sum_{i\in I(3)} \sum_{k\leq V_i^{(3)}} T_{i,k}^{(3)} \geq n(v/v_\alpha + \zeta/2)\,\big] \\ &=: P_1 + P_2 \end{aligned} \tag{68}$$

We estimate the probabilities P_1 and P_2 in two separate Lemmas.

Lemma 4 *There exists a constant $c_2 > 0$, such that for all n large enough,*

$$P_2 \le 3\exp\big(-c_2\,\zeta\,n^{1/3+2\delta}\big)$$

Proof: We set $I^+(3) = \{i \in I(3)\,;\, i \ge 0\}$, $I^-(3) = I(3) \setminus I^+(3)$, and write

$$T_n^{(0)} + \sum_{i\in I(3)} \sum_{k\le V_i^{(3)}} T_{i,k}^{(3)} = T_n^{(0)} + \sum_{i\in I^+(3)} T_{i,1}^{(3)} + \sum_{i\in I^-(3)} T_{i,1}^{(3)} + \sum_{i\in I(3)} \sum_{2\le k\le V_i^{(3)}} T_{i,k}^{(3)} \tag{69}$$

The first two summands in (69) correspond to the duration of first visits of double biased blocks which lie to the "right" of the origin. Since we know that on A_1 there are at most Kn^δ left crossings of biased blocks, we expect these two summands to be dominant. We therefore proceed by

$$\begin{aligned} P_2 \;\le\; & \mathbf{P}\big[\,A_1, X_n/n \le v,\; T_n^{(0)} + \sum_{i\in I^+(3)} T_{i,1}^{(3)} \ge n(v/v_\alpha + \zeta/4)\,\big] \\ & + \mathbf{P}\big[\,A_1, X_n/n \le v,\; \sum_{i\in I^-(3)} T_{i,1}^{(3)} + \sum_{i\in I(3)} \sum_{2\le k\le V_i^{(3)}} T_{i,k}^{(3)} \ge \frac{n\zeta}{4}\,\big] =: P_4 + P_5 \end{aligned} \tag{70}$$

We start with the term P_4 and will show that for all n large enough,

$$P_4 \le 2\exp\big(-c_3\,\zeta\,n^{1/3+2\delta}\big)\;, \tag{71}$$

where $c_3 > 0$ is an appropriate constant. For $j \ge 0$ we set $\tilde{B}_j = B_{j-1} \cup B_j$ and $\tau_j = D_j - R_j$, where $R_j = \inf\{t \ge 0\,;\, X_t = j\lfloor n^\theta \rfloor\}$ and $D_j = \inf\{t > R_j\,;\, X_t \notin \tilde{B}_j\}$. Note that $\mathbf{P}$.-a.s., $X_n \to \infty$, cf.[7], hence $\forall j \in \mathbb{N}$: $R_j < \infty$. Recall that $A_1 \cap \{X_n/n \le v\} \subset \{T_U > n\}$, cf. (20). By setting $j^* = \lfloor (nv + Kn^{1/3})/\lfloor n^\theta \rfloor \rfloor$ and $J^* = [0, j^*] \cap \mathbb{Z}$, we can easily see that on $A_1 \cap \{X_n/n \le v\}$

$$T_n^{(0)} + \sum_{i\in I^+(3)} T_{i,1}^{(3)} \le \sum_{j\in J^*} \tau_j \tag{72}$$

At this point it is worthwile to recall that on A_1, the total length of relevant double biased blocks is of the order nv. Therefore we expect the bound in (72) to be rather accurate. Note that, with respect to the annealed measure $\mathbf{P}$, the variables τ_j are identically distributed but not independent. However, the family $(\tau_j)_{j\in 2\mathbb{N}}$ (as well as $(\tau_{j+1})_{j\in 2\mathbb{N}}$) *is* an independent family of random variables. Hence, by using Chebyshev's inequality,

$$\begin{aligned} P_4 \;\le\; & \mathbf{P}\big[\sum_{j\in J^*} \tau_j \ge n(v/v_\alpha + \zeta/4)\big] \le 2\mathbf{P}\big[\sum_{j\in J^*\cap 2\mathbb{Z}} \tau_j \ge \frac{n}{2}(v/v_\alpha + \zeta/4)\big] \\ \le\; & 2\exp\big(-\lambda\frac{n}{2}(v/v_\alpha + \zeta/4)\big)\,\Big(\mathbf{E}\big[e^{\lambda\,\tau_0}\big]\Big)^{(j^*+1)/2} \end{aligned} \tag{73}$$

where $\lambda > 0$. It follows now that for each $\varepsilon' \in (0,1)$ and n large enough,

$$P_4 \le 2\exp\Big(-\lambda\frac{n}{2}(v/v_\alpha+\zeta/4)\Big)\,\Big(\mathbf{E}[\,e^{\lambda\tau_0}\,]\Big)^{\frac{v}{2}(1+\varepsilon')n^{1-\theta}} \tag{74}$$

In order to estimate the exponential moment occurring in (74), we will use results for the (under $\mathbf{P}$) slightly different stopping time $T_n = \inf\{t \ge 0\,;\, X_t = \lfloor n^\theta \rfloor\}$ with mean $m_n = \mathbf{E}[T_n]$. Using the notation $\lambda = \overline{\lambda}/n^{2\theta}$ and Solomon's result: $m_n/n^\theta \to 1/v_\alpha$, as $n \to \infty$ (cf. [7]) an elementary calculation shows

$$P_4 \le 2\exp\Big(-\frac{\overline{\lambda}\zeta}{16}n^{1-2\theta}\Big)\,\mathbf{E}\big[\,\exp\big(\frac{\overline{\lambda}}{n^\theta}\big(\frac{\tau_0-m_n}{n^\theta}\big)\big)\,\big]^{vn^{1-\theta}} \tag{75}$$

The next lemma, whose proof we defer to the Appendix , provides the necessary estimate on the exponential moment in (75).

Lemma 5 *For parameters as in (29) and $\overline{\lambda} = \pi^2/128$, we have for all n large enough*

$$\mathbf{E}\big[\,\exp\big(\frac{\overline{\lambda}}{n^\theta}\big(\frac{\tau_0-m_n}{n^\theta}\big)\big)\,\big] \le 1 + n^{-(1/3+2\delta)} \tag{76}$$

Putting together the estimates (75) and (76), we arrive at (71) with $c_3 = \overline{\lambda}/32$.

We now turn to the term P_5. We claim that for all n large enough

$$P_5 = \mathbf{P}\big[\,A_1,\, X_n/n \le v,\, \sum_{i\in I^-(3)} T_{i,1}^{(3)} + \sum_{i\in I(3)}\ \sum_{2\le k\le V_i^{(3)}} T_{i,k}^{(3)} \ge \zeta\, n/4\,\big] \le \exp\big(-c_4\,\zeta\, n^{1/3+2\delta}\big) \tag{77}$$

for an appropriate constant $c_4 > 0$. In order to show (77), we follow closely the steps (32)-(49). Since there are some differences in the arguments, we will give a reasonably detailed proof. We start by introducing the set of indices for which the various $T_{i,k}^{(3)}$-s occurring in (77) are strictly positive:

$$\mathcal{Q} = \big\{(i,k) \in (I^-(3)\times\{1,2,...\}) \cup (I^+(3)\times\{2,3,...\})\,;\, T_{i,k}^{(3)} > 0\big\}$$

We claim that, on the set A_1,

$$0 \le Q = |\mathcal{Q}| \le 2Kn^\delta \tag{78}$$

Indeed, the lower bound is obvious. In order to establish the upper bound, we observe that every right crossing (except the first one) of a biased block has to be preceded by a left crossing of the same block. Moreover, if the block is contained in $(-\infty, 0)$, the first crossing itself is a left crossing (note that the walk starts at 0). Since the number of all left crossings of biased blocks is bounded by Kn^δ, (78) follows. The next step is to equip $\mathcal{Q}$ with the historical order, which we achieve by enumerating all the pairs of indices $(i,k) \in \mathcal{Q}$ by attaching an index $l = 1, ..., m$ to them. Thus, on the set $\{Q = m\}$, we have

$$0 < R_{i_1,k_1}^{(3)} < D_{i_1,k_1}^{(3)} \le R_{i_2,k_2}^{(3)} < D_{i_2,k_2}^{(3)} \le \ldots \le R_{i_l,k_l}^{(3)} < D_{i_l,k_l}^{(3)} \le \ldots \le R_{i_m,k_m}^{(3)} < D_{i_m,k_m}^{(3)} \wedge n \tag{79}$$

Recall that the sequence $(k_l)_{1\le l\le m}$ is completely determined by the (i_l)-s, cf. (39). By using Chebyshev's inequality, we find

$$\begin{aligned} P_5 &= \mathbf{P}\big[\,A_1,\, X_n/n \le v,\, \sum_{(i,k)\in\mathcal{Q}} T^{(3)}_{i,k} > \zeta\, n/4\big] \\ &\le \exp(-\lambda\,\zeta\, n/4)\mathbf{E}^\alpha\big[\,\mathbf{E}^\omega[\exp(\lambda \sum_{(i,k)\in\mathcal{Q}} T^{(3)}_{i,k})\,;X_n/n \le v,\, A_1]\,\big] \end{aligned} \tag{80}$$

The next step is to use the Markov property (with respect to $\mathbf{P}^\omega$) and derive estimates uniformly in ω. To this end, for fixed ω, we partition the set $A_1 \cap \{X_n/n \le v\} \cap \mathbf{\Omega}^\omega$ (where $\mathbf{\Omega}^\omega = \{(\omega, w)\,;\, w \in W\}$) according to the values of Q and $(i_l)_{1\le l\le Q}$. Our goal is to estimate the following expression:

$$\mathbf{E}^\omega\big[\,\exp(\lambda \sum_{(i,k)\in\mathcal{Q}} T^{(3)}_{i,k});\, Q = m,\ (i_l)_{1\le l\le m} = (\bar{i}_l)_{1\le l\le m},\ X_n/n \le v,\ A_1\,\big]\,, \tag{81}$$

where $(\bar{i}_l)_{1\le l\le m}$ is a fixed sequence of indices. The number of all such fixed sequences with the property

$$\Big(A_1 \cap \{X_n/n \le v\} \cap \mathbf{\Omega}^\omega\Big) \cap \{(i_l)_{1\le l\le m} = (\bar{i}_l)_{1\le l\le m}\} \ne \emptyset\,,$$

is bounded by 3^{2Kn^δ}. Indeed, for fixed ω, there are at most 3 possible values of i_1, namely 0, and the indices of the closest double biased blocks to the right and to the left, respectively. Similarily, given $i_1, ..., i_l$, there are only 3 possibilities for the value of i_{l+1}. Hence, the number $\mathcal{N}$ of sets into which our set has been partitioned is bounded by

$$\mathcal{N} \le 3^{2Kn^\delta} \tag{82}$$

By conditioning successively on the stopping times $R^{(3)}_{\bar{i}_1,\bar{k}_1}, ..., R^{(3)}_{\bar{i}_m,\bar{k}_m}$ (as we did in (43)-(46)) we find that the expression in (81) is bounded by

$$\prod_{1\le l\le m} \max_{x\in\mathcal{C}_{\bar{i}_l}} \mathbf{E}^\omega_x\big[\exp(\lambda D^{(3)}_{\bar{i}_l,1})\big]\,, \tag{83}$$

where $\mathcal{C}_{\bar{i}_l}$ stands for the center of the double biased block $\mathcal{B}_{\bar{i}_l}$. We estimate the exponential moment occurring in (83) by using the same stochastic comparison inequality as in (48) which gives the following bound:

$$\sup_{\omega\in\Omega}\ \sup_{x\in\mathcal{C}_{\bar{i}_l}} \mathbf{E}^\omega_x\big[\exp(\lambda D^{(3)}_{\bar{i}_l,1})\big] \le \sup_{x\in I} \mathbf{E}^f_x[\exp(\lambda T_I)]\,, \tag{84}$$

where T_I is the exit time from the interval $I = [-2\lfloor n^\theta\rfloor, 2\lfloor n^\theta\rfloor]$. Using (80), the combinatorial factor (82), (83) and (84), we have

$$P_5 \le 3^{2Kn^\delta}\ \exp(-\lambda\,\zeta\, n/4)\ \sup_{x\in I} \mathbf{E}^f_x[\exp(\lambda T_I)]^{2Kn^\delta} \tag{85}$$

Finally, we apply Lemma 3 in (85) with $\rho = 1/2$ and $\lambda = \pi^2/(32n^{2\theta})$ which gives (77) with $c_4 = \pi^2/256$. This finishes the proof of Lemma 4. □

We now come to the investigation of P_1 in (68). In view of Lemma 4, Proposition 2.2 follows once we have shown the next lemma.

Lemma 6 *There exists a constant $c_5 > 0$ such that for all n large enough,*

$$P_1 = \mathbf{P}\big[\, A_1, X_n/n \le v,\ \sum_{i\in I(2)} \sum_{k\le V_i^{(2)}} T_{i,k}^{(2)} \ge n\zeta/2 \,\big] \le \exp(-c_5 \frac{\zeta}{r^2}\, n^{1/3}) \tag{86}$$

Proof: Let us first emphasize the similarity between the times occurring in the terms P_5 and P_1. The center of a double biased block corresponds in an obvious way to a short fair region squeezed in between two biased blocks. Recall that the visits in double biased blocks begin by traversing the center and end at a full crossing of one of the biased blocks, while in short fair regions the visits begin at entering that (short fair) region and end by crossing one of the attached biased blocks. In order to control the probability that $\sum_{i\in I(2)} \sum_{k\le V_i^{(2)}} T_{i,k}^{(2)}$ is large, we use again the control provided by left crossings of the biased blocks attached to the short fair regions. Since our arguments follow exactly the arguments we used to control the term P_5, we refrain from a detailed proof and indicate only the differences. The combinatorial complexity occurring here can be estimated as follows. Since on the set A_1 there are at most Kn^δ fair blocks, the number of short fair regions is also bounded by the same number. On the other hand, the number of all visits of short fair regions can be controlled by the number of all such regions (this estimates the number of first right-crossings) plus two times the number of all left crossings of biased blocks, as in (78). Therefore, on the set A_1, we have at most $3Kn^\delta$ visits of short fair regions. This gives the combinatorial bound 3^{3Kn^δ}, replacing the factor in (82). Expression (83) is now replaced by

$$\prod_{1\le l\le m} \max_{x\in\overline{\mathcal{F}}_{\bar{i}_l}^{(2)}} \mathbf{E}_x^\omega\big[\exp(\lambda D_{\bar{i}_l,1}^{(2)})\big] \tag{87}$$

The exponential moment can be controlled in the same way as in (84) (or (50)), except that our interval is now longer. Indeed, $|\overline{\mathcal{F}}_{\bar{i}_l}^{(2)}| \le rn^{1/3} + 2\lfloor n^\theta \rfloor \le 2rn^{1/3}$, for n large enough. By using the interval $I = [-2rn^{1/3}, 2rn^{1/3}]$, applying Lemma 3 with $\rho = 1/2$ (which gives $\lambda = \pi^2/(32r^2n^{2/3})$) we finally have for n large enough

$$P_1 \le 3^{3Kn^\delta}\ \chi(1/2)^{3Kn^\delta}\ \exp(-\frac{\pi^2}{2^6}\,\frac{\zeta}{r^2}\, n^{1/3}) \tag{88}$$

This completes the proof of Lemma 6 and that of Proposition 2.2. □

Finally, the proof of Theorem 2 follows immediately from (30), (28), (31) and (67).

Appendix

Here we give the proof of Lemma 5. We recall some definitions and the precise statement of the lemma. Since we want to emphasize the n-dependence of the variable τ_0, we prefer to use the more suggestive notation D_{n^θ} for it. More precisely, we set $D_{n^\theta} = \inf\{t \geq 0\,;\, X_t \notin \{-\lfloor n^\theta \rfloor - 1,\ \lfloor n^\theta \rfloor\}\}$. Recall the definitions $T_{n^\theta} = \inf\{t \geq 0\,;\, X_t = \lfloor n^\theta \rfloor\}$ and $m_{n^\theta} = \mathbf{E}[\,T_{n^\theta}\,]$.

Lemma 5 *For parameters as in (29), $\beta = 3\delta$ and $\overline{\lambda} = \pi^2/128$, we have for all n large enough*

$$\mathbf{E}\Big[\,\exp\Big(\frac{\overline{\lambda}}{n^\theta}\Big(\frac{D_{n^\theta} - m_{n^\theta}}{n^\theta}\Big)\Big)\Big] \leq 1 + n^{-(\beta+\theta)} \tag{89}$$

Proof: Set $Z = (D_{n^\theta} - m_{n^\theta})/n^\theta$, $\lambda = \overline{\lambda}/n^\theta$, $\alpha = 5\theta/4$ and $\kappa = 1/v_\alpha$. The exponential moment of Z will be splitted up as follows:

$$\begin{aligned}
\mathbf{E}[\,e^{\lambda Z}\,] &\leq 1 + \lambda \int_0^{n^{-\beta}} e^{\lambda u}\mathbf{P}[Z > u]\,du + \lambda \int_{n^{-\beta}}^{\kappa} e^{\lambda u}\mathbf{P}[Z > u]\,du + \\
&\quad + \lambda \int_{\kappa}^{n^\alpha} e^{\lambda u}\mathbf{P}[Z > u]\,du + \lambda \int_{n^\alpha}^{\infty} e^{\lambda u}\mathbf{P}[Z > u]\,du \\
&= 1 + I_1 + I_2 + I_3 + I_4
\end{aligned} \tag{90}$$

Obviously, $I_1 \leq 2\overline{\lambda} n^{-(\beta+\theta)}$. We now turn to I_3. We have

$$I_3 \leq \exp(\lambda n^\alpha)\,\mathbf{P}[Z > \kappa] \tag{91}$$

Using that $m_{n^\theta}/n^\theta \to 1/v_\alpha$, and $D_{n^\theta} \leq T_{n^\theta}$, we find for n large enough:

$$\mathbf{P}[Z > \kappa] \leq \mathbf{P}\big[\,T_{n^\theta} > n^\theta\big(\kappa + 1/(2v_\alpha)\big)\big] \leq \mathbf{P}\big[\,X_{\lfloor n^\theta 3/(2v_\alpha)\rfloor} < n^\theta\,\big] \tag{92}$$

We now apply the estimate in the last line of the proof of Lemma 4.1 in [2], by choosing $v = v(u)$ and $B = B(u)$ there, as follows. The parameter v is given by $v = (2/3)v_\alpha$ and $B = (1 - v/v_\alpha)^{1/3}\Big(\frac{\pi^2}{64|\log\langle\rho\rangle|}\Big)^{1/3}$. Note that by this choice we have $v \in (0, v_\alpha)$ and $1/v - 1/v_\alpha = 1/(2v_\alpha)$. By setting $c_\alpha = \Big(|\log\langle\rho\rangle|\pi/8\Big)^{2/3}$, we have for n large enough,

$$\mathbf{P}\big[\,X_{\lfloor n^\theta 3/(2v_\alpha)\rfloor} < n^\theta\,\big] \leq \exp\big(-n^{\theta/3}c_\alpha/4\big) \tag{93}$$

Coming back to (91) we find that for n large enough $I_3 \leq \exp(-c_\alpha\, n^{\theta/3}/8)$. We proceed by estimating I_4. Using (48) and (52) once again with $\varepsilon' = 1/2$, we obtain for all n large enough, and every $u \geq n^\alpha$,

$$\mathbf{P}\big[\,Z > u\,\big] \leq \mathbf{P}\big[\,D_{n^\theta} > u\,n^\theta\,\big] \leq c'(1/2)\exp\Big(-\frac{u}{n^\theta}\frac{\pi^2}{64}\Big) \tag{94}$$

Using this, we find

$$I_4 \le c'(1/2)\,\lambda \int_{n^\alpha}^{\infty} \exp(u\lambda - \frac{u\pi^2}{16n^\theta})\,du \;\le c'(1/2)\,\exp\big(-n^{\theta/4}\frac{\pi^2}{128}\big) \tag{95}$$

It remains to estimate I_2. We will need the following fact: For $\delta \in (0, 1/24)$, $\beta = 3\delta$ and for each $\gamma > 0$

$$\limsup_{n\to\infty} n^\gamma \mathbf{P}[\, Z > n^{-\beta}\,] < \infty \tag{96}$$

Let us postpone the proof of (96) and proceed with estimating I_2. Using (96) with $\gamma = 2\beta$, we have for some $c > 0$ and for every n large enough

$$I_2 \le 2\overline{\lambda}\,\kappa n^{-\theta}\,\mathbf{P}[\, Z > n^{-\beta}\,] \le cn^{-(\theta+2\beta)}\ , \tag{97}$$

Thus, putting toghether all our estimates we arrive at (89).

It remains to show (96). Set $M = n^\theta$ and $\beta' = \beta/\theta$. Note that $\beta' < 3/7$. Then, for all n large enough

$$\mathbf{P}[\, Z > n^{-\beta}\,] \le \mathbf{P}[\, D_M - \mathbf{E}[T_M] > M^{1-\beta'}\,] \tag{98}$$

Note that by the choice of δ, $1 - \beta' > 0$. Setting $\gamma' = \gamma/\theta$, we see that (96) follows, once we have shown

$$\limsup_{M\to\infty} M^{\gamma'}\,\mathbf{P}[\, D_M - \mathbf{E}[\,T_M\,] > M^{1-\beta'}\,] < \infty \tag{99}$$

For M large, we divide $\mathbb{N}$ into blocks $I_i = [x_i,\, x_i + \lfloor c\log M\rfloor)$, $(i \ge 0)$, where $x_i = i\,\lfloor c\log M\rfloor$ and $c = (1+\gamma')/|\log\langle\rho\rangle|$. As next we employ a comparison technique described in Lemma 2.5, of [2]. On $\mathbf{\Omega}$, we define another process $(Y_t)_{t\ge 0}$ and hitting times $\tilde{\tau}_i = \inf\{t \ge 0\,;\, Y_t = x_i\}$, where the only difference between (X_t) and (Y_t) is that for $t \ge \tilde{\tau}_i$, $i \ge 0$, the process (Y_t) is reflected at x_{i-1}. Note that the variables $\overline{\tau}_i = \tilde{\tau}_i - \tilde{\tau}_{i-1}$, $i \ge 1$, are well defined (except on a set of zero measure) and identically distributed under $\mathbf{P}$. Note that T_M is stochastically larger than $\sum_{1\le i\le N} \overline{\tau}_i$, where $N = \lfloor M/\lfloor c\log M\rfloor\rfloor$. Hence,

$$\mathbf{P}[\, D_M - \mathbf{E}[\,T_M\,] > M^{1-\beta'}\,] \le \mathbf{P}[\, D_M - \sum_{1\le i\le N} \mathbf{E}[\,\overline{\tau}_i\,] > M^{1-\beta'}\,] \tag{100}$$

Consider the set $G = \{\max_{1\le i\le N} L_i \le \lfloor c\log M\rfloor\}$, where L_i denotes the length of the maximal left excursion from x_i (up to time ∞) of the walk (X_t). We have

$$\mathbf{P}[\, D_M - \sum_{1\le i\le N} \mathbf{E}[\,\overline{\tau}_i\,] > M^{1-\beta'}\,] \;\;\le\;\; \mathbf{P}[\,\{D_M - \sum_{1\le i\le N} \mathbf{E}[\,\overline{\tau}_i\,] > M^{1-\beta'}\} \cap G\,] + \mathbf{P}[\, G^c\,] \tag{101}$$

Using the fact that on the set G, $D_M = \sum_{i=1}^N \overline{\tau}_i$, and applying Lemma 2.2 in [2], we obtain

$$\mathbf{P}[\, D_M - \mathbf{E}[\,T_M\,] > M^{1-\beta'}\,] \le \mathbf{P}[\,\sum_{i=1}^{N}(\overline{\tau}_i - E(\overline{\tau}_i)) > M^{1-\beta'}\,] + \frac{M^{-\gamma'}}{\langle\rho\rangle(1-\langle\rho\rangle)} \tag{102}$$

In order to estimate the first term on the r.h.s. of (102), we need to bound the t-th moment of the variable $\overline{\tau}_1$ for an appropriate $t > 0$. Denote by τ_1 the first time the RWRE hits the point $[c \log M]$. Given a small parameter $\varepsilon > 0$ and $\omega \in \Omega$, we set for all $x \in \mathbb{Z}$

$$\omega_x(\varepsilon) = \begin{cases} \omega_x & ; \quad \omega_x > 1/2 \\ 1/2 - \varepsilon & ; \quad \omega_x = 1/2 \end{cases}$$

and look at the the RWRE with respect to the measure on the path space $\mathbf{P}_\varepsilon = \int \mathbf{P}^{\omega(\varepsilon)}[\,\cdot\,]\,\mathbf{P}(d\omega)$. We then have

$$\mathbf{E}[\,\overline{\tau}_1^t\,] \le \mathbf{E}[\,\tau_1^t\,] \le \mathbf{E}_\varepsilon[\,\tau_1^t] \le \lfloor c \log M \rfloor^t \, C_t \tag{103}$$

for a constant $0 < C_t < \infty$, where we used the estimate (11) in [2]. Note that this estimate can be applied here if we choose ε small enough (which ensures that the parameter s in Theorem 1.1 of [2] is larger than our t.) Recall that $(\overline{\tau}_{2i})$ are i.i.d. variables under $\mathbf{P}$. We can now apply Corollary 1.8 from Nagaev [5] with $t > 2 \vee (1+\gamma')/(1-\beta')$, obtaining thereby

$$\begin{aligned} & \mathbf{P}\Big[\,\sum_{i=1}^{N} (\overline{\tau}_i - \mathbf{E}[\,\overline{\tau}_i\,]) > M^{1-\beta'}\,\Big] \le 2\,\mathbf{P}\Big[\,\sum_{i=1}^{N/2} (\overline{\tau}_{2i} - \mathbf{E}[\,\overline{\tau}_{2i}\,]) > \frac{1}{2}\,M^{1-\beta'}\,\Big] \\ \le \;\; & 2^t (1+2/t)^t \, N \,\mathbf{E}[\,\overline{\tau}_1^t\,]\, M^{-t(1-\beta')} + 2\,\exp\Big(- \frac{M^{2(1-\beta')}}{e^t (2+t) N \mathbf{E}[\,\overline{\tau}_1^2\,]}\Big) \\ \le \;\; & 2^t (1+2/t)^t \, 2 C_t (c \log M)^{t-1} \, M^{-t(1-\beta')+1} \; + \; 2\,\exp\Big(- \frac{1}{e^t(2+t)} \, \frac{M^{1-2\beta'}}{C_2\,(c \log M)}\Big) \end{aligned} \tag{104}$$

Recalling that that $t > 2 \vee (1+\gamma')/(1-\beta')$ and $1 - 2\beta' > 1/7$, we easily conclude from (104) that

$$\lim_{M \to \infty} M^{\gamma'}\,\mathbf{P}\Big[\,\sum_{i=1}^{N} (\overline{\tau}_i - E(\overline{\tau}_i)) > M^{1-\beta'}\,\Big] = 0 \tag{105}$$

This, together with (102) shows (99) and the proof of Lemma 5 is complete. □

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