On roots of random polynomials

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May 14, 1995. Revised October 13, 1995.

Abstract We study the distribution of the complex roots of random polynomials of degree n with i.i.d. coefficients. Using techniques related to Rice's treatment of the real roots question, we derive, under appropriate moment and regularity conditions, an exact formula for the average density of this distribution, which yields appropriate limit average densities. Further, using a different technique, we prove limit distributions results for coefficients in the domain of attraction of the stable law.

AMS 1991 Subject classification: Primary 34F05. Secondary 26C10, 30B20.
KEYWORDS: Random polynomials, complex roots, domain of attraction of the stable law.

*The work of this author was partially supported by the Russian Foundation for Fundamental Research, grant 94-01-00301, and by grants R36000 and R36300 of the International Scientific Foundation.
†The work of this author was done while he visited MIT, under support from NSF grant 9302709–DMS.

1 Introduction

Let $\{a_j\}_{j=0}^{\infty}$ denote a sequence of i.i.d. random variables. Let $P_n(z) = \sum_{j=0}^{n} a_j z^j$ denote the random polynomial of order n defined by the sequence $\{a_j\}$. After initial attempts, by Littlewood and Offord [16], to evaluate the order of the mean number of real roots, Kac [13] has computed, in the case that the $\{a_j\}$ are standard Normal, the distribution of the real zeros of $P_n(z)$. His results were extended in various directions, most notably to the non-Gaussian case, by Erdös and Offord, Stevens, Logan and Shepp, and Ibragimov and Maslova (see [3],[18] for a bibliography). See also [6] for an integral geometric derivation of Kac's formula and an updated account of this question.

Our interest in this paper is to explore the analogue question for the distribution of the roots in the *complex plane*. To state our results, we need to introduce a bit of notation. Let Ω denote a Borel measurable subset of $\mathbf{R}^2$. Let $N_n(\Omega)$ denote the number of complex roots of $P_n(z)$ in Ω. We let $\nu_n(r) = N_n(\{z : |z| < r\})$ and $\bar{\nu}_n(r) = N_n(\{z : |z| > r\})$. When referring to the average distribution of the complex zeros, we will mean the evaluation of $E(\nu_n(r))$. Note that since the law of $\nu_n(r)$ is identical to the law of $\bar{\nu}_n(1/r)$, it follows that $E(\nu_n(r)) = E(\bar{\nu}_n(1/r))$. The computation of the average distribution of zeros was originally studied by Hammersley [10], who derived an exact (albeit complicated) formula for it in the Gaussian case. An early theorem of Sparo and Sur, which refines earlier results by Polya, (c.f. [3, Page 174]), implies that not only does $\nu_n(r)/n \to_{n\to\infty} 0$ in probability for any $r < 1$ but also, letting $N_n(\alpha,\beta) = N_n(\{z : \alpha \leq \arg z < \beta\})$ denote the number of roots of $P_n(z)$ in a sector $0 \leq \alpha \leq \theta \leq \beta < 2\pi$, one has under mild conditions that $N_n(\alpha,\beta)/n \to_{n\to\infty} (\alpha - \beta)/2\pi$ in probability. For an earlier, and probably first, version of the angular distribution result, see [8], whereas for a refinement of this result, see [2].

In a recent paper, Shepp and Vanderbei ([18]) derive, in the case of Normal $\{a_j\}$, an exact expression for the average distribution of the roots as well as limit results as $n \to \infty$. In particular, they give precise estimates of the way in which, as $n \to \infty$, about $n - 2\log n/\pi$ of the zeros concentrate on the unit circle, uniformly in the angle, whereas $2\log n/\pi$ real roots concentrate at ± 1.

The technique of proof of Shepp and Vanderbei is based on an argument principle to compute the average distribution of the zeros, using the Gaussian law of $\{a_j\}$ in order to reduce the question to the evaluation of a function of 4 (correlated) Gaussian random variables. This technique does not seem amenable to handling distributions other than the Normal one.

We extend the results of Shepp and Vanderbei in two different directions. Using an approach based on Jensen's formula, we present in Theorem 1 limit average distribution results for arbitrary i.i.d. coefficients in the domain of attraction of the stable law. These may be thought as global limit theorems. Using an argument closer in spirit to the point of view adopted by Rice [17] in his attempts to handle the real case, we derive in Theorem 2 a local limit theorem for the density of zeros for i.i.d. coefficients which possess finite sixth moment and a bounded density. This approach yields as a by–product a simple derivation of the main result of [18].

Our main result concerning the average distribution of zeros is the following:

Theorem 1 *Let $\{a_j\}_{j=1}^{\infty}$ be i.i.d. random variables whose common distribution G belongs to the* *domain of attraction of an α–stable law. Then, for any $0 \le s < \infty$,*

$$\lim_{n\to\infty} E\left(\frac{\nu_n(\exp(-s/n))}{n}\right) = \frac{1-e^{-\alpha s}(1+\alpha s)}{\alpha s(1-e^{-\alpha s})} \triangleq F(\alpha s)\,. \tag{1}$$

The following is immediate from our previous discussion:

Corollary 1 *Under the assumptions of Theorem 1, for $0 \le s < \infty$,*

$$\lim_{n\to\infty} E\left(\frac{\bar{\nu}_n(\exp(+s/n))}{n}\right) = F(\alpha s)\,. \tag{2}$$

In particular,

$$\lim_{n\to\infty} E\left(\frac{\nu_n(\exp(s/n))-\nu_n(\exp(-s/n))}{n}\right) = 1-2F(\alpha s) = \frac{1+e^{-\alpha s}}{1-e^{-\alpha s}} - \frac{2}{\alpha s}\,. \tag{3}$$

We recall (c.f. [11] for the results quoted here) that the distribution G on the real line belongs to the domain of attraction of a stable law with exponent α if, for suitable A_n, B_n,

$$\lim_{n\to\infty} P(B_n^{-1}\sum_{j=1}^{n} a_j - A_n < x) = \mathcal{F}(x)\,, \tag{4}$$

where $\mathcal{F}$ possesses the characteristic function

$$f(t) = \exp\{-c|t|^{\alpha}(1+K(\alpha,\beta,t))\}\,,$$

with $K(\alpha,\beta,t) = i\beta \mathrm{sign} t \tan(\pi\alpha/2)$ if $\alpha \ne 1$ and $K(1,\beta,t) = 2i\beta\mathrm{sign}t(\log|t|)/2\pi$. This happens iff the characteristic function of G, denoted $g(t)$, satisfies in a neighborhood of the origin the relationship

$$g(t) = \exp\{i\gamma t - c|t|^{\alpha}(1+K(\alpha,\beta,t))h(t)\}\,,$$

where $h(t)$ is slowly varying in the sense of Karamata. Further, the norming factors B_n^{-1} may be defined as the roots of the equation $B_n^{-\alpha}h(B_n^{-1}) = n^{-1}$. In particular, $B_n = n^{1/\alpha}h(n)$ for some slowly varying $h(t)$.

The proof of Theorem 1, based on Jensen's formula, is given in Section 3. The following argument however helps explain why we are interested in the scale n^{-1}. By Jensen's formula (see [15, Pg. 14]),

$$\int_0^r \frac{\nu_n(t)}{t}dt + \log|a_0| = \frac{1}{2\pi}\int_0^{2\pi} \log|P_n(re^{i\theta})|\,.$$

Hence, letting $r = 1 - K\log n/n$, assuming that $|E\log|a_0|\,| < \infty$, and using Jensen's inequality, one obtains that $E\nu_n(r) \le n/K + o(n)$, suggesting that the scale n^{-1} is indeed the meaningful one.

We next turn to describe a local limit result concerning the average density of zeros. Since the asymptotic distribution of real roots is well understood, we will concentrate here on the zeros in the complex plane. Thus, let Ω be a measurable subset of $\mathbf{R}^2$ which does not intersect the real line. We will compute explicitly a function $h_n(r,\theta)$ such that

$$E(\nu_n(\Omega)) = \int_\Omega h_n(r,\theta)drd\theta = \int_\Omega h_n(z)dz\,.$$

Here and throughout, z will stand for complex variables whereas r,θ stand for their polar representation, with $z = re^{i\theta}$.

An explicit computation of $h_n(r,\theta)$ is possible in the case of Normal coefficients (see (15) below). Our main interest however is the proof of the:

Theorem 2 *Assume $\{a_j\}$ possess a bounded density and absolute moments of sixth order. Then, for $r = 1 - x/n$, x fixed, and $\theta \in (0,\pi)$ fixed, one has*

$$\lim_{n\to\infty} n^{-2}h_n(r,\theta) = \frac{\left(\int_0^x e^{-2y}dy\right)\left(\int_0^x y^2e^{-2y}dy\right) - \left(\int_0^x ye^{-2y}dy\right)^2}{\pi x^2\left(\int_0^x e^{-2y}dy\right)^2} = \frac{\left(1-\left(\frac{x}{\sinh(x)}\right)^2\right)}{4\pi x^2}\,,$$

and the convergence is uniform in compact subsets of $[0,\infty)\times(0,\pi)$.

Note that Theorem 2 is consistent with the predictions of Theorem 1.

Remark The i.i.d. assumption as well as the precise assumptions on the coefficients $\{a_j\}$ can be relaxed. However, the computation of limiting average densities of zeros for stable random variables $\{a_j\}$ or for random variables in the domain of attraction of the stable law seems more delicate, and the technique we use does not seem appropriate.

Due to its relative simplicity, we first present, in Section 2, the proof of Theorem 2, together with the precise computation of the Gaussian case. Sections 3 and 4 are devoted to the proof of Theorem 1.

2 Density results

To explain our approach to the density question, let $z = re^{i\theta}$, and let

$$X_1 \equiv X_1^n(r,\theta) = \sum_{j=0}^n a_j r^j \cos(j\theta)\,, \quad X_2 \equiv X_2^n(r,\theta) = \sum_{j=0}^n a_j r^j \sin(j\theta) \tag{5}$$

denote the real and imaginary parts of $P_n(z)$. Let J denote the Jacobian of the (random) transformation $(r,\theta) \to (X_1,X_2)$, and let $p_{r,\theta}$ denote the density (at $(0,0)$) of the random vector $X = (X_1^n(r,\theta), X_2^n(r,\theta))$. Then (see Adler [1, pg. 97]), the average density of the complex roots satisfies, *everywhere but on the real axis*, the formula

$$h_n(r,\theta) = E(|\det J|\,|\,X_1^n = X_2^n = 0)p_{r,\theta}\,.$$

By a straight forward computation, one checks that

$$\det J = \sum_{j,k=0}^{n} jka_ka_jr^{j+k-1}\cos((j-k)\theta) = \frac{1}{r}\left(\left(\sum_{j=0}^{n} ja_jr^j\cos(j\theta)\right)^2 + \left(\sum_{j=0}^{n} ja_jr^j\sin(j\theta)\right)^2\right) \geq 0\,. \tag{6}$$

Thus, the evaluation of h_n reduces to the computation of the expectation of the absolute value of a quadratic form of i.i.d. random variables, conditioned on two linear combinations thereof. That is,

$$h_n(r,\theta) = E(\sum_{j,k=0}^{n} jka_ka_jr^{j+k-1}\cos((j-k)\theta)\,|\,X_1^n = X_2^n = 0)p_{r,\theta}\,. \tag{7}$$

We remark that while our interest is primarily in the *complex* roots, and we will make assumptions that will imply that $\theta \neq 0$ in (7), one could handle also the real roots by a similar study. The approach of Rice, alluded to in the abstract, consists of looking at the (one dimensional) map $r \to X_1$ with $\theta = 0$, and computing its derivative. Since the results of that analysis are well documented, we do not consider it here.

While (7) is valid in great generality, its evaluation is not always easy. The computation in (7) is greatly simplified in the Gaussian case, which is presented in Section 2.2 below, recovering the results of [18].

2.1 Proof of Theorem 2

Let $\{a_j\}$ be a sequence of i.i.d. random variables which are normalized such that $E(a_j) = 0$ and $Ea_j^2 = 1$. Let $\pi > \theta_0 > 0$ and $x_0 > 0$ be given (fixed throughout the derivation). If $1 \geq r = 1 - \frac{x}{n} \geq 1 - x_0/n$ and $\theta_0 \leq \theta \leq \pi - \theta_0$, we write that $(r,\theta) \in \mathcal{B}_0$. Of course, $\mathcal{B}_0$ depends on x_0, θ_0 but we will not spell out this dependence in our notations. Note that since complex roots come in pairs and since the distribution of the zeros is invariant under the transformation $r \to r^{-1}$ (because the $\{a_j\}$ are independent), it is enough to study this distribution for $(r,\theta) \in [0,1]\times(0,\pi)$.

By (7), one needs to compute the expectation

$$E(a_ja_k|X_1^n(r,\theta) = X_2^n(r,\theta) = 0)\,.$$

Note that, for $j \neq k$,

$$\begin{aligned} X_1^n(r,\theta) &= a_jr^j\cos(j\theta) + a_kr^k\cos(k\theta) + X_1^{j,k}(r,\theta) \\ X_2^n(r,\theta) &= a_jr^j\sin(j\theta) + a_kr^k\sin(k\theta) + X_2^{j,k}(r,\theta)\,. \end{aligned}$$

Here, $X_1^{j,k} \equiv X_1^{j,k}(r,\theta)$ and $X_2^{j,k} \equiv X_2^{j,k}(r,\theta)$ are independent of (a_j, a_k). Let $\overline{X}_{1,n} := n^{-1/2}X_1^{j,k}$, $\overline{X}_{2,n} := n^{-1/2}X_2^{j,k}$. One has

$$B_n = \operatorname{cov}(\overline{X}_{1,n}, \overline{X}_{2,n}) = \begin{pmatrix} \frac{1}{n}\sum_{\ell\neq j,k} r^{2\ell}\cos^2(\ell\theta) & \frac{1}{2n}\sum_{\ell\neq j,k} r^{2\ell}\sin(2\ell\theta) \\ \frac{1}{2n}\sum_{\ell\neq j,k} r^{2\ell}\sin(2\ell\theta) & \frac{1}{n}\sum_{\ell\neq j,k} r^{2\ell}\sin^2(\ell\theta) \end{pmatrix}$$

$$:= \begin{pmatrix} \frac{1}{2n}\sum_{\ell\neq j,k} r^{2\ell} & 0 \\ 0 & \frac{1}{2n}\sum_{\ell\neq j,k} r^{2\ell} \end{pmatrix} + V_n\,.$$

With $z = re^{i\theta}$, it holds that

$$|V_n(\ell,m)| \leq \frac{2}{n} + \frac{1}{n}|\sum_{j=0}^{n} z^{2j}|\,,\ \ell,m=1,2. \tag{8}$$

Hence, for any $\theta_0 > 0$ and $x_0 > 0$, $\sup_{n,(r,\theta)\in\mathcal{B}_0} nV_n < \infty$. Also, for $x_0 \geq x \geq 0$, one has for n large enough that

$$\frac{1-e^{-x}}{5x} \leq \frac{\sum_{\ell\neq j,k} r^{2\ell}}{2n} \leq \frac{1}{2}\,.$$

Hence, for such n, B_n is non-degenerate. Let

$$\gamma = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n} r^{2j}\,,\ Q(\alpha,\beta) = \frac{\alpha^2+\beta^2}{\gamma}\,.$$

Let $p_{\overline{X}_{1,n},\overline{X}_{2,n}}(\alpha,\beta)$ denote the density of the random variables $\overline{X}_{1,n},\overline{X}_{2,n}$ at (α,β). It follows after some algebra from standard local limit theorems using Edgeworth expansions (see, e.g., [4, Corollary 19.4], and note that the conditions needed to apply the result are satisfied in our case) that, for $(r,\theta)\in\mathcal{B}_0$,

$$\sup_{\alpha,\beta}|p_{\overline{X}_{1,n},\overline{X}_{2,n}}(\alpha,\beta) - \frac{e^{-Q(\alpha,\beta)}}{\pi\gamma} - \frac{P_{1,n}(\alpha,\beta)e^{-Q(\alpha,\beta)}}{\sqrt{n}\pi\gamma} - \frac{P_{2,n}(\alpha,\beta)e^{-Q(\alpha,\beta)}}{n\pi\gamma}| = O(n^{-3/2})\,,$$

where the bound in the RHS is independent of j,k, $P_{1,n}$, $P_{2,n}$ are polynomials of order 3 and 4 with minimal order 1 and 2, respectively, and with coefficients independent of j,k and of order O(1).

Observe that, with $j\neq k$,

$$p_{r,\theta}E(a_ja_k|X_1^n = X_2^n = 0) = E\left(a_ja_kp_{X_1^n,X_2^n}((0,0)|a_j,a_k)\right) = \frac{1}{n}E\left(a_ja_kp_{\overline{\mathrm{X}}_{1,n},\overline{\mathrm{X}}_{2,n}}(-\alpha_{jk},-\beta_{jk})\right)\,, \tag{9}$$

where

$$\alpha_{jk} = (a_jr^j\cos(j\theta) + a_kr^k\cos(k\theta))/\sqrt{n}\,,\ \beta_{jk} = (a_jr^j\sin(j\theta)+a_kr^k\sin(k\theta))/\sqrt{n}\,.$$

By our moment assumption on the a_j and the fact that $E(a_j)=0$, it follows that

$$|p_{r,\theta}E(a_ja_k|X_1^n = X_2^n = 0) - \frac{E(a_ja_ke^{-Q(\alpha_{jk},\beta_{jk})})}{n\pi\gamma}| = O(n^{-5/2})\,, \tag{10}$$

where the RHS is uniform in $(r,\theta)\in\mathcal{B}_0$ and in j,k. (The control of the term $P_{1,n}$ in the last estimate is due to the fact that the correlation contributed by Q is of order $n^{-1/2}$ at most). Note however that, again uniformly in $\mathcal{B}_0$, $\sum_{j,k=0}^n jkr^{j+k}|\cos((j-k)\theta)| = O(n^4)$, and hence the error term in (10) contribute to $h_n(r,\theta)$, $O(n^{3/2})$ at most. On the other hand, again for $j \neq k$,

$$Q(\alpha_{jk}, \beta_{jk}) = \frac{a_j^2 r^{2j} + a_k^2 r^{2k} + 2a_j a_k r^{j+k}\cos((j-k)\theta)}{\gamma n}.$$

Hence, for $j \neq k$,

$$\begin{aligned}
&E(a_j a_k e^{-Q(\alpha_{jk},\beta_{jk})}) = E(a_j a_k (e^{-Q(\alpha_{jk},\beta_{jk})} - 1)) \\
&= E(a_j a_k (e^{-Q(\alpha_{jk},\beta_{jk})} - 1)\mathbf{1}_{\{a_j^2 + a_k^2 > n\}}) \\
&\quad -\frac{1}{\gamma n} E(\mathbf{1}_{\{a_j^2+a_k^2 \le n\}} a_j a_k (a_j^2 r^{2j} + a_k^2 r^{2k} + 2a_j a_k r^{j+k}\cos((j-k)\theta)) \\
&\quad +O(n^{-2}) \\
&= -\frac{2}{\gamma n} r^{j+k}\cos((j-k)\theta) + O(n^{-2}),
\end{aligned}$$

where the $O(n^{-2})$ term is uniform in $\mathcal{B}_0$. Thus, with the $O(\cdot)$ terms again uniform in $\mathcal{B}_0$,

$$\begin{aligned}
&p_{r,\theta} \sum_{j,k=0, j\neq k}^n jkr^{j+k}\cos((j-k)\theta) E(a_j a_k | X_1^n = X_2^n = 0) \\
&= -\frac{2}{\pi\gamma^2 n^2}\sum_{j,k=0}^n jkr^{2(j+k)}\cos^2((j-k)\theta) + O(\frac{1}{n^3}\sum_{j,k=0}^n jkr^{2(j+k)}|\cos((j-k)\theta)|) + O(n) \\
&= -\frac{2}{\pi\gamma^2 n^2}\sum_{j,k=0}^n jkr^{2(j+k)}\cos^2((j-k)\theta) + O(n) \\
&= -\frac{1}{\pi\gamma^2 n^2}\sum_{j,k=0}^n jkr^{2(j+k)} + O(n),
\end{aligned} \tag{11}$$

where the last equality follows from the same estimate as (8).

Similarly, for $j = k$, using that

$$|E(a_j^2(e^{-a_j^2 r^{2j}/n} - 1)| \le \frac{c}{n}, \tag{12}$$

one concludes that

$$p_{r,\theta}\sum_{j=0}^n j^2 r^{2j} E(a_j^2 | X_1^n = X_2^n = 0) = \frac{\sum_{j=0}^n j^2(1-\frac{x}{n})^{2j}}{\pi\gamma n} + O(n). \tag{13}$$

Combining (7) and (9)–(13), we conclude that

$$h_n(r,\theta) = \frac{\sum_{j=0}^n j^2(1-\frac{x}{n})^{2j}}{\pi\gamma n} - \frac{\left(\sum_{j=0}^n j(1-\frac{x}{n})^{2j}\right)^2}{\pi\gamma^2 n^2} + O(n^{3/2}).$$

But,

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n}(1-\frac{x}{n})^{2j} = \frac{1}{x}\int_0^x e^{-2y}dy = \frac{1-e^{-2x}}{2x},$$

whereas

$$\lim_{n\to\infty} n^{-3}\sum_{j=0}^{n} j^2(1-\frac{x}{n})^{2j} = \frac{\int_0^x y^2e^{-2y}dy}{x^3} = \frac{1-e^{-2x}-2(x^2e^{-2x}+xe^{-2x})}{4x^3},$$

and

$$\lim_{n\to\infty} n^{-4}\left(\sum_{j=0}^{n} j(1-\frac{x}{n})^{2j}\right)^2 \frac{(\int_0^x ye^{-2y}dy)^2}{x^4} = \frac{(1-e^{-2x}-2xe^{-2x})^2}{16x^4}.$$

The statement of the theorem follows. □

2.2 The Gaussian case

Let the sequence $\{a_j\}$ consist of i.i.d. standard Normal random variables. We suppress in this subsection the superscript n from X_i^n. Conditioned on $X_1 = X_2 = 0$, the joint law of the sequence $\{a_j\}_{j=0}^n$ is again Gaussian, of zero mean, and of covariance

$$R_c = I - E(aX^T)(E(XX^T))^{-1}E(Xa^T).$$

Here, $a = (a_0, \ldots, a_n)$ and v^T denotes the transpose of a vector/matrix v. Let $\Delta = \det(EXX^T) = E(X_1^2)E(X_2^2) - (E(X_1X_2))^2$, and let

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = (E(XX^T))^{-1}.$$

Note that

$$E((aX^T)B)_j = r^j[b_{11}\cos(j\theta) + b_{12}\sin(j\theta), b_{12}\cos(j\theta) + b_{22}\sin(j\theta)],$$

and hence, after some algebra,

$$\begin{aligned}&\left(E(aX^T)(E(XX^T))^{-1}E(Xa^T)\right)_{jk} = \\ &r^{j+k}\left(\frac{b_{11}+b_{22}}{2}\cos((j-k)\theta) + \frac{b_{11}-b_{22}}{2}\cos((j+k)\theta) + b_{12}\sin((j+k)\theta)\right). \qquad (14)\end{aligned}$$

Note that

$$\begin{aligned}\Delta(b_{11}+b_{22}) &= E(X_1^2)+E(X_2^2) = \sum_{j=0}^{n} r^{2j} = \sum_{j=0}^{n}|z|^{2j} \\ \Delta(b_{11}-b_{22}) &= E(X_2^2)-E(X_1^2) = -\sum_{j=0}^{n} r^{2j}\cos(2j\theta) = -\mathrm{Re}\sum_{j=0}^{n} z^{2j} \\ 2\Delta b_{12} &= \sum_{j=0}^{n} r^{2j}\sin(2j\theta) = \mathrm{Im}\sum_{j=0}^{n} z^{2j},\end{aligned}$$

and thus, using the fact that in the Gaussian case, $p_{r,\theta} = p^G_{r,\theta} = (2\pi\sqrt{\Delta})^{-1}$, (7) transforms to (with G in h^G_n standing for "Gaussian"),

$$h^G_n(r,\theta)drd\theta = \frac{rdrd\theta\left(\sum_j j^2 r^{2j}\right)}{2\pi\Delta^{1/2}r^2}$$
$$- \frac{rdrd\theta\left(\sum_{j,k=0}^n jkr^{2(j+k)}\cos^2((j-k)\theta)\right)\left(\sum_{j=0}^n |z|^{2j}\right)}{4\pi\Delta^{3/2}r^2}$$
$$+ \frac{rdrd\theta\left(\sum_{j,k=0}^n jkr^{2(j+k)}\cos((j-k)\theta)\left(\cos((j+k)\theta)\mathrm{Re}\left(\sum_{j=0}^n z^{2j}\right) - \sin((j+k)\theta)\mathrm{Im}\left(\sum_{j=0}^n z^{2j}\right)\right)\right)}{4\pi\Delta^{3/2}r^2}.$$

Going back to Cartesian coordinates, and letting $h^G_n(z)dz = h^G_n(r,\theta)drd\theta$, some simple algebra leads now to

$$\begin{aligned} h^G_n(z) &= \frac{\sum_{j=0}^n j^2|z|^{2j}}{2\pi\Delta^{1/2}|z|^2} + \frac{\left(\sum_{j=0}^n j|z|^{2j}\right)\left((\sum_{j=0}^n jz^{2j})(\sum_{j=0}^n z^{2j}) + (\sum_{j=0}^n j(z^*)^{2j})(\sum_{j=0}^n (z^*)^{2j})\right)}{8\pi\Delta^{3/2}|z|^2} \\ &\quad - \frac{(\sum_{j=0}^n |z|^{2j})(\sum_{j=0}^n j|z|^{2j})^2}{8\pi\Delta^{3/2}|z|^2} - \frac{(\sum_{j=0}^n |z|^{2j})|\sum_{j=0}^n jz^{2j}|^2}{8\pi\Delta^{3/2}|z|^2}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Delta &= \sum_{j,k=0}^n r^{2(j+k)}[\cos^2 j\theta\sin^2(k\theta) - \cos(j\theta)\cos(k\theta)\sin(j\theta)\sin(k\theta)] \\ &= \sum_{j,k=0}^n |z|^{2(j+k)}\frac{\sin^2((k-j)\theta)}{2} \\ &= \frac{(\sum_{j=0}^n |z|^{2j})^2}{4} - \frac{|\sum_{j=0}^n z^{2j}|^2}{4}. \end{aligned} \tag{16}$$

These expressions are easily seen to coincide with those given in [18], which form the basis for the asymptotic analysis there. In particular, we recall that it was shown in [18] that

$$\lim_{n\to\infty} h^G_n(z) = \frac{\sqrt{(1-|z|^2)^{-2} - |1-z^2|^{-2}}}{\pi(1-|z|^2)}.$$

3 Proof of Theorem 1

By Jensen's formula, for any $r > 0$,

$$\int_0^r \frac{\nu_n(u)}{u}du = \frac{1}{2\pi}\int_0^{2\pi}\log|P_n(re^{i\theta})|d\theta - \log|P_n(0)|.$$

Let $r = \exp(-s/n)$. Making the change of variables $u = \exp(-v/n)$, one gets

$$\int_s^\infty \frac{\nu_n(e^{-v/n})}{n}dv = \frac{1}{2\pi}\int_0^{2\pi}\log|P_n(r(s)e^{i\theta})|d\theta - \log|P_n(0)|.$$

Hence, since $\nu_n(r)$ is invariant to a multiplication of all coefficients by $1/B_n$, one gets, for any $s_0 \geq 0$,

$$\int_{s_0}^{\infty} \frac{\nu_n(e^{-v/n})}{n} dv = \frac{1}{2\pi}\int_0^{2\pi} \log |B_n^{-1} P_n(r(s)e^{i\theta})| d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log |B_n^{-1} P_n(r(s_0)e^{i\theta})| d\theta \,.$$

The main technical step required for the proof of the theorem is the following lemma:

Lemma 1 *There exists a constant R, independent of s, such that*

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} E \log |B_n^{-1} P_n(r(s)e^{i\theta})| d\theta = h(s) \,,$$

where $h(s) = \frac{1}{\alpha}\log\left(\frac{1-e^{-\alpha s}}{s}\right) + R$.

Equipped with Lemma 1, let us complete the proof of Theorem 1. Let

$$f_n(v) = E\left(\frac{\nu_n(e^{-v/n})}{n}\right) \,.$$

Note that $f_n(\cdot)$ is a bounded monotone function. Thus, by Helly's theorem, one may find a subsequence n_j such that $f_{n_j}(\cdot)$ converges pointwise to a limit $f(\cdot)$. Lemma 1 then implies that $\int_{s_0}^{s} f(v)dv = h(s) - h(s_0)$, and hence the convergence along subsequences may be replaced by a full convergence. But then,

$$\lim_{n\to\infty} E\left(\frac{\nu_n(e^{-v/n})}{n}\right) = f(v) = h'(v) \,.$$

This yields Theorem 1. □

Proof of Lemma 1 : Let X_1^n, X_2^n be as in (5), with $r = r(s) = \exp(-s/n)$, and recall that

$$|P_n(re^{i\theta})|^2 = (X_1^n)^2 + (X_2^n)^2 \,.$$

(We will suppress the dependence of X_i^n on r, θ in our notations). The crucial observation needed for the proof of Lemma 1 is contained in the:

Lemma 2 *For almost all $\theta \in [0, 2\pi]$, the distribution of the $\mathbf{R}^2$ vector $X_n(\theta, s) = B_n^{-1}(X_1^n, X_2^n)$ converges as $n \to \infty$ to the distribution with the characteristic function*

$$g_\alpha(t_1, t_2) = \exp\left(-c_\alpha |t_1^2 + t_2^2|^{\alpha/2}(1 - e^{-\alpha s})/(\alpha s)\right) \,, \tag{17}$$

with

$$c_\alpha = \frac{4\Gamma(\alpha)}{\alpha 2^\alpha [\Gamma(\alpha/2)]^2} \,.$$

Proof of Lemma 2 : Assume first that $\{a_j\}$ possess an α-stable law, with characteristic function

$$g(t) = \exp\left(i\gamma t - |t|^\alpha(1 + K(\alpha,\beta,t))\right),$$

and normalizing constant $B_n = n^{1/\alpha}$ (see (4) for the definitions). We assume for simplicity $\alpha \neq 1$ and $\gamma = 0$, the general case being similar. Let $g^n_\alpha(t_1,t_2)$ denote the characteristic function of X_n. Then, making the change of coordinate $(t_1,t_2) \to (\rho,\psi)$, one gets

$$g^n_\alpha(\rho,\psi) = \exp\left(-\frac{\rho^\alpha}{n}\sum_{j=0}^{n}|r^j\cos(j\phi+\psi)|^\alpha(1+i\beta\mathrm{sign}(\cos(j\phi+\psi))\right).$$

Since for almost all θ, the empirical distribution of the sequence $x_j = j\theta/2\pi$ converges weakly to the uniform distribution (due to ergodicity, see [5, Pg. 294]), one gets that

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}|\cos(j\phi+\psi)|^\alpha(1+i\beta\mathrm{sign}\cos(j\phi+\psi)) \quad &\to_{n\to\infty} \quad \frac{1}{2\pi}\int_0^{2\pi}|\cos x|^\alpha(1+i\beta\mathrm{sign}\cos x)dx \\ &= \quad \frac{1}{2\pi}\int_0^{2\pi}|\cos x|^\alpha dx\,.\end{aligned}$$

Hence, by Abel summation,

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|r^j\cos(j\phi+\psi)|^\alpha\,(1+i\beta\mathrm{sign}(\cos(j\phi+\psi))) = \\ &\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}j(r^{\alpha j}-r^{\alpha(j+1)})\left(\frac{1}{j}\sum_{k=0}^{j}|\cos(k\phi+\psi)|^\alpha\,(1+i\beta\mathrm{sign}(\cos(k\phi+\psi)))\right) + \\ &\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|\cos(j\phi+\psi)|^\alpha\,(1+i\beta\mathrm{sign}(\cos(j\phi+\psi)))\,e^{-\alpha s} = \\ &\frac{1}{2\pi}\int_0^{2\pi}|\cos x|^\alpha dx\left(\frac{1-e^{-\alpha s}}{\alpha s}\right).\end{aligned}$$

It follows that

$$\lim_{n\to\infty} g_n(\rho,\psi) = \exp\left(-\rho^\alpha c_\alpha\left(\frac{1-e^{-\alpha s}}{\alpha s}\right)\right).$$

This completes the proof in the α-stable case.

Assume next that the sequence $\{a_j\}$ is in the domain of attraction of an α-stable law, with norming constant B_n and characteristic function $g(t)$. Recall that (see [11, Chapter 2.6]), a distribution G on the real line with characteristic function $g(t)$ belongs to the domain of attraction of the stable law in (4) if and only if

$$g(t) = \exp(i\gamma t - |t|^\alpha(1+K(\alpha,\beta,t))h(t)), \tag{18}$$

The proof now proceeds exactly as before, with the norming constants B_n chosen as described below (4). □

Equipped with Lemma 2, we may return to the proof of Lemma 1. Recall that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} E\log|B_n^{-1}P_n(r(s)e^{i\theta})|d\theta = \lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})\,. \tag{19}$$

Suppose that one could now interchange the limit and expectation operation. Denoting by $p_\alpha(x_1,x_2)$ the density of the law associated with the characteristic function $g_\alpha(t_1,t_2)$ (such a density exists since $g_\alpha \in L_1(\mathbf{R}^2)$). An explicit computation reveals that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\log\sqrt{x_1^2+x_2^2}p_\alpha(x_1,x_2)dx_1dx_2 = h(s)\,.$$

Thus, all that remains is to prove the uniform integrability of the integrand in the left hand side of (19). That is, we need to prove that, uniformly in n,

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})\mathbf{1}_{\{|\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})|>N\}}d\theta = 0\,. \tag{20}$$

It is straightforward to check (using [11, Theorem 2.6.4]) that for any $\lambda<\alpha$,

$$\sup_n E(|B_n^{-1}X_i^n|^\lambda) < \infty\,.$$

Thus, the upper tail in (20) poses no problem, and the proof of Lemma 1 is reduced to the proof of the

Lemma 3

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})\mathbf{1}_{\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})<-N}d\theta = 0\,. \tag{21}$$

Since it is not particularly illuminating, we provide the proof of Lemma 3 in the Section 4. □

4 Proof of Lemma 3

We recall that the concentration function of a distribution A on the real line with characteristic function $a(t)$ is defined as

$$Q(\tau) = \sup_x A\{[x, x+\tau]\}\,.$$

It holds (c.f. [9, Pg. 292]) that $Q(\tau)\le C\tau\int_{-\tau^{-1}}^{\tau^{-1}}|a(t)|dt$, with C independent of the distribution A. In this section, C denotes a constant whose value may change from line to line but is independent of n, N. We use in the sequel Q to denote the concentration function of the distribution of $B_n^{-1}X_1^n$. Then,

$$\frac{1}{2\pi}\int_0^{2\pi} P(|X_1^n|<e^{-k})d\theta \quad\le\quad \frac{1}{2\pi}\int_0^{2\pi} Q(e^{-k+1})d\theta \tag{22}$$

$$\leq \frac{Ce^{-k+1}}{2\pi}\int_0^{2\pi} d\theta \int_{-e^{1-k}}^{e^{1-k}} \prod_{j=0}^{n} |g\left(\frac{tr^j\cos(j\theta)}{B_n}\right)|d\theta$$

$$\leq \frac{Ce^{-k+1}}{2\pi}\int_0^{2\pi} d\theta \int_{-e^{1-k}}^{e^{1-k}} \exp\left(-\frac{1}{2}\sum_{j=0}^{n}\left(1-\left|g\left(\frac{tr^j\cos(j\theta)}{B_n}\right)\right|^2\right)\right).$$

Note that for sufficiently small $|t|$ one has that $|g(t)|^2 = \exp(-2|t|^\alpha h(t))$, with $h(\cdot)$ slowly varying. Hence, for k such that $e^{-k} > cB_n^{-1}$, we have that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} P(|X_1^n| < e^{-k})d\theta &\leq \frac{Ce^{1-k}}{2\pi}\int_0^{2\pi} d\theta \int_{-e^{1-k}}^{e^{1-k}} \exp(-|t|^\alpha B_n^{-\alpha}\sum_{j=1}^{n}|\cos(j\theta)|^\alpha r^{j\alpha} h(tr^j\cos(j\theta)/B_n))dt \\ &\leq Ce^{-k}, \end{aligned} \tag{23}$$

where we have used the fact that $h(B_n^{-1})/B_n^\alpha < C/n$.

We show below that there exist positive constants ℓ, q, C such that

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\tau)d\theta \leq C(n^{-3} + n^\ell \tau^q). \tag{24}$$

Note that (24) is trivial when the random variables $\{a_i\}$ possess a bounded density.

Assuming (24), we conclude the proof of Lemma 3: Note that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})\mathbf{1}_{\{|\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})|>N\}}d\theta \leq \\ &\sum_{k=\log N}^{[n^{1/8}]}\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}|X_1^n|)\mathbf{1}_{\{-k+1\leq\log(B_n^{-1}|X_1^n|)<-k\}}d\theta + \\ &\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}|X_1^n|)\mathbf{1}_{\{\log(B_n^{-1}|X_1^n|)<-n^{1/8}\}}d\theta.\end{aligned}$$

Using (23), we find that

$$\sum_{k=\log N}^{[n^{1/8}]}\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}|X_1^n|)\mathbf{1}_{\{-k+1\leq\log(B_n^{-1}|X_1^n|)<-k\}}d\theta \leq C(n^{-1/4}+\sum_{k=N}^{\infty}ke^{-k}), \tag{25}$$

whereas, using (24) and the bound

$$E\int_0^{2\pi}\log^2|P_n(re^{i\theta})|d\theta \leq (Cn)^2,$$

which is due to the integrability of the function $\log^2 x$ at zero and the assumption $P(a_j=0)=0$,

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} E\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})\mathbf{1}_{\log(B_n^{-1}\sqrt{(X_1^n)^2+(X_2^n)^2})<-n^{1/8}}d\theta \\ \leq\ &\frac{1}{2\pi}\int_0^{2\pi} E^{1/2}|\log^2|P_n(re^{i\theta})||\mathrm{Prob}^{1/2}(|B_n^{-1}P_n(re^{i\theta})|<e^{-n^{1/8}})d\theta \\ \leq\ &Cn\left(\frac{1}{2\pi}\int_0^{2\pi} Q(e^{-n^{1/8}})d\theta\right)^{1/2} \leq C(1/\sqrt{n}+n^{(\ell+1)/2}e^{-qn^{1/8}/2})\end{aligned} \tag{26}$$

Thus, Lemma 3 follows from (25), (25) and (26) once we prove (24).

To this end, denote by $\tilde{G}$ the distribution with characteristic function $|g(t)|^2$. Then, from (22), for some constants a, b,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} Q(\tau)d\theta &\leq \frac{C\tau}{2\pi}\int_0^{2\pi}\int_{-1/\tau}^{1/\tau}\exp(-\frac{1}{2}\int_{-\infty}^{\infty}\sum_{j=1}^{n}(1-\cos\left(\frac{txr^j\cos(j\theta)}{B_n}\right))\tilde{G}(dx))dt \\ &\leq \frac{C\tau}{2\pi}\int_0^{2\pi}\int_{-1/\tau}^{1/\tau}\exp(-C\int_{-\infty}^{\infty}\sum_{j=1}^{n}(1-\cos\left(\frac{txr^j\cos(j\theta)}{B_n}\right))G(dx))dt\,,\end{aligned}$$

where G is some distribution concentrated on $[-b,-a]\cup[a,b]$ and $C = C(a,b)$. Using Jensen's inequality, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\tau)d\theta \leq \sup_{a\leq|x|\leq b}\frac{CB_n\tau}{2\pi}e^{-cn}\int_{-1/\tau B_n}^{1/\tau B_n}dt\int_0^{2\pi}\exp(c\sum_{j=1}^{n}\cos(txr^j\cos(j\theta))d\theta\,. \tag{27}$$

Let

$$\Gamma_1(x,n) = \{\theta\in[0,2\pi] : \sum_{j=1}^{n}\cos(txr^j\cos(j\theta)) \leq n/2\}\,,$$

and $\Gamma_2(x,n) = [0,2\pi]\setminus\Gamma_1(x,n)$. Since the integral over Γ_1 in (27) yields a bound much better than that in (24), our main task is now to estimate the Lebesgue measure of $\Gamma_2 = \Gamma_2(x,n)$. Note that

$$\begin{aligned}&\int_0^{2\pi}|\sum_{j=1}^{k}\cos(txr^j\cos(j\theta))|^6 d\theta = O(n^3)\\ +&\sum_{s\neq j_1\neq j_2\neq j_3}\int_0^{2\pi}\cos^3(txr^s\cos(s\theta))\cos(txr^{j_1}\cos(j_1\theta))\cos(txr^{j_2}\cos(j_2\theta))\cos(txr^{j_3}\cos(j_3\theta))d\theta\\ &+\dots\\ &+\sum_{j_1\neq j_2\neq\cdots\neq j_6}\int_0^{2\pi}\cos(txr^{j_1}\cos(j_1\theta))\cos(txr^{j_2}\cos(j_2\theta))\cdots\cos(txr^{j_6}\cos(j_6\theta))d\theta\\ \leq\;& C(n^3+n^{\ell+6}/t^q)\,,\end{aligned}$$

for some ℓ, q positive. Hence,

$$\int_0^{2\pi}\mathbf{1}_{\{|\sum_{j=1}^{k}\cos(txr^j\cos(j\theta))|>n/2\}}d\theta \leq \frac{2^6}{n^6}\int_0^{2\pi}|\sum_{j=1}^{k}\cos(txr^j\cos(j\theta))|^6 d\theta \leq C(n^{-3}+n^\ell t^{-q})\,.$$

Substituting in (27), we find that

$$\frac{1}{2\pi}\int_0^{2\pi} Q(\tau)d\theta \leq C(e^{-cn/2}+n^{-3}+n^\ell\tau^q)\,,$$

which is more than enough to imply (24). □

Acknowledgment We thank Larry Shepp for raising our interest in this question, making [18] available to us, and for giving us the opportunity to meet. O.Z. thanks Moshe Porat and Sharon Urieli for originally bringing the problem to his attention, Nir Cohen for helpful discussions, Alan Edelman for helpful discussions and for making [6] available to us, and Robert Adler for the reference to [1].

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