LOCAL ASYMPTOTICS FOR CONTROLLED MARTINGALES

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Abstract. We consider controlled martingales with bounded steps where the controller is allowed at each step to choose the distribution of the next step, and where the goal is to hit a fixed ball at the origin at time n. We show that the algebraic rate of decay (as n increases to infinity) of the value function in the discrete setup coincides with its continuous counterpart, provided a reachability assumption is satisfied. We also study in some detail the uniformly elliptic case and obtain explicit bounds on the rate of decay. This generalizes and improves upon several recent studies of the one dimensional case, and is a discrete analogue of a stochastic control problem recently investigated in Armstrong and Trokhimtchouck (2010).

1. Introduction

Consider a family of (possibly multi-dimensional) martingales $\{M_n\}_{n\geq 0}$ in discrete time, with $M_0 = 0$, equipped with their natural filtration $\mathcal{F}_n$. What is the maximal probability that, at time n, the martingale is in a prescribed set? Similarly, what is the minimal probability?

Such questions are naturally framed in the language of control theory, and have recently received attention from several authors, see [1, 2, 5]. It follows from the analysis in [5] that, even in the one dimension with $|M_{n+1} - M_n| \leq 1$ and $\mathbb{E}\left[(M_{n+1} - M_n)^2 | \mathcal{F}_n\right] \geq \lambda > 0$ a.s., it is not true that $\mathbb{P}\left[|M_n| \leq 1\right] \sim n^{-1/2}$, in general, as one may naively expect.

In a continuous time (diffusion) setup, the analogous question is older; see [7, 6] for early work in his direction. A rather complete solution was given recently by [2], through the analysis of the associated dynamic programming equation (which turns out to be a fully nonlinear parabolic PDE). In particular, it is shown there that if $\{X_t^u\}$ is a controlled diffusion process in $\mathbb{R}^d$ with zero drift, where the control u is the instantaneous diffusion coefficient which is restricted to be uniformly elliptic and bounded above, then there exists $\alpha > 0$ such that

$$\sup_u \mathbb{P}\left[|X_t^u| \leq 1\right] \sim t^{-\alpha}.$$

The exponent α is determined by the solution to a nonlinear eigenvalue problem and typically we have $\alpha < d/2$.

Our goal in this paper is to provide a similar analysis of the discrete time setup. Our analysis builds on [7] but requires significant modifications. We present here

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two corollaries of our main result, Theorem 2.7. In what follows, $|\cdot|$ denotes the Euclidean norm.

Corollary 1.1 (Uniformly elliptic martingales). *Fix $\lambda \in (0,1]$ and $R \geq \sqrt{2d}$ and let $\mathfrak{M}_{d,\lambda,R}$ denote the collection of laws of discrete time martingales of the form*

$$M_n = \sum_{i=1}^{n} \Delta_i \in \mathbb{R}^d$$

satisfying

$$|\Delta_i| \leq R \quad a.s. \tag{1.1}$$

and

$$\lambda \leq \inf_{v\in\mathbb{R}^d,\,|v|=1} \mathbb{E}\left[\tfrac{1}{2}(\Delta_i\cdot v)^2 \,|\, \mathcal{F}_{i-1}\right] \leq \sup_{v\in\mathbb{R}^d,\,|v|=1} \mathbb{E}\left[\tfrac{1}{2}(\Delta_i\cdot v)^2 \,|\, \mathcal{F}_{i-1}\right] \leq 1 \quad a.s. \tag{1.2}$$

Then there exists a constant $\alpha = \alpha(d,\lambda) > 0$ such that,

$$\begin{aligned} 0 < \liminf_{n\to\infty} n^\alpha \sup_{\mathbb{P}\in\mathfrak{M}_{d,\lambda,R}} \mathbb{P}\left[|M_n| \leq \sqrt{d}R\right] & \\ \leq \limsup_{n\to\infty} n^\alpha \sup_{\mathbb{P}\in\mathfrak{M}_{d,\lambda,R}} \mathbb{P}\left[|M_n| \leq \sqrt{d}R\right] &< \infty. \end{aligned} \tag{1.3}$$

As we will see, the exponent α in Corollary (1.1) satisfies

$$\frac{d\lambda}{2} \leq \alpha(d,\lambda) \leq \frac{(d-1)\lambda}{2} + \frac{1}{2} \tag{1.4}$$

and each of the two inequalities in (1.4) is an equality if and only if $\lambda = 1$. Notice in particular that this implies that $\alpha(d,\lambda) < d/2$ if $\lambda < 1$, which means that the quantity $\sup_{\mathbb{P}\in\mathfrak{M}_{d,\lambda,R}} \mathbb{P}\left[|M_n| \leq \sqrt{2d}R\right]$ decays at a slower rate than for a simple random walk. It was previously observed in [5] in the discrete setup for $d = 1$ that $\alpha < 1/2$ if λ is sufficiently small. We actually obtain the stronger statement that $\alpha < d/2$ for general controlled, uniformly elliptic martingales, provided that the set of controls has at least two elements. Both the latter statement as well as the bounds (1.4) were proved in [2, (3.20)] in the continuum framework, and they apply in our discrete setup since, as we will see, our exponent α is the same as the one corresponding to the minimal Pucci operator from [2].

It is also of interest to study the behavior of the exponent $\alpha(d,\lambda)$ as $\lambda \to 0$. Here the estimate (1.4) is not very sharp on either side, and it turns out that, except for a possible sub-algebraic correction, $\alpha(d,\lambda) \sim \lambda^{1/4}$. Precisely, for each $\delta \in (0,1/4)$, there exist constants $C(d,\delta) > 1$ and $c(d,\delta) > 0$ such that, for every $\lambda \in (0,1]$,

$$c\lambda^{1/4+\delta} \leq \alpha(d,\lambda) \leq C\lambda^{1/4-\delta}. \tag{1.5}$$

In particular, $\alpha(d,\lambda) \to 0$ as $\lambda \to 0$ and

$$\lim_{\lambda\to 0} \limsup_{n\to\infty} \frac{|\log \sup_{\mathbb{P}\in\mathfrak{M}_{d,\lambda,R}} \mathbb{P}\left[|M_n| \leq \sqrt{d}R\right]|}{\log n} = 0\,,$$

which was previously proved for $d = 1$ in [5]. The interpretation is that, for a controlled martingale, the quantity $\mathbb{P}\left[|M_n| \leq \sqrt{d}R\right]$ may decay at an arbitrarily slow (algebraic) rate in n provided that the set of controls is sufficiently rich. The bounds (1.5) are new and follow from test function calculations in Section 4.

In our second corollary, we consider a non-uniformly elliptic martingale, answering a question communicated to us by Y. Peres; after the work on this paper was completed, we learnt of an independent, different proof of the corollary, due to Lee, Peres and Smart.

Corollary 1.2. *Let M_n be an $\mathbb{R}^d$-valued martingale adapted to a filtration $\mathcal{F}_n$ with $X_0 = 0$ which satisfies, for some $\lambda \in (0,1]$,*

$$\mathbb{P}\left[|M_{n+1} - M_n| \leq 1\ \right] = 1$$

and

$$\mathbb{E}\left[|M_{n+1} - M_n|^2 \,|\, \mathcal{F}_n\right] = \lambda^2.$$

Then there exists $C(\lambda) > 0$ such that

$$\mathbb{P}\left[|M_n| \leq 1\right] \leq Cn^{-1/2}.$$

Note that the exponent 1/2 in Corollary 1.2 is sharp in every dimension, as exhibited by the local CLT for a simple random walk in one of the coordinate directions.

In the next section, we state our precise assumptions and the main result, Theorem 2.7, the proof of which comes in Section 3. The proofs of Corollaries 1.1 and 1.2 come in Section 4, as well as a discussion of how to estimate α and the proofs of (1.4) and (1.5).

2. Setup and main results

2.1. **Notation and assumptions.** Throughout the paper, we work in dimension $d \geq 1$. For $r > 0$ and $x \in \mathbb{R}^d$, we let $B_r(x)$ denote the open ball of radius r centered at $x \in \mathbb{R}^d$. We also set $B_r := B_r(0)$.

Definition 2.1. For each $R \geq 1$, we define $\mathcal{M}_R(\mathbb{R}^d)$ to be the family of centered Borel probability measures supported on B_R. That is, for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ and with X the canonical random variable on $\mathbb{R}^d$, we have

$$\mathbb{E}_\mu\left[X\right] = 0 \tag{2.1}$$

and

$$\mathbb{P}_\mu\big[\,|X| \leq R\big] = 1. \tag{2.2}$$

We also set, for each $\lambda \in (0,1]$ and $R \geq \sqrt{2d}$,

$$\mathcal{E}_{\lambda,R}(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}_R(\mathbb{R}^d) \,:\, \lambda I_d \leq \frac{1}{2}\mathbb{E}_\mu\left[XX^t\right] \leq I_d \right\}. \tag{2.3}$$

Here I_d denotes the $d \times d$ identity matrix, and if A and B are symmetric matrices, then we write $A \leq B$ in the case that $B - A$ is nonnegative definite.

Given a subset $\mathcal{P} \subseteq \mathcal{M}_R(\mathbb{R}^d)$ (the *control*) and a point $x \in \mathbb{R}^d$, we introduce the family of controlled martingales $\{(X_n, u_n)\}_{n\geq 0}$ with $X_n \in \mathbb{R}^d$, $X_0 = x$, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, so that the control $u_n \in \mathcal{P}$ is $\mathcal{F}_n$ measurable and, conditioned on $\mathcal{F}_n$, $X_{n+1} - X_n$ is distributed according to u_n. With an abuse of notation, we denote by

$\mathcal{P}_n$ the class of *admissible controls*, that is those sequences $\mathbf{u} = (u_1, \ldots, u_n)$ satisfying the above restrictions, and we let $\mathbb{P}^x$ denote the law of the sequence $\{(X_n, u_n)\}_{n\geq 0}$. In this setup, we are interested in the evaluation, for fixed $\delta > 0$, of the quantity

$$\sup_{\mathbf{u}\in\mathcal{P}_n} \mathbb{P}^x \left[X_n \in \overline{B}_\delta\right]. \tag{2.4}$$

Remark 2.2. It is natural to also consider the dual problem, i.e., the quantity

$$\inf_{\mathbf{u}\in\mathcal{P}_n} \mathbb{P}^x \left[X_n \in \overline{B}_\delta\right]. \tag{2.5}$$

The analysis required is similar and we comment on it in Section 2.3 below.

We next introduce the value function, which satisfies the dynamic programming equation.

Definition 2.3 (The value function w)**.** Given $\mathcal{P} \subseteq \mathcal{M}_R(\mathbb{R}^d)$ and $\delta > 0$, we define a function $w : \mathbb{R}^d \times \mathbb{N} \to \mathbb{R}$ by setting

$$w(x, 0) := \begin{cases} 1 & \text{if } x \in \overline{B}_\delta, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{B}_\delta, \end{cases}$$

and then defining $w(\cdot, n)$ inductively by

$$w(x, n+1) := \sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho \left[w(x + X, n)\right].$$

It is clear that $w(x, n)$ equals the expression in (2.4).

Our interest lies in the asymptotic behavior of $w(x, n)$ for large n. We prove our main result under two additional assumptions, stated below. These assumptions can be quickly checked for large classes of examples, as we show in Section 4. Before stating these, we first introduce some further notation.

Definition 2.4 (The operator F^-)**.** Given $\mathcal{P} \subseteq \mathcal{M}_R(\mathbb{R}^d)$, we define the operator F^- on the space $C(\mathbb{R}^d)$ by

$$F^- \left[\phi\right](x) := \phi(x) - \sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho \left[\phi(X + x)\right]. \tag{2.6}$$

We extend the definition of F^- to merely locally bounded functions ϕ by setting

$$F^- \left[\phi\right](x) := \phi(x) - \sup_{\rho\in\mathcal{P}} \sup_{\psi\in C(\mathbb{R}^d),\, \psi\leq\phi} \mathbb{E}_\rho \left[\psi(X + x)\right]$$

By abuse of notation, we also use F^- to denote the functions $\mathbb{S}^d \to \mathbb{R}$ (here, $\mathbb{S}^d$ denotes non-negative definite d-by-d matrices) given by $M \mapsto F^- \left[\phi_M\right]$, where ϕ_M is any quadratic function with Hessian $M \in \mathbb{S}^d$, that is, we define

$$F^-(M) := -\frac{1}{2} \sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho \left[X \cdot MX\right]$$

Note that, by (2.1), $F[\phi]$ is unchanged if we add an affine function to ϕ. In particular, for every $M \in \mathbb{S}^d$, $p \in \mathbb{R}^d$, $a \in \mathbb{R}$ and quadratic $\phi(x) := x \cdot Mx + p \cdot x + a$, we see that $F^- \left[\phi\right] = F(M)$.

In general, F^- is a concave, (possibly) degenerate elliptic, fully nonlinear operator. In the case that $\mathcal{P} \subseteq \mathcal{E}_{\lambda,R}(\mathbb{R}^d)$ for some $\lambda > 0$, then F^- is uniformly elliptic. In the case that $\mathcal{P} = \mathcal{E}_{\lambda,R}(\mathbb{R}^d)$, the operator F^- coincides with the minimal Pucci operator with ellipticity constants λ and 1 (as defined in [2]).

In the rest of the paper, except in Section 2.3 and Section 4, we write $F = F^{-}$. To aid our computations, we note that, for all $t \geq 0$ and locally bounded functions $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$, we have:

$$F[t\phi] = tF[\phi] \tag{2.7}$$

and

$$F[\phi] + F[\psi] \leq F[\phi + \psi]. \tag{2.8}$$

We next present our two assumptions.

Assumption 2.5 (F admits a self-similar solution)**.** There exist $\alpha > 0$, $\sigma \in (0, 1]$ and a solution $\Phi \in C^2(\mathbb{R}^d \times (0, \infty))$ of the fully nonlinear (possibly degenerate) parabolic partial differential equation

$$\partial_t \Phi + F(D^2\Phi) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \tag{2.9}$$

which satisfies:

$$\Phi > 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \tag{2.10}$$

$$\Phi(\sqrt{\lambda}x, \lambda t) = \lambda^{-\alpha}\Phi(x, t) \quad \text{for every } \lambda > 0, \tag{2.11}$$

and Φ decays like a Gaussian up to $C^{2,\sigma}$: that is, there exist constants $a > 0$ and $K > 1$ such that

$$\Phi(\cdot, 1) \in C^{2,\sigma}(\mathbb{R}^d) \tag{2.12}$$

and, for every $x \in \mathbb{R}^d$,

$$\|\Phi(\cdot, 1)\|_{C^{2,\sigma}(B_1(x))} \leq K \exp\left(-a|x|^2\right). \tag{2.13}$$

Assumption 2.6 (Behavior of w up to finite times)**.** We have:
(i) For every $r > 0$, there exists $N_0(r) > 1$ such that, for every $N \geq N_0(r)$,

$$\inf\left\{w(x, N) \,:\, |x| \leq r\sqrt{N \log N}\right\} > 0. \tag{2.14}$$

(ii) For every $r > 1$ and $n \in \mathbb{N}$, there exists $C(r, n) > 1$ such that

$$w(x, n) \leq C \exp\left(-r\frac{|x|}{\sqrt{n}}\right) \quad \text{for every } x \in \mathbb{R}^d. \tag{2.15}$$

Note that, in our setup, (2.15) is always satisfied, due to Azuma's inequality [3].

2.2. Local asymptotics for w.

We next present the main result of the paper.

Theorem 2.7. *(i) Assume that Assumptions 2.5 and 2.6(ii) hold. Then*

$$\sup_{r>0} \limsup_{n\to\infty} \sup_{x\in B_{r\sqrt{n}}} \frac{w(x, n)}{\Phi(x, n)} < +\infty. \tag{2.16}$$

(ii) Assume that Assumptions 2.5 and 2.6(i) hold. Then

$$0 < \inf_{r>0} \liminf_{n\to\infty} \inf_{x\in B_{r\sqrt{n}}} \frac{w(x, n)}{\Phi(x, n)} \tag{2.17}$$

Observe that, in view of (2.10) and (2.11), the inequalities (2.16) and (2.17) together imply that, for every $r > 0$,

$$0 < \liminf_{n\to\infty} \inf_{x\in B_{r\sqrt{n}}} n^\alpha w(x,n) \le \limsup_{n\to\infty} \sup_{x\in B_{r\sqrt{n}}} n^\alpha w(x,n) < \infty.$$

These can be compared to the conclusions of Corollaries 1.1 and 1.2.

Given the conclusion of Theorem 2.7, it is natural to expect a stronger statement to hold, namely a full local limit theorem for w: that is, for some constant $L > 0$,

$$\sup_{r>0} \limsup_{n\to\infty} \sup_{x\in B_{r\sqrt{n}}} \left| \frac{w(x,n)}{\Phi(x,n)} - L \right| = 0. \tag{2.18}$$

While our setup may be a bit too general for (2.18), we do expect it to hold, for instance, in the uniformly elliptic setting ($\mathcal{P} \subseteq \mathcal{E}_{\lambda,R}$). Indeed, this is relatively easy to obtain from Theorem 2.7 and the test functions in Section 3, provided we have at our disposal some regularity theory for uniformly parabolic finite difference equations (which we would apply to w). We could not find such a result matching our situation. We speculate that one could derive it from adaptations of known techniques, however developing such a regularity theory would take us too far astray from the focus of this paper, and so we do not prove (2.18).

2.3. **Minimal probabilities.** As discussed in Remark 2.2, it is natural to consider the optimal control problem (2.5) instead of (2.4), with associated value function v satisfying the dynamic programming equation

$$v(x,n+1) = \inf_{\rho\in\mathcal{P}} \mathbb{E}_\rho v(x+X,n), \quad v(x,0) = w(x,0).$$

The analysis is similar, with the operator F^- replaced by the operator

$$F^+[\phi](x) := \phi(x) - \inf_{\rho\in\mathcal{P}} \sup_{\psi\in C(\mathbb{R}^d),\, \psi\le\phi} \mathbb{E}_\rho[\psi(X+x)],$$

and a similar definition for $F^+(M)$, $M \in \mathbb{S}^d$. In the analysis, the relations

$$F^+[-\phi] = -F^-[\phi], \quad F^+[\phi] \ge F^-[\phi]$$

and

$$F^+[\phi] + F^-[\psi] \le F^+[\phi+\psi] \le F^+[\phi] + F^+[\phi]$$

come in handy. Using now $F = F^+$ and replacing w by v in Assumption 2.6, one then obtains Theorem 2.7 for v.

We remark that, in some natural situations, the assumption (2.14) holds for w but not for v. For an example, see the case examined in Section 4.3, that is the setup of Corollary 1.2 with $d \ge 2$. In that situation, one can use, when at $x \ne 0$, controls in a direction tangential to the sphere centered at the origin and passing through x, to conclude that $v(x,n) = 0$ if $|x| > 2\sqrt{d}\delta$ and $n \ge 0$. Thus, Assumption 2.6(i) does not hold for v.

3. Proof of Theorem 2.7

In this section we prove the local limit theorem for the value function w. We proceed by presenting some lemmas needed in the argument, beginning with some basic properties of the finite difference equation.

Recall that the equation satisfied by w is

$$w(x,n+1) = \sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho\left[w(x+X,n)\right]. \tag{3.1}$$

It can be written in the equivalent form

$$w(x,n+1) - w(x,n) + F\left[u(\cdot,n)\right](x) = 0 \tag{3.2}$$

which is an explicit finite difference scheme for the (continuum) parabolic equation

$$w_t + F(D^2 w) = 0. \tag{3.3}$$

We first record the fact that the scheme is in fact *consistent* with (3.3).

Lemma 3.1. *There exists $C(R) > 0$ such that, for every $\sigma \in (0,1]$, $\varphi \in C^{2,\sigma}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\left|F\left[\varphi\right](x) - F(D^2\varphi(x))\right| \le C\left[D^2\varphi\right]_{C^\sigma(B_1(x))}.$$

Proof. It is enough to consider the case $x = 0$. For $\epsilon > 0$ let $\rho^\epsilon, \rho_\epsilon$ be such that

$$\begin{aligned}
\sup_{\rho\in\mathbb{P}} \mathbb{E}_\rho[\varphi(X) - \varphi(0)] &\le \mathbb{E}_{\rho^\epsilon}[\varphi(X) - \varphi(0)] + \epsilon, \\
\sup_{\rho\in\mathbb{P}} \mathbb{E}_\rho[X\cdot D^2\varphi(0)X] &\le \mathbb{E}_{\rho_\epsilon}[X\cdot D^2\varphi(0)X] + \epsilon.
\end{aligned}$$

We have

$$\begin{aligned}
F\left[\varphi\right](0) - F(D^2\varphi(0)) &= \sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho\left[\varphi(X) - \varphi(0)\right] - \frac{1}{2}\sup_{\rho\in\mathcal{P}} \mathbb{E}_\rho\left[X\cdot D^2\varphi(0)X\right] \\
&\le \epsilon + \mathbb{E}_{\rho^\epsilon}\left[\varphi(X) - \varphi(0)\right] - \frac{1}{2}\mathbb{E}_{\rho^\epsilon}\left[X\cdot D^2\varphi(0)X\right] \\
&\le \epsilon + \sup_{\rho\in\mathcal{P}}\left(\mathbb{E}_\rho\left[\varphi(X) - \varphi(0)\right] - \frac{1}{2}\mathbb{E}_\rho\left[X\cdot D^2\varphi(0)X\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
F\left[\varphi\right](0) - F(D^2\varphi(0)) &\ge \mathbb{E}_{\rho_\epsilon}\left[\varphi(X) - \varphi(0)\right] - \frac{1}{2}\mathbb{E}_{\rho_\epsilon}\left[X\cdot D^2\varphi(0)X\right] - \frac{\epsilon}{2} \\
&\ge -\inf_{\rho\in\mathcal{P}}\left(\mathbb{E}_\rho\left[\varphi(X) - \varphi(0)\right] - \frac{1}{2}\mathbb{E}_\rho\left[X\cdot D^2\varphi(0)X\right]\right) - \frac{\epsilon}{2}.
\end{aligned}$$

Together with the arbitrariness of ϵ these imply

$$\left|F\left[\varphi\right](0) - F(D^2\varphi(0))\right| \le \sup_{\rho\in\mathcal{P}}\left|\mathbb{E}_\rho\left[\varphi(X) - \varphi(0) - \frac{1}{2}X\cdot D^2\varphi(0)X\right]\right|.$$

Using the centering condition and then Taylor's formula, we find that, for any $\rho \in \mathcal{P}$,

$$\begin{aligned}
&\left| \mathbb{E}_\rho \left[\varphi(X) - \varphi(0) - \frac{1}{2} X \cdot D^2\varphi(0) X \right] \right| \\
&\qquad \leq \mathbb{E}_\rho \left[\left| \varphi(X) - \varphi(0) - X \cdot D\varphi(0) - \frac{1}{2} X \cdot D^2\varphi(0) X \right| \right] \\
&\qquad \leq \sup_{y \in \overline{B}_R} \left| \varphi(y) - \varphi(0) - y \cdot D\varphi(0) - \frac{1}{2} y \cdot D^2\varphi(0) y \right| \\
&\qquad \leq C \left[D^2 \varphi \right]_{C^\sigma(B_1)}.
\end{aligned}$$

Note that we used both (2.1) and (2.2) in the third line and then Taylor's formula in the last line above. ☐

We next check that the finite difference scheme (3.2) is *monotone*, i.e., that it satisfies a comparison principle.

Lemma 3.2. *Assume $u, v : \mathbb{R}^d \to \mathbb{R}$ are locally bounded and satisfy, for each $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,*

$$\begin{cases} u(x, n+1) - u(x, n) + F\left[u(\cdot, n)\right](x) \leq 0, \\ v(x, n+1) - v(x, n) + F\left[v(\cdot, n)\right](x) \geq 0, \\ u(x, 0) \leq v(x, 0). \end{cases}$$

Then $u \leq v$ in $\mathbb{R}^d \times \mathbb{N}$.

Proof. Using the form (3.1) rather than (3.2), we observe that, for every $x \in \mathbb{R}^d$,

$$u(x, 1) \leq \sup_{\rho \in \mathcal{P}} \mathbb{E}_\rho \left[u(x + X, 0) \right] \leq \sup_{\rho \in \mathcal{P}} \mathbb{E}_\rho \left[v(x + X, 0) \right] \leq v(x, 1).$$

The lemma now follows by induction. ☐

The proof of Theorem 2.7 requires a test function calculation, similar to the one in [2, Lemma 4.4]. The result is summarized in the following lemma.

Lemma 3.3. *Fix $\beta > 0$ and consider the function*

$$\Psi(x, t) := t^{-\beta} \exp \left(-\beta \left(1 + \frac{|x|^2}{t} \right)^{1/2} \right). \tag{3.4}$$

Then there exist $C(d, R, \beta) > 1$ and $c(d, R, \beta) > 0$ such that, for every $x \in \mathbb{R}^d$ and $t \geq C$,

$$\Psi(x, t+1) - \Psi(x, t) + F\left[\Psi(\cdot, t)\right](x) \geq c t^{-1} \Psi(x, t) \cdot \begin{cases} -C & \text{if } |x| \leq C\sqrt{t}, \\ \dfrac{|x|}{\sqrt{t}} & \text{if } |x| \geq C\sqrt{t}. \end{cases} \tag{3.5}$$

Proof. We split the computation into three steps: first we estimate $\partial_t \Psi + F^-(D^2\Psi)$ from below and in the last two steps we show by approximation that this cannot be too much different from the finite difference scheme. Throughout, C and c denote positive constants which depend only on (d, R, β) and may vary in each occurrence.

Step 1. We estimate $\partial_t \Psi + F(D^2\Psi)$ from below. We compute

$$\partial_t \Psi(x,t) = -\beta t^{-1}\Psi(x,t)\left(1 - \left(1+\frac{|x|^2}{t}\right)^{-1/2}\frac{|x|^2}{2t}\right), \tag{3.6}$$

$$D\Psi(x,t) = -t^{-1/2}\Psi(x,t)\left(\beta\left(1+\frac{|x|^2}{t}\right)^{-1/2}\right)\frac{x}{\sqrt{t}} \tag{3.7}$$

and

$$\begin{aligned} D^2\Psi(x,t) = -\beta t^{-1}\Psi(x,t)\Bigg(&\left(1+\frac{|x|^2}{t}\right)^{-1/2} I_d - \beta\left(1+\frac{|x|^2}{t}\right)^{-1}\frac{x\otimes x}{t} \\ &- \left(1+\frac{|x|^2}{t}\right)^{-3/2}\frac{x\otimes x}{t}\Bigg). \end{aligned} \tag{3.8}$$

Using that

$$\frac{|x|^2}{t} I \geq \frac{x\otimes x}{t},$$

we may discard the first and third terms in parentheses to obtain

$$D^2\Psi(x,t) \leq \beta^2 t^{-1}\left(\left(1+\frac{|x|^2}{t}\right)^{-1}\frac{x\otimes x}{t}\right)\Psi(x,t).$$

Inserting this expression into the operator F and using (2.7) and (2.8), we obtain

$$F(D^2\Psi(x,t)) \geq -\beta^2 t^{-1}\left(1+\frac{|x|^2}{t}\right)^{-1}\frac{|x|^2}{t}\Psi(x,t) \geq -\beta^2 t^{-1}\Psi(x,t).$$

It follows that

$$\begin{aligned} \partial_t\Psi(x,t) + F(D^2\Psi(x,t)) &\geq \beta t^{-1}\Psi(x,t)\left(\left(1+\frac{|x|^2}{t}\right)^{-1/2}\frac{|x|^2}{2t} - (1+\beta)\right) \\ &\geq \beta t^{-1}\Psi(x,t)\left(\left(1+\frac{|x|^2}{t}\right)^{-1/2}\frac{|x|^2}{2t} - C\right) \\ &\geq ct^{-1}\Psi(x,t)\cdot\begin{cases} -C & \text{if } |x|\leq C\sqrt{t}, \\ \dfrac{|x|}{\sqrt{t}} & \text{if } |x|\geq C\sqrt{t}. \end{cases} \end{aligned}$$

Step 2. In preparation to evaluate Ψ on the finite difference scheme by comparing to Step 1, we estimate $|D^3\Psi|$ and $\partial_t^2\Psi$. The claims are: for all $x\in\mathbb{R}^d$ and $t\geq 1$,

$$\left|\partial_t^2\Psi(x,t)\right| \leq Ct^{-2}\left(1+\frac{|x|}{\sqrt{t}}\right)\Psi(x,t) \tag{3.9}$$

and

$$\left|D^3\Psi(x,t)\right| \leq Ct^{-1}\Psi(x,t) + Ct^{-3/2}\left(1+\frac{|x|}{\sqrt{t}}\right)\Psi(x,t). \tag{3.10}$$

Differentiating (3.6) yields

$$\partial_t^2\Psi(x,t) = \beta t^{-2}\Psi(x,t)\left(1+\left(1+\frac{|x|^2}{t}\right)^{-1/2}\frac{|x|^2}{t}+\left(1+\frac{|x|^2}{t}\right)^{-3/2}\frac{|x|^4}{t^2}\right.$$
$$\left.+\beta\left(1-\left(1+\frac{|x|^2}{t}\right)^{-1/2}\frac{|x|^2}{t}\right)\right),$$

from which we get (3.9). To prove (3.10), we must differentiate (3.8). Define $M(x,t)$ to be the matrix in the parentheses in (3.8), so that

$$|D^3\Psi(x,t)| \le Ct^{-1}\Psi(x,t)|DM(x,t)| + Ct^{-1}|D\Psi(x,t)||M(x,t)|.$$

It is easy to check that, for $x\in\mathbb{R}^d$ and $t\ge 1$,

$$|M(x,t)| + |DM(x,t)| \le C.$$

Using this and (3.7), we obtain (3.10).

Step 3. We evaluate Ψ on the finite difference scheme. From (3.9) we have, for every $(x,t)\in\mathbb{R}^d\times(1,\infty)$,

$$\Psi(x,t+1)-\Psi(x,t) \ge \partial_t\Psi(x,t) - Ct^{-2}\left(1+\frac{|x|}{\sqrt{t}}\right)\Psi(x,t)$$

and, by Lemma 3.1 and (3.10),

$$F\left[\Psi(\cdot,t)\right](x) \ge F(D^2\Psi(x,t)) - Ct^{-1}\Psi(x,t) - Ct^{-3/2}\left(1+\frac{|x|}{\sqrt{t}}\right)\Psi(x,t).$$

Putting these together, we finally obtain that, for every $x\in\mathbb{R}^d$ and $t\ge C$

$$\begin{aligned}
&\Psi(x,t+1)-\Psi(x,t)+F\left[\Psi(\cdot,t)\right](x)\\
&\qquad \ge \partial_t\Psi(x,t)+F(D^2\Psi(x,t)) - Ct^{-1}\Psi(x,t) - Ct^{-3/2}\Psi(x,t)\frac{|x|}{\sqrt{t}}\\
&\qquad \ge ct^{-1}\Psi(x,t)\cdot\begin{cases} -C & \text{if } |x|\le C\sqrt{t},\\ \dfrac{|x|}{\sqrt{t}} & \text{if } |x|\ge C\sqrt{t}.\end{cases}
\end{aligned}$$

This is (3.5). □

To prepare for the proof of Theorem 2.7, we must perform a second computation to show that, up to a suitable error, Φ is a solution of the finite difference equation. In fact, we bend Φ slightly in order to make it a strict subsolution or supersolution of (3.2) in the region $|x|\lesssim\sqrt{t}$. This computation is summarized in the following two lemmas.

Lemma 3.4. *Assume Assumption 2.5 holds. For each $\theta>0$, define*

$$\Phi_\theta(x,t) := \exp\left(\frac{1}{\theta}t^{-\theta}\right)\Phi(x,t).$$

Then Φ_θ satisfies, for some $C(d,R,\theta,\sigma,a,\alpha)>1$,

$$\begin{aligned}(3.11)\quad & \Phi_\theta(x,t+1)-\Phi_\theta(x,t)+F\left[\Phi_\theta(\cdot,t)\right](x) \\ & \qquad \leq -t^{-1-\theta}\Phi(x,t)+Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{2t}\right) \quad \text{in } \mathbb{R}^d\times(0,\infty).\end{aligned}$$

Lemma 3.5. *Assume Assumption 2.5 holds. For each $\theta>0$, define*

$$\Phi_{-\theta}(x,t):=\exp\left(-\frac{1}{\theta}t^{-\theta}\right)\Phi(x,t).$$

Then $\Phi_{-\theta}$ satisfies, for some $C(d,R,\theta,\sigma,a,\alpha)>1$,

$$\begin{aligned}(3.12)\quad & \Phi_{-\theta}(x,t+1)-\Phi_{-\theta}(x,t)+F\left[\Phi_{-\theta}(\cdot,t)\right](x) \\ & \qquad \geq t^{-1-\theta}\Phi(x,t)-Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{2t}\right) \quad \text{in } \mathbb{R}^d\times(0,\infty).\end{aligned}$$

Proof of Lemma 3.4. Similar to the proof of Lemma 3.3, we first insert Φ_θ into the continuum equation, estimate this from above, and then transfer the estimate by approximation to the finite difference equation. Throughout, C and c denote positive constants which may vary in each occurrence and depend only on $(d,R,\theta,\sigma,a,\alpha)$.

Step 1. We evaluate $\partial_t\Phi_\theta+F(D^2\Phi_\theta)$. We compute:

$$\partial_t\Phi_\theta(x,t)=\exp\left(\frac{1}{\theta}t^{-\theta}\right)\left(-t^{-1-\theta}\Phi(x,t)+\partial_t\Phi(x,t)\right) \tag{3.13}$$

and

$$D^2\Phi_\theta=\exp\left(\frac{1}{\theta}t^{-\theta}\right)D^2\Phi(x,t).$$

Using (2.7), we find, for every $x\in\mathbb{R}^d$ and $t>0$,

$$\partial_t\Phi_\theta(x,t)+F(D^2\Phi_\theta(x,t))=-\exp\left(\frac{1}{\theta}t^{-\theta}\right)t^{-1-\theta}\Phi(x,t)\leq -t^{-1-\theta}\Phi(x,t). \tag{3.14}$$

Step 2. We estimate the quantity $[D^2\Phi_\theta(\cdot,t)]_{C^{0,\sigma}B_R(x))}$. The assumptions (2.11) and (2.12) imply that, for every $(x,t)\in\mathbb{R}^d\times(0,\infty)$,

$$\begin{aligned}\left[D^2\Phi(\cdot,t)\right]_{C^{0,\sigma}(B_R(x))} &= t^{-1-\alpha-\sigma/2}\left[D^2\Phi(\cdot,1)\right]_{C^{0,\sigma}(B_R(x/\sqrt{t}))} \\ &\leq Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{t}\right)\end{aligned}$$

and therefore, for every $(x,t)\in\mathbb{R}^d\times(1,\infty)$,

$$\begin{aligned}\left[D^2\Phi_\theta(\cdot,t)\right]_{C^{0,\sigma}(B_R(x))} &= \exp\left(\frac{1}{\theta}t^{-\theta}\right)\left[D^2\Phi(\cdot,t)\right]_{C^{0,\sigma}(B_R(x))} \\ &\leq Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{t}\right).\end{aligned} \tag{3.15}$$

Step 3. We estimate the quantity $|\partial_t^2\Phi_\theta|$. The claim is:

$$\partial_t^2\Phi_\theta(x,t)\leq Ct^{-2-\alpha}\exp\left(-\frac{a|x|^2}{t}\right). \tag{3.16}$$

It is convenient to use self-similarity (2.11) to relate the time differences to spatial ones, in view of the assumption (2.13). First, differentiating the self-similarity relation yields

$$\partial_t \Phi(x,t) = -\frac{1}{2} t^{-1} x \cdot D\Phi(x,t) - \alpha t^{-1} \Phi(x,t)$$

and

$$\partial_t^2 \Phi(x,t) = \alpha(\alpha+1) t^{-2} \Phi(x,t) + \left(\alpha + \frac{3}{4} \right) t^{-2} x \cdot D\Phi(x,t) + \frac{1}{4} t^{-2} x \cdot D^2 \Phi(x,t) x.$$

Using (2.11) again and then (2.13), we estimate

$$\begin{aligned} |\partial_t \Phi(x,t)| &\leq C t^{-1-\alpha} \left(\Phi\left(\tfrac{x}{\sqrt{t}}, 1 \right) + \frac{|x|}{\sqrt{t}} \left| D\Phi\left(\tfrac{x}{\sqrt{t}}, 1 \right) \right| \right) \\ &\leq C t^{-1-\alpha} \left(1 + \frac{|x|}{\sqrt{t}} \right) \exp\left(-\frac{a|x|^2}{t} \right) \\ &\leq C t^{-1-\alpha} \exp\left(-\frac{a|x|^2}{2t} \right) \end{aligned}$$

and

$$\begin{aligned} \left| \partial_t^2 \Phi(x,t) \right| &\leq C t^{-2-\alpha} \left(\Phi\left(\tfrac{x}{\sqrt{t}}, 1 \right) + \frac{|x|}{\sqrt{t}} \left| D\Phi\left(\tfrac{x}{\sqrt{t}}, 1 \right) \right| + \frac{|x|^2}{t} \left| D^2\Phi\left(\tfrac{x}{\sqrt{t}}, 1 \right) \right| \right) \\ &\leq C t^{-2-\alpha} \left(1 + \frac{|x|^2}{t} \right) \exp\left(-\frac{a|x|^2}{t} \right) \\ &\leq C t^{-2-\alpha} \exp\left(-\frac{a|x|^2}{2t} \right). \end{aligned}$$

In view of the fact that

$$\partial_t^2 \Phi_\theta(x,t) = \exp\left(\frac{1}{\theta} t^{-\theta} \right) \left(\partial_t^2 \Phi(x,t) - 2t^{-1-\theta} \partial_t \Phi(x,t) + (1+\theta) t^{-2-\theta} \Phi(x,t) \right),$$

we obtain

$$\begin{aligned} \partial_t^2 \Phi_\theta(x,t) &\leq C \left(\left| \partial_t^2 \Phi(x,t) \right| + t^{-1-\theta} \left| \partial_t \Phi(x,t) \right| + t^{-2-\theta} \Phi(x,t) \right) \\ &\leq C t^{-2-\alpha} \exp\left(-\frac{a|x|^2}{t} \right). \end{aligned}$$

This is (3.16).

Step 4. We complete the proof using Lemma 3.1 combined with (3.14), (3.15) and (3.16). We have:

$$\begin{aligned} &\Phi_\theta(x,t+1) - \Phi_\theta(x,t) + F\left[\Phi_\theta(\cdot,t) \right](x) \\ &\qquad \leq \partial_t \Phi(x,t) + F\left(D^2 \Phi(x,t) \right) - C t^{-2-\alpha} \exp\left(-\frac{a|x|^2}{2t} \right) \\ &\qquad\qquad - C t^{-1-\alpha-\sigma/2} \exp\left(-\frac{a|x|^2}{t} \right) \\ &\qquad \leq -t^{-1-\theta} \Phi(x,t) + C t^{-1-\alpha-\sigma/2} \exp\left(-\frac{a|x|^2}{2t} \right), \end{aligned}$$

as desired. □

Proof of Lemma 3.5. The proof is essentially the same as that of Lemma 3.4, with only minor modifications coming from the change in sign of θ in the definition of $\Phi_{-\theta}$. The details are omitted. □

Using the above Lemmas 3.4 and 3.5 together with Assumptions 2.5 and 2.6, we now give the proof of Theorem 2.7.

Proof of (2.17) *(lower bound).* The proof is based on the fact that after a long time, and for appropriate choices of the parameters, the function

$$\zeta(x,t) := \Phi_\theta(x,t) - s\Psi(x,t) \tag{3.17}$$

is a subsolution of the finite difference equation after a large time. Here Ψ and Φ_θ are as in Lemma 3.3 and 3.4, respectively, and $s > 1$ is a large constant to be selected below. Once we show this, the lower bound (2.17) follows easily from Lemma 3.2.

Throughout the proof, C and c denote positive constants which may vary in each occurrence and depend only on $(d, R, \sigma, a, \alpha, \delta)$. When constants depend on other parameters, we will denote it in the notation, e.g. a constant depending on the above parameters *and* on r will be denoted $C(r)$.

We first fix the parameters (with the exception of s). With $\alpha > 0$ and $\sigma > 0$ as in Assumption 2.5, we first select $\theta > 0$ in the definition of Φ_θ such that

$$0 < \theta < \frac{\sigma}{2}. \tag{3.18}$$

We then take $\beta > 0$ in the definition of Ψ to satisfy

$$\alpha + \theta < \beta < \alpha + \frac{\sigma}{2}. \tag{3.19}$$

Step 1. We show that there exists $T(s) > 1$ such that ζ defined as in (3.17) satisfies, for every $x \in \mathbb{R}^d$ and $t \geq T$,

$$\zeta(x,t+1) - \zeta(x,t) + F\left[\zeta(\cdot,t)\right](x) \leq 0. \tag{3.20}$$

According to (2.7) and (2.8), for every $t > 0$,

$$F\left[\zeta(\cdot,t)\right](x) \leq F\left[\Phi_\theta(\cdot,t)\right](x) - sF\left[\Psi(\cdot,t)\right](x).$$

Then according to (3.5), (3.11) and Assumption 2.5,

$$\begin{aligned} &\zeta(x,t+1) - \zeta(x,t) + F\left[\zeta(\cdot,t)\right](x) \\ &\qquad \leq -t^{-1-\alpha-\theta}\Phi\left(\tfrac{x}{\sqrt{t}},1\right) + Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{2t}\right) \\ &\qquad\quad - cst^{-1-\beta}\Psi\left(\tfrac{x}{\sqrt{t}},1\right)\cdot\begin{cases} -C & \text{if } |x| \leq C\sqrt{t}, \\ \frac{|x|}{\sqrt{t}} & \text{if } |x| \geq C\sqrt{t}. \end{cases} \end{aligned} \tag{3.21}$$

Now we simply note that if t is large enough, then by (3.18) and (3.19) we have, for every $|y| \leq C$,

$$t^{-1-\alpha-\theta}\Phi(y,1) \geq Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|y|^2}{2}\right) + Cst^{-1-\beta}\Psi\left(y,1\right), \tag{3.22}$$

while on the other hand, for every $|y| \geq C$, the condition (3.19) and the fact that $\Psi(\cdot, 1)$ has fatter tails than a Gaussian ensures that, for large enough t,

$$|y|t^{-1-\beta}\Psi(y,1) \geq Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|y|^2}{2}\right). \tag{3.23}$$

These inequalities imply that, for large enough t, the right side of (3.21) is nonpositive in $\mathbb{R}^d$, as claimed. In terms of s, we see that it suffices to take

$$T(s) := Cs^{1/(\beta-\alpha-\theta)}, \tag{3.24}$$

where C, according to our convention, is a constant that depends on $(d, R, \sigma, a, \alpha)$ only (in particular, it does not depend on s).

Step 2. We complete the proof of the lower bound. Denote $N := \lceil T \rceil$, and observe that, in view of (3.24),

$$\begin{aligned}
\{y \in \mathbb{R}^d \,:\, \zeta(y,N) \geq 0\} &= \left\{y \in \mathbb{R}^d \,:\, \Phi_\theta(y,N) \geq s\Psi(y,N)\right\} \\
&\subseteq \left\{y \in \mathbb{R}^d \,:\, N^{-\alpha}\exp\left(-\frac{a|y|^2}{N}\right) \geq csN^{-\beta}\exp\left(-\frac{\beta|y|}{\sqrt{N}}\right)\right\} \\
&\subseteq \left\{y \in \mathbb{R}^d \,:\, \exp\left(-\frac{a|y|^2}{N}\right) \geq cN^{-\theta}\exp\left(-\frac{\beta|y|}{\sqrt{N}}\right)\right\} \\
&\subseteq \left\{y \in \mathbb{R}^d \,:\, |y| \leq C\sqrt{N\log N}\right\}.
\end{aligned}$$

(In the second inclusion, we used that $s \geq N^{\beta-\alpha-\theta}$.)

Take r as equal to C of the last display (recall that C does not depend on s). Let $N_0(r)$ be as in part (i) of Assumption 2.6, and choose s large enough so that $N > N_0(r)$. Note that, given the function $N_0(\cdot)$, $s = s(d, R, \sigma, a, \alpha)$, and thus $N = N(d, R, \sigma, a, \alpha, \delta)$. Then,

$$\inf\left\{w(y,N) \,:\, y \in \mathbb{R}^d,\ \zeta(y,N) \geq 0\right\} > 0.$$

Since $\sup_{\mathbb{R}^d}\zeta(\cdot,N) \leq C$ and $w \geq 0$ in $\mathbb{R}^d$, we obtain

$$\zeta(\cdot,N) \leq Cw(\cdot,N) \quad \text{in } \mathbb{R}^d.$$

According to (3.20) and Lemma 3.2,

$$\zeta(x,n) \leq Cw(x,n) \quad \text{for every } x \in \mathbb{R}^d,\ n \geq N.$$

The lower bound in (2.17) now follows, since

$$\begin{aligned}
\inf_{r>0}\liminf_{t\to\infty}\inf_{|x|\leq r\sqrt{t}}\frac{\zeta(x,t)}{\Phi(x,t)} &\geq \inf_{r>0}\liminf_{t\to\infty}\inf_{|x|\leq r\sqrt{t}}\frac{\Phi_\theta(x,t)}{\Phi(x,t)} - s\sup_{r>0}\limsup_{t\to\infty}\sup_{|x|\leq r\sqrt{t}}\frac{\Psi(x,t)}{\Phi(x,t)} \\
&\geq \liminf_{t\to\infty}\exp\left(\frac{1}{\theta}t^{-\theta}\right) \\
&\qquad - s\sup_{r>0}\limsup_{t\to\infty}t^{\alpha-\beta}\left(\sup_{|y|\leq r}\Psi(y,1)\right)\left(\inf_{|y|\leq r}\Phi(y,1)\right)^{-1} \\
&= 1.
\end{aligned}$$

Note that we used that $\beta > \alpha$, from (3.19). $\square$

Proof of (2.16) *(upper bound).* The proof is similar to (and even somewhat easier than) that of the lower bound. Instead of (3.17) we use the function

$$\xi(x,t) := \Phi_{-\theta}(x,t) + \Psi(x,t), \tag{3.25}$$

where Ψ and $\Phi_{-\theta}$ are as in Lemma 3.3 and 3.5, respectively. The goal is to show that ξ is a supersolution of the finite difference equation after a large time. Then we apply Lemma 3.2 to conclude, as above. The choices of the parameters θ and β as well as the convention for the constants C and c are the same as in the proof of the lower bound.

Step 1. We show that there exists $T > 1$ such that ξ satisfies, for every $x \in \mathbb{R}^d$ and $t \geq T$,

$$\xi(x,t+1) - \xi(x,t) + F\left[\xi(\cdot,t)\right](x) \geq 0. \tag{3.26}$$

Using (2.7) and (2.8) we find that, for every $t > 0$,

$$F\left[\xi(\cdot,t)\right](x) \geq F\left[\Phi_{-\theta}(\cdot,t)\right](x) + F\left[\Psi(\cdot,t)\right](x).$$

By (3.5) and (3.12),

$$\begin{aligned}
&\xi(x,t+1) - \xi(x,t) + F\left[\xi(\cdot,t)\right](x) \\
&\quad \geq t^{-1-\alpha-\theta}\Phi\left(\tfrac{x}{\sqrt{t}},1\right) - Ct^{-1-\alpha-\sigma/2}\exp\left(-\frac{a|x|^2}{2t}\right) \\
&\qquad + ct^{-1-\beta}\Psi\left(\tfrac{x}{\sqrt{t}},1\right)\cdot\begin{cases} -C & \text{if } |x| \leq C\sqrt{t}, \\ \frac{|x|}{\sqrt{t}} & \text{if } |x| \geq C\sqrt{t}. \end{cases}
\end{aligned}$$

By the choice of parameters, i.e., (3.18) and (3.19), we have for sufficiently large t that (3.22) holds for every $|y| \leq C$ and (3.23) holds for every $|y| \geq C$. Together these yield the claim.

Step 2. We complete the proof of the upper bound. Select T, s, r as in Step 1. By the definition of Ψ and Assumption 2.6, we have

$$w(x,N) \leq C\exp\left(-\beta\frac{|x|}{\sqrt{N}}\right) \leq C\Psi(x,N) \leq C\xi(x,N).$$

(As T depends only on (d,R,σ,a,α), the constant C, which depends on T, satisfies our convention for dependence on parameters.) By (3.26) and Lemma 3.2,

$$w(x,n) \leq C\xi(x,n) \quad \text{for every } x \in \mathbb{R}^d,\ n \geq N.$$

We thus conclude the proof of the upper bound by observing that

$$\begin{aligned}
\sup_{r>0}\limsup_{t\to\infty}\sup_{|x|\leq r\sqrt{t}}\frac{\xi(x,t)}{\Phi(x,t)} &\leq \sup_{r>0}\limsup_{t\to\infty}\sup_{|x|\leq r\sqrt{t}}\frac{\Phi_\theta(x,t)}{\Phi(x,t)} + \sup_{r>0}\limsup_{t\to\infty}\sup_{|x|\leq r\sqrt{t}}\frac{\Psi(x,t)}{\Phi(x,t)} \\
&\leq \limsup_{t\to\infty}\exp\left(-\frac{1}{\theta}t^{-\theta}\right) \\
&\qquad + \sup_{r>0}\limsup_{t\to\infty}t^{\alpha-\beta}\left(\sup_{|y|\leq r}\Psi(y,1)\right)\left(\inf_{|y|\leq r}\Phi(y,1)\right)^{-1} \\
&= 1.
\end{aligned}$$

We note once again that we used that $\beta > \alpha$, by (3.19). $\square$

The proof of Theorem 2.7 is now complete.

4. Existence of self-similar profiles

In this section, we show that Assumptions 2.5 and 2.6 hold for a wide class of examples and we indicate methods for computing (or at least estimating) α.

4.1. **A nonlinear principal eigenvalue problem.** In our search for Φ, we may use the self-similarity relation to reformulate the parabolic equation as an elliptic one. By differentiating (2.11), we have

$$\partial_t \Phi(x,t) = -\frac{1}{2} t^{-1} x \cdot D\Phi(x,t) - \alpha t^{-1} \Phi(x,t)$$

and substituting this into (2.9) yields

$$F(D^2\Phi(x,t)) - \frac{1}{2} t^{-1} x \cdot D\Phi(x,t) = \alpha t^{-1}\Phi(x,t) \quad \text{in } \mathbb{R}^d \times (0,\infty).$$

Using (2.7) and (2.11) again to change to the variable $y = x/\sqrt{t}$, we may eliminate the time variable. We get

$$F(D^2\Phi(y,1)) - \frac{1}{2} y \cdot D\Phi(y,1) = \alpha \Phi(y,1) \quad \text{in } \mathbb{R}^d. \tag{4.1}$$

This is a principal eigenvalue problem: the unknowns α and $\Phi(\cdot,1)$ are the principal eigenpair. If we can solve it, then we may recover the full function Φ via the self-similarity relation. While the domain $\mathbb{R}^d$ is unbounded, the drift term makes the problem well-posed in the uniformly elliptic setting (see [2]). We investigate this in more detail in the next subsection.

4.2. **Uniformly elliptic martingales: the proof of Corollary 1.1.** In this subsection we verify that Assumptions 2.5 and 2.6 hold for both F^- and F^+ in the uniformly elliptic case, that is, under the additional hypothesis

$$\mathcal{P} \subseteq \mathcal{E}_{\lambda,R}(\mathbb{R}^d) \quad \text{for some } \lambda > 0,\ R \geq \sqrt{2d}. \tag{4.2}$$

We use [2, Theorem 1.1] and the Evans-Krylov theorem [4, Theorem 6.6] to show that Assumption 2.5 holds, while we verify Assumption 2.6(i) by a pathwise construction.

With $F = F^+$ or $F = F^-$, the results of [2] imply the existence of $\alpha > 0$ and $\Phi \in C(\mathbb{R}^d \times (0,\infty))$ satisfying (2.9) in the weak viscosity sense as well as (2.10) and (2.11). It is also proved that Φ is unique, provided we impose the normalization $\Phi(0,1) = 1$, and that the function $\Phi(\cdot,1)$ satisfies (4.1) and, for some constants $K_0(d,\lambda) > 1$ and $a(d,\lambda) > 0$,

$$|\Phi(x,1)| \leq K_0 \exp\left(-2a|x|^2\right). \tag{4.3}$$

To check Assumption 2.5, we have left to show that the Evans-Krylov theorem and (4.3) imply the stronger bound (2.13). This is handled in the following lemma.

Lemma 4.1. *Let F, λ, α, Φ, K_0 and $a > 0$ be as above. Then there exist $\sigma(d,\lambda) \in (0,1]$ and $C(d,\lambda,\alpha,K_0,a) > 0$ such that, for every $x \in \mathbb{R}^d$,*

$$\|\Phi(\cdot,1)\|_{C^{2,\sigma}(B_1(x))} \leq C \exp\left(-a|x|^2\right).$$

Proof. Throughout the argument, C denotes a positive constant depending only on $(d, \lambda, \alpha, K_0, a)$ which may vary in each occurrence. As mentioned above, the function $\varphi(x) := \Phi(x, 1)$ satisfies the equation

$$F(D^2\varphi) - \frac{1}{2}x \cdot D\varphi = \alpha\varphi \quad \text{in } \mathbb{R}^d. \tag{4.4}$$

For the moment, we must interpret (4.4) in the weak viscosity sense (as defined in [4]), although we will see shortly that φ is C^2 and therefore (4.4) can be understood in the classical sense.

The equation (4.4) possesses a local length scale arising from the competition between the gradient term and the diffusive term. Since the gradient term is stronger for larger $|x|$, in order to apply local elliptic regularity estimates to φ near $x \in \mathbb{R}^d$, it is natural to rescale the equation in some way which depends on $|x|$. We perform the rescaling by introducing the variable $y = x/r$ and denoting $\varphi_r(y) := \varphi(x)$, where $0 < r \leq 1$. In terms of φ_r, the equation (4.4) takes the form

$$F(D^2\varphi_r) - \frac{1}{2}r^2 y \cdot D\varphi = \alpha r^2 \varphi_r \quad \text{in } \mathbb{R}^d. \tag{4.5}$$

Notice that the first-order coefficient is uniformly bounded and smooth for every $|y| \lesssim r^{-2}$. The interior gradient Hölder estimate for uniformly elliptic equations therefore implies that $\varphi_r \in C^{1,\sigma}(B_4(y_0))$ for some $\sigma(d, \lambda) \in (0, 1]$ and, for every $0 < r \leq 1$ and $y_0 \in \mathbb{R}^d$ with $|y_0| \leq r^{-2}$,

$$\|\varphi_r\|_{C^{1,\sigma}(B_4(y_0))} \leq C\left(1 + \alpha r^2\right)\|\varphi_r\|_{L^\infty(B_8(y_0))}. \tag{4.6}$$

The standard reference for this estimate for equations with no gradient dependence is [4, Theorem 8.3], and the argument there can be adapted in a straightforward manner to handle equations with gradient dependence and, in particular, (4.5). Alternatively, we refer to [8] for a statement with hypotheses covering our case.

We may now re-express (4.5) as

$$F(D^2\varphi_r) = f \quad \text{in } \mathbb{R}^d$$

where, in view of (4.6), the function $f(y) := \frac{1}{2}r^2 y \cdot D\varphi(y) + \alpha r^2 \varphi_r(y)$ satisfies, for each $0 < r \leq 1$ and $|y_0| \leq r^{-2}$,

$$\|f\|_{C^{\sigma}(B_4(y_0))} \leq C\left(1 + \alpha r^2\right)\|\varphi_r\|_{L^\infty(B_8(y_0))}.$$

As F is either convex or concave, the Evans-Krylov theorem (c.f. [4, Theorem 6.6]) yields (after redefining $\sigma(d, \lambda)$ to be smaller, if necessary) that $\varphi_r \in C^{2,\alpha}(B_2(y_0))$ and gives the estimate

$$\begin{aligned} \|\varphi_r\|_{C^{2,\sigma}(B_2(y_0))} &\leq C\left(\|\varphi_r\|_{L^\infty(B_4(y_0))} + \|f\|_{C^\sigma(B_4(y_0))}\right) \\ &\leq C\left(1 + \alpha r^2\right)\|\varphi_r\|_{L^\infty(B_8(y_0))}. \end{aligned} \tag{4.7}$$

We now reverse the scaling to express (4.7) in terms of φ. For a fixed $x_0 \in \mathbb{R}^d$, set $r := \frac{1}{4}(1 + |x_0|)^{-1}$ and $y_0 := x_0/r$. Note that $|y_0| < r^{-2}$ and that we have

$$\|\varphi\|_{L^\infty(B_{2r}(x_0))} = \|\varphi_r\|_{L^\infty(B_2(y_0))} \quad \text{and} \quad \|\varphi\|_{C^{2,\sigma}(B_{2r}(x_0))} \leq r^{-2-\sigma}\|\varphi_r\|_{C^{2,\sigma}(B_2(y_0))}.$$

We therefore obtain from (4.7) that

$$\|\varphi\|_{C^{2,\sigma}(B_{2r}(x_0))} \leq Cr^{-2-\sigma}\left(1 + \alpha r^2\right)\|\varphi\|_{L^\infty(B_{2r}(x_0))}.$$

According to [2], the exponent α depends only on (d,λ). Applying (4.3) and using the fact that $B_{8r}(x_0) \subseteq \mathbb{R}^d \setminus B_{3|x_0|/4}$ for $|x_0| > 4$, we obtain

$$\|\varphi\|_{C^{2,\sigma}(B_{2r}(x_0))} \leq C\left(1+|x_0|\right)^{2+\sigma} \exp\left(-2a\cdot\frac{9}{16}|x_0|^2\right) \leq C\exp\left(-\frac{17}{16}a|x_0|^2\right).$$

The previous estimate implies

$$\|\varphi\|_{C^{2,\sigma}(B_1(x_0))} \leq C\exp\left(-a|x_0|^2\right). \qquad \square$$

Checking Assumption 2.6 in the uniformly elliptic case is straight forward. As we previously mentioned, the upper bound (2.15) follows from Azuma's inequality. To check (2.14), it is enough to consider a simple random walk. That is, we take $\{\Delta_i\}_{i\geq 0}$ as a sequence of i.i.d. random variables so that $\mathbb{P}(\Delta_0 = \pm\sqrt{2d}e_i) = 1/2d$ for e_i the standard unit vectors in $\mathbb{R}^d$. Let y_x denote an element of $\sqrt{2d}\mathbb{Z}^d$ with minimal norm $|x-y_x|$. Then, with $|y_x|_1$ denoting the ℓ^1 norm of $y_x/\sqrt{2d}$, we have, for every $n > |y_x|_1$,

$$\mathbb{P}(|M_n - y_x| \leq \sqrt{2}d) \geq \left(\frac{1}{2d}\right)^n.$$

This immediately implies (2.14). This completes the proof of Corollary 1.1.

4.3. **The proof of Corollary 1.2.** In this subsection we consider the particular case of martingales whose increments are bounded with norm of constant second moment. That is, for some $\lambda \in (0,1]$, we set

$$\mathcal{P} := \left\{\rho \in \mathcal{M}_1(\mathbb{R}^d) \,:\, \mathbb{E}_\rho\left[|X|^2\right] = \lambda\right\}.$$

We easily check that the operator F^- can be expressed by

$$F^-(M) = -\frac{1}{2}\lambda\cdot(\text{largest eigenvalue of } M).$$

By a direct computation, we find that $\varphi(x) := \exp\left(-\frac{1}{2\lambda}|x|^2\right)$ is an exact, smooth solution of

$$F^-(D^2\varphi(x)) - \frac{1}{2}x\cdot D\varphi(x) = \frac{1}{2}\varphi(x) \quad \text{in } \mathbb{R}^d.$$

In particular, Assumption 2.5 holds for $F = F^-$ with $\alpha = \frac{1}{2}$ and

$$\Phi(x,t) := t^{-\frac{1}{2}}\exp\left(-\frac{|x|^2}{2\lambda t}\right).$$

Azuma's inequality implies that Assumption 2.6(ii) holds. We therefore obtain Corollary 1.2 as a consequence of Theorem 2.7(i).

4.4. **Estimating the exponent α: the proof of (1.4) and (1.5).** As we have seen, finding the exponent α and self-similar profile Φ is equivalent to solving a nonlinear eigenvalue problem. This is of course difficult in general, both analytically and computationally. Even for particular examples like $\mathcal{P} = \mathcal{E}_{\lambda,R}$, in which case F^- and F^+ are the minimal and maximal Pucci operators, respectively, we do not believe it is possible to give a closed form expression for α or Φ (although for rotationally invariant operators like these, the problem can be reduced to an ODE in the radial variable, which greatly reduces its complexity).

Fortunately, it is more tractable to estimate α. This can be done by exhibiting explicit β for which there exist subsolutions and supersolutions of the equation

$$F(D^2g) - \frac{1}{2}x \cdot Dg = \beta g \quad \text{in } \mathbb{R}^d. \tag{4.8}$$

Let X denote the space

$$X := \left\{ g \in C^2(\mathbb{R}^d) \ : \ \text{there exists} \ \ a > 0 \ \text{ such that } 0 \leq g(x) \leq \exp\left(-a|x|^2\right) \right\}$$

and set $X_+ := X \cap \{g > 0\}$. The following formulas for α were proved in [2]:

$$\alpha = \sup \left\{ \beta > 0 \, : \, \exists g \in X_+ \ \text{ satisfying } \ F(D^2g) - \frac{1}{2}x \cdot Dg \geq \beta g \quad \text{in } \mathbb{R}^d \right\}. \tag{4.9}$$

and

$$\alpha = \inf \left\{ \beta > 0 \, : \, \exists g \in X_+ \ \text{ satisfying } \ F(D^2g) - \frac{1}{2}x \cdot Dg \leq \beta g \quad \text{in } \mathbb{R}^d \right\}. \tag{4.10}$$

This allows us to bound α from below (respectively, above) by exhibiting a supersolution (respectively, subsolution) of (4.8) with an explicit β.

In the case that $\mathcal{P} = \mathcal{E}_{\lambda,R}$, test functions were found in [2] that give the bounds

$$\frac{d\lambda}{2} \leq \alpha(F^-) \leq \frac{(d-1)\lambda}{2} + \frac{1}{2} \leq \frac{(d-1)}{2\lambda} + \frac{1}{2} \leq \alpha(F^+) \leq \frac{d}{2\lambda}, \tag{4.11}$$

where, for each of these inequalities, equality holds only if $\lambda = 1$. in particular, $\alpha(F^-) < \frac{d}{2} < \alpha(F^+)$ if $\lambda < 1$. A more general fact along these lines was shown in [2, Example 3.12], namely that if $\mathcal{P} \subseteq \mathcal{E}_{\lambda,R}$, then $\alpha(F^-) < \frac{d}{2} < \alpha(F^+)$ unless $\mathcal{P}$ is a singleton set, that is, unless the controller has no actual control and the martingale is just a simple random walk.

In the next lemma, we use (4.9) and (4.10) to prove the bounds (1.4) as promised in the introduction. To aid our computation, we remark that for $\mathcal{P} = \mathcal{E}_{\lambda,R}$, the operator F^- can be expressed for each $M \in \mathbb{S}^d$ as

$$\begin{aligned} F^-(M) = -\lambda \cdot &(\text{sum of the negative eigenvalues of } M) \\ &- (\text{sum of the positive eigenvalues of } M)\,. \end{aligned} \tag{4.12}$$

Lemma 4.2. *In the case that $\mathcal{P} = \mathcal{E}_{\lambda,R}$ and $F = F^-$, for every $\delta > 0$, there exist $C(d,\delta) > 1$ and $c(d,\delta) > 0$ such that*

$$c\lambda^{1/4+\delta} \leq \alpha \leq C\lambda^{1/4-\delta}.$$

Proof of Lemma 4.2, upper bound. Fix $p \in (0, 1/2)$ and parameters $a, b > 0$ to be selected below, and consider the test function

$$\varphi(x) := \exp\left(-\frac{1}{2}a|x|^2 - b\left(\lambda + |x|^2\right)^{p/2} \right). \tag{4.13}$$

According to (4.10), it suffices to show, for appropriate choices of a and b, that φ satisfies

$$F^-(D^2\varphi(x)) - \frac{1}{2}x \cdot D\varphi(x) \leq C\lambda^{p/2}\varphi \quad \text{in } \mathbb{R}^d. \tag{4.14}$$

Here and throughout the rest of the argument, C denotes a positive constant depending only on (p, d) which may vary in each occurrence.

Step 1. We compute the first two derivatives of φ and estimate $F^-(D^2\varphi(x))$ from above. We have

$$D\varphi(x) = -\varphi(x)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)x$$

and

$$\begin{aligned} D^2\varphi(x) = {} & \varphi(x)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)^2 x\otimes x \\ & + \varphi(x)\left(bp(2-p)\left(\lambda + |x|^2\right)^{p/2-2}\right)x\otimes x \\ & - \varphi(x)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)I. \end{aligned} \tag{4.15}$$

Discarding some of the terms coming from expanding the square on the first term on the right in the expression for $D^2\varphi(x)$ above, we find that

$$\begin{aligned} D^2\varphi(x) \geq M(x) := {} & \varphi(x)\left(a^2 + bp(2-p)\left(\lambda + |x|^2\right)^{p/2-2}\right)x\otimes x \\ & - \varphi(x)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)I. \end{aligned}$$

Observe that

$$\text{eigenvalues of } M(x) = \varphi(x)\cdot\begin{cases} E(x) & \text{with multiplicity } 1, \\ -a - bp\left(\lambda + |x|^2\right)^{p/2-1} & \text{with multiplicity } d-1. \end{cases}$$

where

$$\begin{aligned} E(x) & = a^2|x|^2 - a + bp(2-p)\left(\lambda + |x|^2\right)^{p/2-2}|x|^2 - bp\left(\lambda + |x|^2\right)^{p/2-1} \\ & = a^2|x|^2 - a + bp(1-p)\left(\lambda + |x|^2\right)^{p/2-1} - bp(2-p)\lambda\left(\lambda + |x|^2\right)^{p/2-2}. \end{aligned}$$

Using (4.12), we get

$$\begin{aligned} & F^-(D^2\varphi(x)) \leq F^-(M(x)) \\ & \quad = \varphi(x)\cdot\begin{cases} \lambda(d-1)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right) - \lambda E(x) & \text{if } E(x)\leq 0, \\ \lambda(d-1)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right) - E(x) & \text{if } E(x) > 0. \end{cases} \end{aligned} \tag{4.16}$$

Step 2. We study the set where $E(x)$ is positive. The claim is that, for $a \geq 1$, there exists $C > 1$ such that $b \geq C$ implies

$$E(x) \geq \frac{1}{2}a|x|^2 + \frac{1}{2}bp|x|^2\left(1 + \frac{|x|^2}{\lambda}\right)^{p/2-1} \geq 0 \quad \text{for every } |x| > C\sqrt{\lambda}, \tag{4.17}$$

This follows from the following three facts, each of which is easy to check:

$$|x|^2 \geq \frac{2}{a} \quad \implies \quad \frac{1}{2}a^2|x|^2 \geq a,$$

$$|x|^2 \leq \frac{2}{a} \ \text{ and } \ b \geq C \quad \implies \quad \frac{1}{4}bp(1-p)\left(\lambda + |x|^2\right)^{p/2-1} \geq a,$$

$$|x|^2 \geq C\lambda \quad \implies \quad \frac{1}{4}bp(1-p)\left(\lambda + |x|^2\right)^{p/2-1} \geq bp(2-p)\lambda\left(\lambda + |x|^2\right)^{p/2-2}.$$

Step 3. We check (4.14) for $|x| \geq C\sqrt{\lambda}$. Note that the estimate (4.17) says precisely that

$$E(x)\varphi(x) \geq -\frac{1}{2}x \cdot D\varphi(x) \quad \text{for every } |x| \geq C\sqrt{\lambda}.$$

This therefore allows us to absorb the gradient term on the left side of (4.14). Using (4.16) and taking now $a := 1$, we find that:

$$\begin{aligned} F^-(D^2\varphi(x)) - \frac{1}{2}x \cdot D\varphi(x) &\leq \varphi(x)\lambda(d-1)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right) \\ &\leq \varphi(x)\lambda(d-1)\left(a + bp\lambda^{p/2-1}\right) \\ &\leq C\lambda^{p/2}\varphi(x). \end{aligned}$$

Step 4. We check (4.14) in the set $|x| \leq C\sqrt{\lambda}$. Here we get the estimate (4.14) differently, since the gradient term does not hurt us:

$$-\frac{1}{2}x \cdot D\varphi(x) \leq \varphi(x) - \varphi(x)\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)|x|^2 \leq C\lambda^{p/2}\varphi(x).$$

A similar estimate yields, for $|x| \leq C\sqrt{\lambda}$,

$$-\lambda E(x)\varphi(x) \leq \lambda\left(a + bp\left(\lambda + |x|^2\right)^{p/2-1}\right)\varphi(x) \leq C\lambda^{p/2}\varphi(x).$$

Returning to (4.16), we obtain that, for $|x| \leq C\sqrt{\lambda}$,

$$F^-(D^2\varphi(x) - \frac{1}{2}x \cdot D\varphi(x) \leq C\lambda^{p/2}\varphi(x).$$

This completes the proof of (4.14). □

Proof of Lemma 4.2, lower bound. We use the same test function φ from the previous argument, except here we take $p \in (1/2, 1]$. The goal is to show that, for appropriate choices of the parameters a and b (here we take them to be very small, depending on p), we have the reverse of (4.14):

$$F^-(D^2\varphi(x)) - \frac{1}{2}x \cdot D\varphi(x) \geq c\lambda^{p/2}\varphi \quad \text{in } \mathbb{R}^d. \tag{4.18}$$

The analysis and computations involved are quite similar to those of the previous argument, and so we leave the details to the reader. □

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