BLOWUP FOR THE BOUSSINESQ EQUATIONS WITH SMOOTH FORCING

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Abstract. Following the multiscale program of Córdoba and Martínez-Zoroa, we construct finite-time blowup for the inviscid Boussinesq system on $\mathbb{R}^2$ with smooth forcing in both equations. The forces belong to $C^\infty(\mathbb{R}^2 \times [0, T_*])$ and are supported in one fixed spatial ball. The initial temperature is smooth and compactly supported, and the initial velocity is zero. The temperature remains bounded, while its gradient norm tends to infinity and the vorticity norm has infinite limsup as $t \uparrow T_*$, both in L^∞.

Contents

1. The problem and the construction

The inviscid Boussinesq equations govern the dynamics of a fluid of non-uniform density within a fixed, constant gravitational field. This work constructs solutions to the Boussinesq equations with smooth, compactly supported forcing terms and smooth initial data, which blow up in finite time in the following sense: the vorticity norm is unbounded along a sequence of preterminal times converging to the finite blow-up time, and the temperature anomaly stays bounded but develops an unbounded gradient. The velocity and pressure are smooth on every closed time interval strictly before blowup. The velocity satisfies the Biot–Savart law and the decay conditions specified below.

This paper develops the program initiated by Diego Córdoba and Luis Martínez-Zoroa, whose groundbreaking multiscale constructions provide its principal intellectual foundation. We believe that the primary intellectual credit for the underlying strategy belongs to them. The present work is a contribution within that program, with a particular debt to their construction for the incompressible porous media (IPM) equation [6] and to the Boussinesq construction of Córdoba, Andrés Laín-Sanclemente and Martínez-Zoroa [5]. We describe the inherited ideas and the adaptations below.

1.1. **The Boussinesq equation.** We consider the forced inviscid Boussinesq system

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = f_\theta, \\ \partial_t u + u\cdot\nabla u + \nabla p = \theta e_2 + f_u, \\ \operatorname{div} u = 0, \end{cases} \qquad \text{on } \mathbb{R}^2. \tag{1.1}$$

Here $e_2 = (0,1)$ is the second, upward canonical vector. The temperature anomaly θ is transported by the velocity field u. A positive temperature anomaly, interpreted as a lower density, produces upward buoyancy through θe_2, opposite to physical, downward gravity. The background hydrostatic contribution is absorbed into the pressure. The forcing terms f_θ, f_u will be chosen to construct the blowup example.

The Boussinesq equation can also be written in vorticity form. Set $J(x_1,x_2) = (-x_2,x_1)$, $\nabla^\perp = J\nabla$, and $\omega = \partial_1 u_2 - \partial_2 u_1$. The vorticity equation is

$$\partial_t\omega + u\cdot\nabla\omega = \partial_1\theta + \operatorname{curl} f_u. \tag{1.2}$$

Write $K = \nabla^\perp\Delta^{-1}$ for the planar Biot–Savart operator. Its Fourier convention and the point-reflection parity used throughout are specified in Subsection 1.5.

For $T>0$, let $\mathcal{Y}_T$ denote the localized smooth Biot–Savart class of pairs (ϑ, v) such that

$$\vartheta, v \in C^\infty(\mathbb{R}^2\times[0,T]), \qquad \vartheta, \operatorname{curl} v \in C^\infty\big([0,T]; C_c^\infty(\mathbb{R}^2)\big),$$
$$\operatorname{div} v = 0, \qquad v = K(\operatorname{curl} v),$$

and, for every integer $j\ge 0$ and spatial multi-index γ,

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^2}\langle x\rangle^{3+|\gamma|}\left|\partial_t^j\partial_x^\gamma v(x,t)\right| < \infty, \qquad \langle x\rangle = (1+|x|^2)^{1/2}. \tag{1.3}$$

Here the compact-support condition means that one compact spatial set contains the supports of ϑ and $\operatorname{curl} v$ throughout $[0,T]$; this set may depend on T.

For smooth compactly supported scalar and vorticity data in the parity class (1.9), with zero initial circulation and velocity given by the Biot–Savart law, the classical local existence and uniqueness theory [3] applies also with prescribed smooth compactly supported forces of the same parity. The resulting solution belongs to $\mathcal{Y}_T$ for a sufficiently small $T>0$. Indeed, bounded velocity and the transport equations preserve compact scalar and vorticity support locally in time. The vorticity equation preserves zero circulation, and evenness of vorticity gives its zero first moments:

$$\int_{\mathbb{R}^2}\omega(x,t)\,dx = 0, \qquad \int_{\mathbb{R}^2} x_i\omega(x,t)\,dx = 0 \quad (i=1,2).$$

These identities also hold for every time derivative of ω. Taylor expansion of the Biot–Savart kernel through its linear term, using these cancellations, gives (1.3).

Fix $A_0\ge 1$, $\lambda_0 = \lceil\sqrt{A_0}/2\rceil$, and a radial $\chi_0\in C_c^\infty(B_{R_0})$ with $R_0>2$, $0\le\chi_0\le 1$, and $\chi_0 = 1$ on B_2. The prescribed data are

$$\theta_{\mathrm{in}}(x) = -\frac{A_0}{\lambda_0}\sin(\lambda_0 x_2)\chi_0(x), \qquad u_{\mathrm{in}} = 0. \tag{1.4}$$

Near the origin, temperature decreases with height, with $\partial_2\theta_{\mathrm{in}}(0) = -A_0$. Thus colder, heavier fluid lies above warmer, lighter fluid, giving a smooth Rayleigh–Taylor-unstable stratification. An upward displacement carries warmer fluid into colder surroundings, so buoyancy reinforces the displacement. This is the instability that amplifies the oscillatory layers in our construction; its amplitude equations are derived in Subsection 1.2.

Theorem 1.1 (Smooth forcing and finite-time blowup)**.** *For the fixed data (1.4), there are odd forces*

$$f_\theta \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}), \qquad f_u \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}; \mathbb{R}^2),$$

and a time $T_* \in (0,\infty)$ such that:

(1) *For every $0 < T < T_*$, the system (1.1) has a unique solution (θ, u, p) with the prescribed initial values $\theta(x,0) = \theta_{\mathrm{in}}(x)$ and $u(x,0) = 0$, satisfying*

$$(\theta, u) \in \mathcal{Y}_T, \qquad p \in C^\infty(\mathbb{R}^2 \times [0,T]), \qquad p(0,t) = 0.$$

The solution has the parity (1.9).

(2) *This solution blows up as $t \uparrow T_*$ in the sense that*

$$\sup_{0 \le t < T_*} \|\theta(t)\|_\infty < \infty, \qquad \lim_{t \uparrow T_*} \|\nabla\theta(t)\|_\infty = \infty, \qquad \limsup_{t \uparrow T_*} \|\omega(t)\|_\infty = \infty, \tag{1.5}$$

where $\omega = \operatorname{curl} u$.

One fixed ball $\overline{B}_{R_F}$, with $R_F < \infty$, contains the supports of θ, u, ω for $0 \le t < T_*$ and of both forces for all $t \in \mathbb{R}$. Assertions about the state (including traces and equations), pressure and uniqueness apply only before blowup; only the forces extend through T_*, with every mixed space-time derivative.

1.2. **How the construction produces growth.** We follow the strategy developed by Córdoba and Martínez-Zoroa [6, Section 1.2] for constructing finite-time singularities in the forced incompressible porous media equation. We build the temperature and velocity by successively adding localized oscillatory perturbations, which we call layers. Each new perturbation grows through an instability created by the temperature gradient already present. Its short wavelength allows its temperature gradient to become large while its temperature amplitude remains small. After controlling the accompanying velocity, this larger gradient provides the background for the next perturbation, introduced at a still smaller spatial scale. We arrange infinitely many such stages within a finite time. The difficulty is to make the temperature gradients accumulate without cancellation while keeping both external forces smooth through the blowup time.

The wave and its equations. First ignore spatial cutoffs and suppose that the fields already constructed have the local form

$$u_{\mathrm{old}}(x,t) = D(t)x, \qquad \theta_{\mathrm{old}}(x,t) = G(t) \cdot x.$$

Here $D(t)$ is the velocity gradient, with $\operatorname{tr} D = 0$, and $G(t)$ is the temperature gradient. The corresponding vorticity ω_{old} is spatially constant. We add a temperature wave ϑ and a vorticity wave ϖ with the same phase:

$$s = \lambda\zeta(t) \cdot x, \qquad \vartheta(x,t) = \Theta(t)\sin s, \qquad \varpi(x,t) = \Omega(t)\cos s.$$

The positive frequency parameter λ is fixed; the nonzero vector $\zeta(t)$ changes with time, so the physical wavevector is $\lambda\zeta(t)$. The real numbers $\Theta(t)$ and $\Omega(t)$ are signed amplitudes. The streamfunction and velocity of this wave are

$$\psi(x,t) = -\frac{\Omega(t)}{\lambda^2|\zeta(t)|^2}\cos s, \qquad v(x,t) = \nabla^\perp\psi = \frac{\Omega(t)}{\lambda|\zeta(t)|^2} J\zeta(t)\sin s;$$

indeed $\Delta\psi = \varpi$. These are added fields: the new temperature and velocity are $\theta_{\mathrm{old}} + \vartheta$ and $u_{\mathrm{old}} + v$.

The velocity v is perpendicular to ζ, whereas both wave gradients are parallel to ζ. Thus the wave does not advect itself:

$$v \cdot \nabla\vartheta = v \cdot \nabla\varpi = 0.$$

Also $v \cdot \nabla\omega_{\mathrm{old}} = 0$. Subtracting the equations for the older fields, the equations for the added wave, with no new forcing, are therefore

$$(\partial_t + Dx \cdot \nabla)\vartheta + v \cdot G = 0, \qquad (\partial_t + Dx \cdot \nabla)\varpi = \partial_1\vartheta.$$

Transporting the phase requires

$$(\partial_t + Dx\cdot\nabla)s = \lambda(\dot\zeta + D^T\zeta)\cdot x = 0.$$

After this cancellation, equating the coefficients of $\sin s$ and $\cos s$ in the two wave equations gives the amplitude equations. Together with phase transport, they form the system

$$\dot\zeta = -D^T\zeta, \qquad \dot\Theta = -\frac{J\zeta\cdot G}{\lambda|\zeta|^2}\Omega, \qquad \dot\Omega = \lambda\zeta_1\Theta.$$

Each equation records one effect: the older velocity turns and stretches the wavevector; the new velocity moves the older temperature; and the horizontal temperature derivative creates new vorticity. The last coefficient uses the laboratory component ζ_1, because the buoyancy direction is fixed.

Growth and return of the vorticity. For a simple growth example, freeze $D = 0$ and $G = -Ae_2$, with $A > 0$. Write $\zeta = r\mathsf{e}(\varphi)$, where $r = |\zeta| > 0$ and $\mathsf{e}(\varphi) = (\sin\varphi, \cos\varphi)$; thus φ measures the angle from the positive x_2-direction toward the positive x_1-direction. The amplitude equations become

$$\frac{d}{dt}\begin{pmatrix}\Theta\\ \Omega\end{pmatrix} = \begin{pmatrix}0 & A\sin\varphi/(\lambda r)\\ \lambda r\sin\varphi & 0\end{pmatrix}\begin{pmatrix}\Theta\\ \Omega\end{pmatrix}.$$

When $0 < \varphi < \pi$, the eigenvalues are $\pm\sqrt{A}\sin\varphi$. On the growing eigenline $\Omega = (\lambda r/\sqrt{A})\Theta$, both amplitudes are multiplied by $e^{\sqrt{A}\sin\varphi\, t}$. We use negative amplitudes, so that the new gradient points in the direction $-\zeta$. At the origin,

$$\nabla\vartheta(0,t) = \lambda\Theta\zeta, \qquad Dv(0,t) = \Omega\, J\mathsf{e}(\varphi)\otimes\mathsf{e}(\varphi),$$

where $(a\otimes b)x = a(b\cdot x)$. The first identity shows why a short wave can have a small temperature amplitude and a large gradient. The second describes the accompanying shear: displacement along $\mathsf{e}(\varphi)$ creates velocity along $J\mathsf{e}(\varphi)$. Its negative coefficient Ω would interfere with the growth of the next wave.

We therefore follow growth by a controlled rotation. The key is to rotate G and ζ together, rather than change φ in the frozen example while keeping $G = -Ae_2$ fixed. A common rotation preserves $J\zeta\cdot G$ and $|\zeta|$, but can change the sign of the laboratory component ζ_1. It can therefore reverse the sign of $\dot\Omega = \lambda\zeta_1\Theta$ without reversing the coefficient that couples Ω to $\dot\Theta$.

With $\Theta < 0$, briefly making $\zeta_1 < 0$ brings the negative vorticity amplitude back toward zero. We select the rotation to end with $\Omega = 0$ and $\zeta_1 = 0$, while retaining the temperature gain. Both amplitude derivatives then vanish for as long as the control holds $\zeta_1 = 0$. This holding interval allows the next wave to grow; it does not freeze all the older fields or the layer's entire future.

The use of an ODE-based ansatz to produce amplification and prepare the next layer is also central to the Boussinesq construction of Córdoba, Laín-Sanclemente and Martínez-Zoroa [5, Sections 2.6, 2.12 and 2.14]. Their pendulum dynamics govern a layer's translation and deformation. Subsection 3.4 compares their evolution with our growth, steering and holding intervals.

From a wave to compact layers. For the actual construction we replace $\sin s$ by a smooth odd periodic profile $F(s)$ that equals s near zero. Let P be its mean-zero periodic primitive, so $P' = F$. The same cancellation and amplitude equations hold with $\vartheta = \Theta F(s)$, $\varpi = \Omega F'(s)$ and $\psi = \Omega P(s)/(\lambda^2|\zeta|^2)$; this is Lemma 3.1. The advantage is that the temperature and velocity are now exactly linear in x near the origin.

We localize the temperature and the streamfunction, as in [5, Section 1.4.3]. With a smooth compactly supported transported envelope $g(x,t)$ equal to one near the origin, the leading fields have the form

$$\vartheta_{\text{lead}} = \Theta F(s)g, \qquad \psi_{\text{lead}} = \frac{\Omega}{\lambda^2|\zeta|^2}P(s)g.$$

Taking the velocity to be $\nabla^\perp\psi_{\text{lead}}$ keeps it compact and divergence free. Where $g = 1$ and $F(s) = s$, both the temperature and velocity retain their exact affine form. Every finer layer is supported within this region of every older layer throughout its lifetime. Thus each new wave sees precisely the affine background used above. The background means the base and all earlier layers, not just the immediately preceding one.

Corrections outside the central region cancel oscillatory localization errors to successively higher orders without changing these dynamics. This implements, for the two Boussinesq equations, the high-order correction strategy of [6, Section 1.2.4]. After smooth activation and correction, denote the compact temperature and streamfunction of layer q by θ_q, ψ_q. The solution is assembled as

$$\theta = \theta^b + \sum_{q\geq 1} \theta_q, \qquad \psi = \psi^b + \sum_{q\geq 1} \psi_q, \qquad u = \nabla^\perp \psi,$$

where θ^b, ψ^b are the base fields. Only finitely many terms are present on any closed time interval before blowup.

Oddness fixes the origin under the velocity flow. We keep the temperature gradients there in one acute cone, so their gains cannot cancel and persist to give the full gradient limit in (1.5). The vorticity lower bound is evaluated at the ends of steering intervals, when the newest amplitude is zero and the older contributions have a common sign. Between these times there can be cancellation, which is why that conclusion is a limsup. At the terminal time the force is controlled by summability: the correction order increases with the layer index, so the force series converges with every mixed derivative. Section 8 chooses the frequencies to meet these requirements and to fit infinitely many stages into a finite time.

1.3. **Related work and the organization of the proof.** Local smooth existence, uniqueness and a blow-up criterion for the Boussinesq equations were studied by Chae and Nam [3].

For singularity formation, the domain and the regularity of the initial data distinguish several results. Elgindi and Jeong [7] construct singularities on sectors: their data have finite energy and are Lipschitz, and are smooth away from the origin. Elgindi and Pasqualotto [8] construct compactly supported $C^{1,\alpha}$ solutions on the plane that become singular; their initial fields can lack radial smoothness at the origin. Chen and Hou [4] obtain blowup from smooth finite-energy data in a setting with boundary. These results concern different combinations of geometry and initial regularity from the present forced construction on the whole plane with smooth data.

Our closest intellectual predecessors are the forced constructions discussed above. For IPM, Córdoba and Martínez-Zoroa [6] constructed smooth finite-energy solutions developing singularities with a compactly supported source in $L^\infty_t C^\infty_x$. Their work supplies not only the layer organization but also the principle of increasing the approximation order as the spatial scale decreases. Remark 1 anticipates improved time regularity, including smoothness of every order in time with further analysis. Alpöge, Buckmaster and Coiculescu [1] develop this strategy for IPM on $\mathbb{T}^2$ with $F \in C^\infty([0,1]\times\mathbb{T}^2)$ and blowup of both density and velocity gradients. We draw on that later development as well; the foundational strategy remains that of Córdoba and Martínez-Zoroa.

For Boussinesq on the plane, Córdoba, Laín-Sanclemente and Martínez-Zoroa [5] construct compactly supported velocity and density that are smooth before blowup, using their multi-layer pendulum ansatz. Their momentum force lies in $C^0_t(C^{1,\alpha}_x \cap L^2_x)$ through the terminal time, for $0 < \alpha < \sqrt{4/3} - 1$. The scalar and curl forces have compact support, whereas their chosen momentum force does not. Remark 5 also allows any prescribed finite number of time derivatives of the scalar and curl forces, after choosing the construction parameters for that order. Their main theorem states divergence of the time integral of the density-gradient norm.

Theorem 1.1 gives a different realization of this program: both forces have one fixed support ball and all mixed derivatives on the closed slab, and the temperature-gradient norm has a full limit of infinity. The dynamical comparison concerns how an ansatz prepares successive layers, not an identification of the two ODE systems. We give self-contained proofs of the Boussinesq identities and estimates needed here, including the two residual equations and the endpoint conclusions.

Section 3 gives the exact affine-wave calculation and constructs the controlled evolution. Its four tasks are to amplify the new layer, return its vorticity amplitude to zero, preserve the earlier layers, and make the stage lengths summable. The cone and deformation estimates give the geometric information needed next. Section 4 constructs compact fields with the required affine regions and retains the prescribed sine datum by a smooth initial interpolation. Section 5 proves uniform derivative bounds in the prescribed finite-order range, as well as bounds of all orders for each fixed layer. Sections 6 and 7 give the finite correction equations, the compact inverse curl, and the estimates for mixed derivatives of the forces under explicit layer hypotheses. Section 8 verifies those hypotheses with one common parameter choice. Section 9 constructs the state before blowup and the force on the closed time interval, and Section 10 proves the two blowup conclusions and uniqueness in the larger finite-energy class X_τ, defined there.

An unnamed constant C may be enlarged while retaining the dependencies specified in its estimate. Unless stated otherwise, it depends only on the fixed data, profiles, design, bound constants and positive gaps introduced in Section 3, and is independent of the layer indices, frequencies, admissible derivative orders and lifetimes. Order-dependent constants and finite-level constants display their orders or levels as subscripts or arguments. Bounds of all orders for each fixed layer are distinguished from uniform finite-order bounds; preliminary lemmas specify their own permitted inputs. Named constants and positive gaps remain fixed once chosen. The order in which they are chosen is specified in Section 8: low-order geometric constants come first, derivative and correction constants next, and frequencies last. The following change of units is made once; subsequent enlargements of constants leave the normalized equation and datum fixed.

1.4. **Reduction to unit base scales.** We carry out the construction with unit base amplitude and frequency. The cutoff remains arbitrary: the change of units alters its support radius, which is not prescribed in the theorem.

Lemma 1.2 (Scaling reduction)**.** *It suffices to construct the conclusions of Theorem 1.1 for the datum $-\sin(X_2)\widetilde{\chi}_0(X)$ and zero velocity, for every radial $\widetilde{\chi}_0 \in C_c^\infty(B_{\widetilde{R}_0})$ with $\widetilde{R}_0 > 2$, $0 \le \widetilde{\chi}_0 \le 1$ and $\widetilde{\chi}_0 = 1$ on B_2.*

Proof. For the original data set $\mu = \lambda_0 \ge 1$, $\nu = \sqrt{A_0}$ and $\widetilde{\chi}_0(X) = \chi_0(X/\mu)$. This cutoff is radial, equals one on $B_{2\mu} \supset B_2$, and is supported in $B_{\mu R_0}$. Given the unit construction, evaluate all tilded fields at $(X,T) = (\mu x, \nu t)$ and set

$$\begin{gathered}\theta = \frac{\nu^2}{\mu}\widetilde{\theta}, \qquad u = \frac{\nu}{\mu}\widetilde{u}, \qquad p = \frac{\nu^2}{\mu^2}\widetilde{p}, \qquad \omega = \nu\widetilde{\omega}, \\ f_\theta = \frac{\nu^3}{\mu}\widetilde{f}_\theta, \qquad f_u = \frac{\nu^2}{\mu}\widetilde{f}_u, \qquad T_* = \frac{\widetilde{T}_*}{\nu}, \quad R_F = \frac{\widetilde{R}_F}{\mu}.\end{gathered} \tag{1.6}$$

Every term of the scalar equation gains the factor ν^3/μ. Every term of the momentum equation, including θe_2, gains ν^2/μ; divergence gains ν and curl gives $\omega = \nu\widetilde{\omega}$; the vorticity residual gains ν^2. Thus both equations keep coefficient one, with the same laboratory direction. The degree-minus-one homogeneity of the multiplier for K gives $K[\nu\widetilde{\omega}(\mu\,\cdot)] = (\nu/\mu)(K\widetilde{\omega})(\mu\,\cdot)$, hence $u = K\omega$. For the compact streamfunctions used by the construction, $\psi = (\nu/\mu^2)\widetilde{\psi}(\mu x, \nu t)$ gives $u = \nabla_x^\perp\psi$ and $\omega = \Delta_x\psi$. Lemma 4.1 gives the same identity without a zero-frequency ambiguity. The initial temperature is exactly (1.4). Support radii divide by μ, and point-reflection parity and $p(0,t) = 0$ are preserved.

For every spatial multi-index γ and integer $b \ge 0$,

$$\begin{aligned}\partial_x^\gamma\partial_t^b f_\theta(x,t) &= \nu^{3+b}\mu^{|\gamma|-1}(\partial_X^\gamma\partial_T^b\widetilde{f}_\theta)(\mu x, \nu t), \\ \partial_x^\gamma\partial_t^b f_u(x,t) &= \nu^{2+b}\mu^{|\gamma|-1}(\partial_X^\gamma\partial_T^b\widetilde{f}_u)(\mu x, \nu t).\end{aligned} \tag{1.7}$$

These finite factors preserve every mixed derivative on the closed terminal slab. The rescaling also preserves global smoothness and compact space-time support whenever extensions of the forces have been chosen. Fixed-ball support also preserves the weighted L^1 and L^∞ force norms in (7.1). State and pressure smoothness transfer on each closed time interval before blowup. The temperature supremum gains ν^2/μ, the gradient norm gains ν^2, and the vorticity norm gains ν; hence all three conclusions of (1.5) transfer, with their stated full-limit and limsup quantifiers.

The rescaling preserves the smooth class $\mathcal{Y}_T$: it preserves smoothness and compact scalar and vorticity support, and $\langle \mu x\rangle$ is comparable to $\langle x\rangle$ for the fixed $\mu > 0$, so every weighted bound (1.3) is preserved.

For uniqueness, consider also the larger class X_τ defined before Proposition 10.2. A competitor on $[0,\tau]$ maps to

$$(\widetilde{\vartheta},\widetilde{v})(X,T) = \left(\frac{\mu}{\nu^2}\vartheta, \frac{\mu}{\nu}v\right)\left(\frac{X}{\mu},\frac{T}{\nu}\right), \qquad 0 \le T \le \nu\tau.$$

Its pressure transforms by $\widetilde{p} = (\mu^2/\nu^2)p$ in the same coordinates. The substitution is valid in distributions and almost everywhere, with the unit forces. In dimension two the L^2 norms of the two components gain respectively μ^2/ν^2 and μ^2/ν; their spatial gradient L^∞ norms gain ν^{-2} and ν^{-1}. Local Lipschitz continuity and strong initial traces are preserved as well. The inverse is (1.6), so this is a bijection from X_τ to $X_{\nu\tau}$ with the corresponding forces and data. It is also a bijection from $\mathcal{Y}_\tau$ to $\mathcal{Y}_{\nu\tau}$ by the support and weighted bounds just checked. Thus uniqueness transfers in either class whenever it is proved for the unit construction. No terminal state or terminal pressure is used. $\square$

Until the final proof of Theorem 1.1, all variables are in these normalized units. We drop tildes, write (x,t) for (X,T), and let χ_0, R_0 denote an arbitrary admissible normalized cutoff and its support radius. In particular the construction datum is

$$\theta_{\text{in}}(x) = -\sin(x_2)\chi_0(x), \qquad u_{\text{in}} = 0. \tag{1.8}$$

We retain $A_0 = \lambda_0 = \sigma_0 = 1$ where a zeroth index is useful in a sum or ratio. The frequency recursion still starts at layer 2. The initial interpolation has duration one in these units; its duration for the original datum is $1/\sqrt{A_0}$, with A_0 as in (1.4).

1.5. **Notation.** Our Fourier convention is $\widehat{h}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}h(x)\,dx$. Thus Δ^{-1} has symbol $-|\xi|^{-2}$ away from zero and $K = \nabla^\perp\Delta^{-1}$ has symbol $-iJ\xi/|\xi|^2$. For the compact streamfunctions used below the zero-frequency issue is resolved exactly by Lemma 4.1.

We work in the point-reflection class

$$\theta(-x,t) = -\theta(x,t), \qquad u(-x,t) = -u(x,t), \qquad \omega(-x,t) = \omega(x,t), \tag{1.9}$$

with odd f_θ, f_u, and normalize pressure by $p(0,t) = 0$. The spatial support and time intervals are specified in the theorem.

2. AI statement

We happily used both Claude and Codex to iterate on our proof. We had, using Claude, our first blowup solution, but not this one, on 8/15/26, and we Lean verified it on 8/22/26. This involved inputting ideas from previous joint work of ours on blowup for the IPM equation following Córdoba–Martínez-Zoroa and a number of other works of ours and others, as well as iteration with the model on various ansätze, eventually leading to blowup in Boussinesq. The first writeup that we produced iterating with Claude was, in our opinion, the worst writeup we had ever seen in the history of mathematics (topped soon after by the writeups for 3d Euler and then for hypodissipative Navier-Stokes). It was then our task to make a presentable and understandable writeup, and we iterated over the weeks on the writeup with Claude and Codex, then via 5.6 Sol. (Throughout our collaboration we have generally input all of our intermediate writeups to Codex for simplification, ideation, and iteration.) After iterating with Claude and Codex on various ideas of ours, including alternative proof architectures, leading to several different proofs, we arrived at the current simplified argument, which we then iterated on using Claude and Codex, first 5.6 Sol and then Astra once we had access, to arrive at the current writeup modulo our hand editing.

3. An exact wave and its controlled growth

We now develop the wave calculation of Subsection 1.2 for the general periodic profile used in the construction, then choose the common rotation that produces growth and returns each new vorticity amplitude to zero. The estimates also follow every older layer, since together they form the background for later additions. Spatial cutoffs and localization errors are treated in Section 4.

3.1. **The two-field calculation.** Let $D(t)$ be a trace-free matrix and $G(t)$ a vector. On a region where the earlier state is $u_{\mathrm{old}} = Dx$, $\theta_{\mathrm{old}} = G\cdot x$, its vorticity is spatially constant. Take a nonzero phase gradient $\zeta(t)$, signed temperature and vorticity amplitudes $\Theta(t), \Omega(t)$, a fixed frequency parameter $\lambda > 0$, and a smooth odd 2π-periodic function F. Let P be its even, mean-zero periodic primitive. With $s = \lambda\zeta\cdot x$ and $\kappa = |\zeta|^2$, put

$$\vartheta = \Theta F(s), \qquad \psi = \frac{\Omega}{\lambda^2\kappa}P(s), \qquad v = \frac{\Omega}{\lambda\kappa}J\zeta F(s), \qquad \varpi = \Omega F'(s). \tag{3.1}$$

Adding these fields to the affine background changes the scalar and vorticity residuals by the increments computed in the following proof.

Lemma 3.1 (Exact affine wave)**.** *The scalar and vorticity residual increments vanish for* (3.1) *if*

$$\dot\zeta = -D^T\zeta, \qquad \dot\Theta = -\frac{J\zeta\cdot G}{\lambda|\zeta|^2}\Omega, \qquad \dot\Omega = \lambda\zeta_1\Theta. \tag{3.2}$$

This is a local wave; its compactly supported version is constructed in Section 4.

Proof. The first equation gives $(\partial_t + Dx\cdot\nabla)s = 0$. Furthermore $J\zeta\cdot\zeta = 0$, so $v\cdot\nabla\vartheta = v\cdot\nabla\varpi = 0$. The scalar increment is $[\dot\Theta + \Omega(J\zeta\cdot G)/(\lambda\kappa)]F(s)$. The vorticity increment is $[\dot\Omega - \lambda\zeta_1\Theta]F'(s)$, since $v\cdot\nabla\omega_{\mathrm{old}} = 0$. Both expressions vanish by (3.2). In particular, no sinusoidal identity was used. $\square$

The sine-wave growth calculation in Subsection 1.2 therefore applies to these profiles as well. We continue to write $\mathsf{e}(\varphi) = (\sin\varphi, \cos\varphi)$. We next specify how all earlier layers determine D and G, and how the common rotation controls the phase component ζ_1.

3.2. **The affine background and amplitude variables.** Use $\alpha(t)$ for the common rotation angle; it is distinct from every streamfunction. Let R_α denote counterclockwise rotation by α. Put $\mathsf{e}_0 = R_\alpha e_2 = \mathsf{e}(-\alpha)$, $w_0 = 1$, and $A_0 = \sigma_0 = 1$. The activation weights $w_j \in [0,1]$, activation times t_j^{in} and unit activation vectors p_j are specified in Subsection 3.3 below. For each activated layer q let

$$\begin{aligned} &G_{<q} = -\mathsf{e}_0 - \sum_{1\le j<q} w_jA_j\mathsf{e}_j, \qquad A_j = -\lambda_j|\zeta_j|\Theta_j, \quad \mathsf{e}_j = \zeta_j/|\zeta_j|,\\ &D_{<q} = \dot\alpha J + \sum_{1\le j<q} w_j\Omega_j\, J\mathsf{e}_j\otimes\mathsf{e}_j,\\ &\dot M_q = D_{<q}M_q, \qquad M_q(t_q^{\mathrm{in}}) = \mathrm{Id}, \qquad \zeta_q = M_q^{-T}p_q. \end{aligned} \tag{3.3}$$

Each amplitude pair solves (3.2) with its own background coefficients. Write its coefficients as a_q, b_q, respectively, so that $\dot\Theta_q = a_q\Omega_q$, $\dot\Omega_q = b_q\Theta_q$. Only finitely many layers are activated at any time before blowup.

The deformation matrix describes the affine transport by the complete background. The phase gradient has a direction and a length; multiplying it by the frequency parameter gives the physical wavevector. The temperature and vorticity amplitudes are scalar coefficients, not the full compact fields after localization and correction.

The feedback that holds the first phase gradient component fixed is

$$\mathcal{F}_j = -\frac{\sum_{i<j} w_i \Omega_i \mathsf{e}_{i,1}(J\mathsf{e}_i \cdot \zeta_j)}{\zeta_{j,2}}, \qquad \mathcal{F}_0 = 0. \tag{3.4}$$

The sum here and below over shears begins at $i = 1$. Indeed transposing (3.3) gives

$$\dot{\zeta}_{j,1} = (\mathcal{F}_j - \dot{\alpha})\zeta_{j,2}. \tag{3.5}$$

Thus $\dot{\alpha} = \mathcal{F}_j - \dot{Z}/\zeta_{j,2}$ prescribes $\zeta_{j,1} = Z$ if their initial values match and $\zeta_{j,2} \neq 0$. This formula retains laboratory gravity: $b_j = \lambda_j \zeta_{j,1}$, not a derivative in a rotating gravity direction.

Let $\rho_j = \log|\zeta_j|$, $\zeta_j = |\zeta_j|\mathsf{e}(\varphi_j)$, set $\varphi_0 = -\alpha$, and write $d_{ij} = \varphi_j - \varphi_i$. Writing $\dot{\zeta}_j = -D^T_{<j}\zeta_j$, obtained from (3.3), in polar coordinates gives

$$\dot{\rho}_j = \sum_{i<j} w_i \Omega_i \sin d_{ij} \cos d_{ij}, \qquad \dot{\varphi}_j = -\dot{\alpha} - \Sigma_j, \quad \Sigma_j = \sum_{i<j} w_i \Omega_i \sin^2 d_{ij}. \tag{3.6}$$

In particular $\dot{d}_{ij} = \Sigma_i - \Sigma_j$: the common rotation disappears from relative angles. This cancellation is useful when differentiating the control, because it prevents the highest derivative of α from reappearing in the first equation used to estimate it.

3.3. **The frequency schedule, activation, and stage intervals.** There are four groups of parameters. Frequencies and material radii control oscillation and support. The seed and the prescribed logarithmic gain control amplitude. The growth scale, growth rate and control derivative rate govern time. Finally, the admissible derivative order determines the finite number of corrections; it is not a layer index or a Fourier mode. The formulas below specify their relations, including the exceptional first-stage values.

Fix $\beta = 1/8$, an integer $Q_* \geq 200$, and $Q_q = Q_* + q$. The frequencies and envelope radii are fixed before solving the ODE:

$$\begin{gathered} \lambda_1 \gg 1, \qquad \lambda_q = \lambda_{q-1}^{Q_q} \quad (q \geq 2), \\ \ell_1 = \frac{s_0}{C_\ell}, \quad C_\ell \geq 4, \qquad \ell_q = \lambda_{q-1}^{-3} \quad (q \geq 2), \\ \Pi_1 = (\log \lambda_1)^{10}, \qquad \Pi_q = \lambda_{q-1}^4 (\log \lambda_q)^{10} \quad (q \geq 2). \end{gathered} \tag{3.7}$$

Here Π_q is the common size and time-derivative scale used for layer q. The small number s_0 specifies the linear interval of F in (4.1). The nondecreasing function $\mathcal{K} : \mathbb{N}_0 \to [2, \infty)$ is defined independently of all frequencies in (7.21). For $\log \lambda_1 \geq \mathcal{K}(0)$, set

$$\begin{aligned} k_q &= \max\{k \in \mathbb{N}_0 : 120k \leq Q_q,\ \mathcal{K}(k) \leq \log \lambda_q\}, \quad & J_q &= 2k_q + 8, \\ \Theta_q^{\mathrm{seed}} &= -\lambda_q^{-k_q - 6}, & L_q &= (k_q + 5 + \beta)\log \lambda_q. \end{aligned} \tag{3.8}$$

Here k_q is the derivative order to be controlled uniformly and J_q is the corresponding number of correction levels. Then $|\Theta_q^{\mathrm{seed}}| e^{L_q} = \lambda_q^{-7/8}$ exactly. The seed is small enough that the force generated by smooth activation is within the required bound. The amplification is stopped at the prescribed temperature size, rather than at a time selected from an approximate growth formula.

Take a smooth nondecreasing η equal to zero on $(-\infty, 0]$ and one on $[1, \infty)$. Fix any $\eta_2 \in C_c^\infty((0,1))$ with $\eta_2 \geq 0$ and $\int \eta_2 = 1$, and write $H_2 = \|\eta_2\|_\infty$. The pulse multiplier c_{p} exceeds the upper barrier $\bar{v}$ for the normalized amplitude ratio in the model for later stages, which includes the preceding layer's response. The parameter Λ_1 compresses the first negative pulse, and s_* is the first insertion angle. These design data are independent of every frequency parameter:

$$c_{\mathrm{p}} = 3, \qquad \bar{v} = 2, \qquad \Lambda_1 \geq 2c_{\mathrm{p}} H_2, \qquad 0 < s_* \leq \tfrac{1}{8}, \qquad c_{\mathrm{p}} \Lambda_1 H_2 \sin s_* \leq \tfrac{1}{4}. \tag{3.9}$$

Here t^b_{q-1} and σ_{q-1} are the end time and growth scale of the preceding stage, defined in items (iii)–(iv) below. Set $t^{\mathrm{in}}_1 = 1$, $t^{\mathrm{in}}_q = t^b_{q-1}$ and $s_1 = s_*$. At activation, for $q \geq 2$ put

$$s_q = L_q \frac{\sigma_{q-2}}{\sigma_{q-1}}, \qquad \Gamma_q = \sigma_{q-1} \sin s_q, \qquad \Lambda_q = L_q. \tag{3.10}$$

The rate Γ_q sets the growth and transition time scales. For $q = 1$ put $\Gamma_1 = \sin s_*$. For every layer use $\mathfrak{h}^{\max}_q = c_{\mathrm{p}} \Lambda_q \sin s_q$. In particular Λ_1 is fixed; it is not the prescribed gain L_1. The previously completed phase gradient is vertical at activation. Choose $p_q = \mathsf{e}(s_q)$, $\Theta_q = \Theta^{\mathrm{seed}}_q$, and $\Omega_q = \sqrt{b_q/a_q}\,\Theta_q$ there. Positivity of a_q, b_q is part of the control estimate, not an additional arbitrary choice. The activation is

$$w_q(t) = \eta(\Gamma_q(t - t^{\mathrm{in}}_q)). \tag{3.11}$$

Activation takes place during the first part of growth. The homogeneous amplitude pair is already nonzero at entry, but the cutoff makes the physical increment vanish there to every order. Once activation is complete, the cutoff remains one. The stopping time below is a temperature-amplitude threshold, not a perpendicularity condition on the phase.

Growth is followed by a transition between the two feedback laws and then by steering. The short negative pulse in the steering law has fixed duration; only its amplitude is selected. The subsequent holding interval overlaps the growth of the next stage.

(i) On growth, $\dot\alpha = \mathcal{F}_{q-1}$, until the first time t^a_q with $-\Theta_q = \lambda_q^{-7/8}$.
(ii) On the transition of length Γ_q^{-1}, ending at t^1_q, use $\dot\alpha = (1 - \eta_b)\mathcal{F}_{q-1} + \eta_b \mathcal{F}_q$, where $\eta_b = \eta(\Gamma_q(t - t^a_q))$.
(iii) The steering ends at $t^b_q = t^1_q + \Gamma_q^{-1}(1 + \Lambda_q^{-1})$. Set

$$\begin{aligned} Z_q(t) &= \zeta_{q,1}(t^1_q)[1 - \eta(\Gamma_q(t - t^1_q))] \\ &\quad - \mathfrak{h}_q |\zeta_q(t^1_q)| \eta_2\big(\Lambda_q \Gamma_q (t - t^1_q - \Gamma_q^{-1})\big), \qquad \dot\alpha = \mathcal{F}_q - \frac{\dot Z_q}{\zeta_{q,2}}. \end{aligned} \tag{3.12}$$

Select the smallest $\mathfrak{h} \in [0, \mathfrak{h}^{\max}_q]$ for which $\Omega_q(t^b_q; \mathfrak{h}) = 0$.
(iv) On the following holding interval, through t^a_{q+1}, use $\dot\alpha = \mathcal{F}_q$. Finally define $\sigma_q^2 = |G_{<q+1}(t^b_q)|$.

Thus the growth scale is the square root of the background-gradient magnitude at the end of steering. It is distinct from the growth rate and from rescaled time. During the holding interval the amplitude pair of this layer is stationary; the common rotation and the evolution of the other layers need not stop.

Theorem 3.2 proves existence of a selected root and smooth continuation across every join before blowup. Selecting a root once per stage does not require differentiating that root as a function of all older data. For the recursively continued stages set

$$T_* := \lim_{q\to\infty} t^b_q = \lim_{q\to\infty} t^{\mathrm{in}}_q, \qquad I_q = [t^{\mathrm{in}}_q, T_*). \tag{3.13}$$

The limit is provisionally allowed to be infinite. Its finiteness follows from the controlled-stage estimates and is used in the assembly argument.

3.4. **Comparison with the pendula ansatz.** The finite-dimensional construction of Córdoba, Laín-Sanclemente and Martínez-Zoroa [5, Sections 2.5–2.6] also selects an evolution of the parameters that amplifies a layer, allows switching while keeping the force within the required bounds, and prepares subsequent layers. For this comparison only, write c_n for the first coordinate of a layer center in their coordinate convention, and use their notation a_n, b_n for its inverse length scales and

B_n for its streamfunction amplitude; these are not our amplitude-system coefficients. Their center and deformation equations are

$$\dot{c}_n = \sum_{m<n} B_m b_m \sin(a_m(c_n - c_m)),$$
$$\frac{d}{dt}\log b_n = \sum_{m<n} B_m a_m b_m \cos(a_m(c_n - c_m)), \qquad \frac{d}{dt}(a_n b_n) = 0.$$

Set $a_* = a_{n-1}(1)$, $b_* = b_{n-1}(1)$ and $B_* = B_{n-1}(1)$. Their ideal model for the relative separation $c_n - c_{n-1}$ retains only the preceding layer's contribution and freezes these coefficients: $\dot{\Xi}_{\mathrm{p}} = B_* b_* \sin(a_* \Xi_{\mathrm{p}})$. For its model inverse scale b_{mod} and entry time t_n, put

$$\tau_{\mathrm{p}} = B_* a_* b_* (t - t_n),$$
$$\varphi_{\mathrm{p}}(\tau_{\mathrm{p}}) = a_* \Xi_{\mathrm{p}}(t), \qquad \mathcal{B}_{\mathrm{p}}(\tau_{\mathrm{p}}) = \frac{b_{\mathrm{mod}}(t)}{b_{\mathrm{mod}}(t_n)}.$$

With their choice $0 < \varphi_{\mathrm{p}}(0) < \pi/2$, the equations become

$$\frac{d\varphi_{\mathrm{p}}}{d\tau_{\mathrm{p}}} = \sin\varphi_{\mathrm{p}}, \qquad \frac{d\log\mathcal{B}_{\mathrm{p}}}{d\tau_{\mathrm{p}}} = \cos\varphi_{\mathrm{p}}, \qquad \mathcal{B}_{\mathrm{p}}(\tau_{\mathrm{p}}) = \frac{\sin\varphi_{\mathrm{p}}(\tau_{\mathrm{p}})}{\sin\varphi_{\mathrm{p}}(0)}.$$

Their initial phase and time scale make the stretch grow and then return. The density pulse is held constant during large deformation and is switched on and off where deformation is small; switching it off leaves vorticity behind. The ideal return is at their terminal time 1, not an exact reset at the next activation. Their full equations are controlled by comparison with this model [5, Sections 2.12, 2.14 and 3].

Here the corresponding preparation is accomplished by controlling the phase component in (3.5). Growth holds the preceding phase component fixed; transition changes the feedback; and (3.12) uses a fixed-duration negative pulse with a selected amplitude to reach $\zeta_{q,1} = \Omega_q = 0$. The subsequent holding interval freezes this amplitude pair while the next layer grows. Thus we retain a temperature gain while returning the newest vorticity amplitude to zero, in contrast to their retained vorticity after the density pulse has ended. Neither the older fields nor the layer's entire future evolution is stationary. Thus our steering and holding implement a related ODE-based preparation principle, with different variables and a different endpoint condition; the pendulum equation is not being substituted for the controlled system (3.2).

3.5. **Controlled-stage estimates.** The estimates are grouped by their later use. Nonvanishing and the cone condition yield persistent temperature growth. The vorticity bound at the end of steering yields the second blowup conclusion. Deformation bounds ensure nesting of supports for the full lifetime, and the propagator estimate controls the corrections relative to the growing amplitude. We keep the first-stage constants separate because its phase-gradient length is fixed, unlike that of the later stages.

Throughout the control proof write $P_j = -\Theta_j$, $r_j = |\zeta_j|$, $\rho_j = \log r_j$, $\vartheta_j = \varphi_j - \varphi_{j-1}$ for $j \geq 2$, $\vartheta_1 = s_*$, and $d_{ij} = \varphi_j - \varphi_i$. The derivative orders may be any sequence $0 \leq k_j \leq Q_j/120$; no correction constant enters this proof. Put $\varpi_j = L_j s_j$ for $j \geq 2$, $X_1 = \Lambda_1\Gamma_1$ and $X_j = L_j\Gamma_j$ for $j \geq 2$.

The subscripted P_j denotes a positive amplitude and ϑ_j an adjacent angle throughout this control argument. The unsubscripted P and ϑ in (3.1) remain the periodic primitive and the temperature field. Here ϖ_j measures the steering angle, and X_j includes the rate of the compressed negative pulse.

We now fix the bounds needed for these four groups of estimates: first vorticity and geometric bounds, then amplification bounds, then time and deformation bounds, and finally the error tolerance. All constants in this section depend only on the fixed design, fixed bound constants and positive gaps, and the indicated norms of low-order derivatives of the profiles. Intermediate constants C are finite maxima of the finitely many estimates below, uniform over layer indices and all trial pulse amplitudes. The lower- and upper-bound constants for vorticity, angles and phase length are

$$
\begin{gathered}
0 < c_W < \tfrac{1}{3}, \quad C_W > 1 + \bar{v}, \quad C_W^{\mathrm{all}} > 1 + c_{\mathrm{p}}, \\
0 < c_\theta < \tfrac{2}{3} < 1 < C_\vartheta, \quad 0 < \zeta_- < 1 < \tfrac{3}{2} < \zeta_+.
\end{gathered} \tag{3.14}
$$

Here c_W is a lower-bound constant and C_W an upper-bound constant; $1/3 - c_W$ is a positive gap. Choose also a positive component lower bound z_* below both ζ_- and $\sqrt{15}/4$, an amplitude lower-bound constant $0 < c_A < \cos(2s_*)$, and $C_\varphi > c_{\mathrm{p}} H_2$. The constant c_A fixes the persistent amplitude lower bound independently of the cone projection. Its conservative restriction $c_A < \cos(2s_*) < 1$ leaves room below the near-one amplitude bound in the proof. For the separate direction estimate we use $c_{\mathrm{cone}} := \cos s_* > 0$, justified by the summed angles below. The constant I_0 in (3.46) measures the first-part gain in the model for later stages. Its admissible coefficient-error tolerance $\delta_0 = \delta_0(c_{\mathrm{p}}, \bar{v})$ is supplied by Lemma 3.9; both are fixed before the bounds below. For the logarithmic gains and squared growth scales choose

$$
\begin{gathered}
0 < G_- < \min(1 + I_0, 1 + \log 2), \qquad G_+ > \max(1 + \bar{v}, 2 + \Lambda_1^{-1}), \\
c_\sigma = e^{G_-}, \qquad C_\sigma = \zeta_+ e^{G_+}.
\end{gathered} \tag{3.15}
$$

These are the constants multiplying the frequency powers in the lower and upper bounds for the squared growth scales. Fix $C_S > 1$ and $C_T = 2C_S$. Fix $C_K > 2\zeta_+\zeta_-^{-2}$, and finite larger constants $\widehat{C}_\Omega, C_\Theta, C_\Pi, C_{\mathrm{seed}}$ for the ratio, amplitude, coefficient-size and amplitude bounds during activation proved below; take $C_\Theta > (1 + \widehat{C}_\Omega)e^{G_+}$ and $C_\Pi > 4(1 + C_W^{\mathrm{all}} + \zeta_+^2 + z_*^{-2})$. For the propagator we fix $C_r \geq 1$ to bound the growth-coordinate change in Lemma 3.14, and $0 < \rho_0 < 1$ so that the transformed amplitude ratio stays in $[-\rho_0, \rho_0]$; C_E, C_D, C_H bound the integrated growth error, later evolution and holding intervals for layers $q \geq 2$. The combined transition/steering and first-hold integrals have fixed majorants

$$
C_{\mathrm{short}} > 4 + c_{\mathrm{p}} + \Lambda_1^{-1}, \qquad C_{\mathrm{hold}} > \sin s_*.
$$

Their defining estimates appear in Subsection 3.7.9, and together give (3.72). All seven constants are fixed at this low-order step, before the error tolerance. The model conclusions and the estimates below approach the displayed model ranges as their errors tend to zero. Choose the error tolerance $\epsilon_0 > 0$ so that the finitely many strict inclusions in these wider bounds hold, including the positive lower bounds for the growth scale, cone projection and vorticity and the improved lower bound for the phase-gradient length $1 - C\epsilon_{\mathrm{sch}} > 1/2$. Finally choose one finite $K_{\mathrm{sch}} \geq 1$ dominating the constants in the low-order estimates that improve the bootstrap bounds, including products of constants and the reciprocals of the gaps between the model ranges and these bounds, and set

$$
0 < \epsilon \leq \epsilon_0, \qquad \epsilon_{\mathrm{sch}} = \epsilon / K_{\mathrm{sch}} < \tfrac{1}{4}. \tag{3.16}
$$

In particular, require $C\epsilon_0/K_{\mathrm{sch}} < \delta_0(c_{\mathrm{p}}, \bar{v})$ and $C'\epsilon_0/K_{\mathrm{sch}} < C_S - 1$, where C, C' are the constants in the model-error and stage-length estimates below, respectively. In making these finite comparisons, coarse bootstrap factors are bounded by fixed numbers, so the constants are computed before ϵ is chosen. Neither K_{sch} nor ϵ depends on a higher-order constant, a common correction majorant, a growth scale from the controlled evolution, or a frequency. The same choices serve the proof at every derivative order.

The theorem concerns the amplitude system (3.2) with the complete older fields and deformation in (3.3), run through the growth, transition, steering and holding intervals of Subsection 3.3. The frequency, radius and lifetime scales are (3.7). Here the seeds and gains in (3.8) may use any integers $0 \leq k_q \leq Q_q/120$; the later choice involving $\mathcal{K}$ is not required. Recall that $P_q = -\Theta_q$ is the positive temperature amplitude, $r_q = |\zeta_q|$ is the phase-gradient length and $A_q = \lambda_q r_q P_q$ is the gradient amplitude. The quantity $\sigma_q^2 = |G_{<q+1}(t_q^b)|$ is the squared growth scale at the end of steering. All these bounds and the schedule tolerance are fixed as in (3.14)–(3.16).

The deformation and normalized amplitude quantities used in the theorem are

$$
\begin{gathered}
K_q = \sup_{I_q} \max(\|M_q\|, \|M_q^{-1}\|), \qquad y_q^0 = (\Theta_q, \sigma_{q-1}\Omega_q/\lambda_q), \qquad Y_q = \sup_{I_q} |y_q^0|, \\
\mathcal{B}_q = \begin{pmatrix} 0 & \widehat{a}_q \\ \widehat{b}_q & 0 \end{pmatrix}, \quad \widehat{a}_q = -\frac{J\zeta_q \cdot G_{<q}}{\sigma_{q-1}|\zeta_q|^2}, \quad \widehat{b}_q = \sigma_{q-1}\zeta_{q,1}.
\end{gathered} \tag{3.17}
$$

Then $(y_q^0)' = \mathcal{B}_q y_q^0$ by (3.2); write $\mathcal{U}_q(t,s)$ for this system's propagator.

Theorem 3.2 (Controlled stages)**.** *There is a finite threshold $\lambda_{\rm lo}$, independent of the correction constants and depending on the preceding fixed data and on Q_*, s_0, C_ℓ, such that all stages exist for $\lambda_1 \ge \lambda_{\rm lo}$. The conclusions are grouped by their later uses; lifetime bounds hold on the evolution obtained by the selected pulse amplitudes unless a shorter interval or a trial family is specified.*

(i) Existence and selection. *Each crossing of the growth threshold is transversal. For every trial pulse amplitude $\mathfrak{h} \in [0, \mathfrak{h}_q^{\max}]$, the stage exists through t_q^b. The endpoint map $\mathfrak{h} \mapsto \Omega_q(t_q^b; \mathfrak{h})$ is continuous, has opposite signs at the parameter endpoints, and has a nonempty compact zero set. The prescribed least root is therefore defined. On the selected evolution $T_* < \infty$, all finite joins are smooth.*

(ii) Amplitudes, growth scales and lengths.

$$
\begin{gathered}
1 - C\epsilon_{\rm sch} \le r_j \le \zeta_+, \quad \zeta_{j,2} \ge z_*, \quad 0 < P_j \le e^{G_+}\lambda_j^{-7/8}, \\
A_j(t) \ge c_A \lambda_j^{1/8} \quad (t \ge t_j^a), \\
c_\sigma \lambda_j^{1/8} < \sigma_j^2 < C_\sigma \lambda_j^{1/8}, \\
\sigma_j \ge \max(\epsilon_{\rm sch}^{-1}\sigma_{j-1}, \sigma_{j-1}^3), \\
|[t_q^{\rm in}, t_q^b]| < C_S/\sigma_{q-2} \quad (q \ge 2), \quad t_1^b - 1 = (L_1 + 2 + \Lambda_1^{-1})/\Gamma_1.
\end{gathered} \tag{3.18}
$$

Here $r_1 = 1$.

(iii) End-of-steering values and persistent geometry and vorticity bounds. *At the end of each steering interval $\zeta_{j,1} = \Omega_j = 0$, and both remain zero through t_{j+1}^a; for $j = 1$ they also remain zero through the second transition. At the end of steering $A_j(t_j^b)/\sigma_j^2 \in [1 - C\epsilon_{\rm sch}, 1]$, and subsequently $A_j/\sigma_j^2 = 1 + O(\epsilon)$. For $t \ge t_{j+1}^b$ the vorticity amplitude is nondecreasing and lies in $[c_W, C_W]\sigma_j$. On the intervening stage of the next layer it lies in $[-C\epsilon_{\rm sch}, C_W]\sigma_j$ for the selected pulse amplitude, and in $[-C\epsilon_{\rm sch}, C_W^{\rm all}]\sigma_j$ for every trial pulse amplitude. For $j \ge 2$ the adjacent angles ϑ_j lie in $[c_\theta, C_\vartheta]s_j$ throughout their lifetimes. Every active temperature direction, including the base, satisfies $\mathsf{e}_j(t) \cdot \mathsf{e}_1(t) \ge c_{\rm cone} = \cos s_*$. At the end of each steering interval t_n^b, $n \ge 2$,*

$$
2\dot\alpha(t_n^b) + \sum_{j \le n} w_j \Omega_j(t_n^b) \ge c_W \sigma_{n-1}. \tag{3.19}
$$

(iv) Deformation. *$K_1 = 1$ and $K_q \le C_K/s_q \le \sigma_{q-1} \le \sqrt{C_\sigma}\lambda_{q-1}^{1/16}$ for $q \ge 2$.*

(v) Activation and relative propagation. *The activation finishes before the threshold crossing and*

$$
|\Theta_q| + |\sigma_{q-1}\Omega_q/\lambda_q| \le C_{\rm seed}\lambda_q^{-k_q-6} \quad (t_q^{\rm in} \le t \le t_q^{\rm in} + \Gamma_q^{-1}). \tag{3.20}
$$

The pair y_q^0 is nonzero and $Y_q \le C_\Theta \lambda_q^{-7/8}$. Its propagator satisfies, with C_R given in (3.72) from the fixed design, bound constants, positive gaps and low-order profile norms,

$$
\|\mathcal{U}_q(t,s)\| \le C_R \frac{|y_q^0(t)|}{|y_q^0(s)|} \qquad (t_q^{\rm in} \le s \le t < T_*). \tag{3.21}
$$

(vi) Lifetime and control speed. *The same threshold gives* $|I_q| \le \Pi_q$ *and the schedule comparisons of Lemma 3.13. A finite constant* C_α*, depending on the fixed design and profile* C^1 *norms, bounds the angle and its first time derivative:* $\sup(|\alpha| + |\dot\alpha|) \le C_\alpha \le \Pi_1/2$.

Each stage uses only the preceding stages and the parameters fixed above. This theorem gives smoothness before T_*; bounds for all derivatives of each fixed layer are proved in Section 5.

3.6. **Local steering models with fixed bounds.** The purpose of the local models is to turn the new vorticity amplitude back to zero without losing the temperature gain. We first prove that the temperature component stays nonzero in the underlying linear system; this justifies the amplitude ratio used in the reduced equations. The first stage has a fixed phase-gradient length. In later stages, the preceding layer also develops the positive vorticity amplitude needed at the end of the stage. The models have free parameters here and are applied to the family obtained by varying the trial pulse amplitude in the proof of Theorem 3.2. In that application we first continue the geometric variables for every trial amplitude, and only then divide the linear pair by its nonvanishing temperature component. The lifetime deformation estimate is a later part of the same proof.

3.6.1. *Nonvanishing of the temperature amplitude.*

Lemma 3.3 (Tracking the growing amplitude ratio)**.** *Let* $a, b > 0$ *be continuous on a compact interval* $[t_0, t_1]$*, and suppose* $u^* = \sqrt{b/a}$ *is* C^1*. Set* $\gamma = \sqrt{ab}$*. For* $0 < \nu < 1$*, assume*

$$\left|\frac{d}{dt}\log u^*\right| \le \nu\gamma.$$

A solution of $\dot u = b - au^2$ *with* $u(t_0)/u^*(t_0) \in [1-\nu, 1+\nu]$ *remains in this interval of relative values and continues throughout* $[t_0, t_1]$.

Proof. The relative variable $r = u/u^*$ obeys $\dot r = \gamma(1-r^2) - r(\log u^*)'$. At the upper and lower boundary, respectively,

$$\dot r\big|_{r=1+\nu} \le \gamma[1-(1+\nu)^2 + \nu(1+\nu)] = -\nu\gamma < 0, \qquad \dot r\big|_{r=1-\nu} \ge \gamma[1-(1-\nu)^2 - \nu(1-\nu)] = \nu\gamma > 0.$$

The first-exit test gives invariance. Since u^* is bounded on the compact interval, this also bounds u and excludes finite-time escape for its scalar equation. ∎

The next lemma gives the nonvanishing estimate that allows us to use an amplitude ratio satisfying a Riccati equation. In this local comparison ℓ denotes the interval length. The integral form also bounds the slope using the total pulse mass, as in (3.30) below.

Lemma 3.4 (Concavity and nonvanishing)**.** *Suppose* $Q \ge 0$ *is continuous on* $[0, \ell]$ *and*

$$\chi'' + Q\chi = 0, \qquad \chi(0) = 1, \qquad \chi'(0) = c \ge 0.$$

If $Q \le q$ *and* $\rho = q(1+c\ell)\ell^2/2 < 1$*, then*

$$\begin{aligned} 1-\rho \le 1 + cy - \tfrac12 q(1+c\ell)y^2 \le \chi(y) \le 1 + cy,& \\ c - q(1+c\ell)y \le \chi'(y) \le c \qquad (0 \le y \le \ell).& \end{aligned} \tag{3.22}$$

Alternatively, writing $m_Q = \int_0^\ell Q$*, the condition* $\rho' = (1+c\ell)\ell m_Q < 1$ *implies*

$$1-\rho' \le \chi(y) \le 1+cy, \qquad c - (1+c\ell)m_Q \le \chi'(y) \le c. \tag{3.23}$$

In both cases χ *is positive and concave on the whole interval.*

Proof. Before a first possible zero, $\chi'' = -Q\chi \le 0$. Thus $\chi' \le c$ and $\chi \le 1 + cy \le 1 + c\ell$ there. The bound $\chi'' \ge -q(1+c\ell)$, integrated twice, gives (3.22). Its positive lower bound excludes the putative zero by continuity. For the integral version use instead

$$\chi'(y) = c - \int_0^y Q(z)\chi(z)\,dz \ge c - (1+c\ell)m_Q.$$

Integrating once more gives the lower bound $1-\rho'$, and excludes a zero in the same way. All endpoint bounds follow by continuity. $\square$

Lemma 3.5 (Continuation of the amplitude ratio)**.** *Let $\dot\Theta = a\Omega$, $\dot\Omega = b\Theta$ have continuous coefficients on $[t_0,t_1]$, with $\Theta(t_0)\neq 0$. If one finite bound for $|\Omega/\Theta|$ holds on every initial subinterval on which Θ is nonzero, then Θ is nonzero throughout $[t_0,t_1]$. If $a>0$, $b\geq 0$, $\Theta(t_0)<0$ and $\Omega(t_0)\leq 0$, then*

$$\Theta(t)\leq\Theta(t_0)<0,\qquad \Omega(t)\leq 0.$$

If also $\Omega(t_0)<0$, then $\Omega(t)<0$ on the whole interval.

Proof. The linear system exists on the compact interval. At a first zero of Θ, the interval-uniform quotient bound would force $\Omega=0$ as well. Uniqueness, run backwards from that time, would make the pair identically zero, contrary to its datum. For the sign assertion, as long as $\Omega\leq 0$ one has $\dot\Theta\leq 0$ and hence $\Theta\leq\Theta(t_0)<0$. If this interval ended before t_1, continuity would keep $\Theta<0$ a little longer, and $\dot\Omega=b\Theta\leq 0$ would prevent exit through $\Omega=0$. The interval therefore reaches t_1. The same inequality makes Ω nonincreasing, giving strict negativity for a strictly negative initial value. $\square$

A short interval permits a second useful comparison, stated directly for the amplitude ratio.

Lemma 3.6 (Nonvanishing during a short steering interval)**.** *On $[t_0,t_0+\ell]$ suppose $0<a\leq\bar a$ and $|b|\leq\bar b$, with $4\ell^2\bar a\bar b\leq 1$. For the linear pair in Lemma 3.5, assume $\Theta(t_0)<0$ and $u_0=\Omega(t_0)/\Theta(t_0)\geq 0$. Then $\Theta<0$ and*

$$-2\bar b(t-t_0)\leq\frac{\Omega(t)}{\Theta(t)}\leq u_0+\bar b(t-t_0).$$

Proof. Before a possible zero of Θ, the quotient u satisfies $\dot u = b-au^2\leq\bar b$. The function $U(t)=-2\bar b(t-t_0)$ is a subsolution because

$$b-aU^2\geq-\bar b-4\bar a\bar b^2\ell^2\geq-2\bar b=\dot U.$$

Indeed $(u-U)'\geq -a(u+U)(u-U)$, so an integrating factor and $u_0-U(t_0)\geq 0$ give $u\geq U$. The two bounds are uniform on all initial intervals before a zero. Lemma 3.5 then excludes that zero. This comparison is an independent short-interval explanation; the model for later stages below uses its own integral estimate. $\square$

3.6.2. *The first-stage steering model.* For the first layer the preceding background has no shear. Its phase gradient therefore has fixed length, and steering reduces to a scalar comparison. The compression chosen here is fixed independently of the long growth interval. This model also accounts for the exceptional first holding interval, whose contribution is bounded but not small.

The first model allows arbitrary data satisfying (3.24). In the construction we take $h=\eta_2$, $H=H_2$ and $\Lambda=\Lambda_1$, with $c_{\mathrm p}=3$ as in (3.9). Fix a smooth nondecreasing step η, equal to zero on $(-\infty,0]$ and one on $[1,\infty)$. In this subsection $h\in C_c^\infty((0,1))$ is any nonnegative profile of unit integral, and $H=\|h\|_\infty$. Thus $H\geq 1$; no prescribed upper bound on H is required. Choose fixed numbers

$$c_{\mathrm p}>1,\qquad \Lambda\geq 2c_{\mathrm p}H. \tag{3.24}$$

In particular $\Lambda>2$. The compression Λ is fixed independently of any subsequent growth length or frequency parameter.

For $0\leq\mu\leq c_{\mathrm p}$ set $\tau_b=1+\Lambda^{-1}$ and

$$z(\tau)=\begin{cases}1-\eta(\tau), & 0\leq\tau\leq 1,\\ -\mu\Lambda h(\Lambda(\tau-1)), & 1\leq\tau\leq\tau_b,\end{cases}\qquad P_{\tau\tau}=zP,\quad P(0)=P_\tau(0)=1. \tag{3.25}$$

Lemma 3.7 (First-stage steering with a short negative pulse)**.** *For the system and data* (3.25) *under* (3.24)*, the following assertions hold. For every trial* μ*,* $P > 0$ *on the whole steering interval. The quotient* $v = P_\tau/P$ *satisfies*

$$\frac{1}{1+\tau} \le v(\tau) \le 1 \quad (0 \le \tau \le 1), \qquad \log 2 \le \int_0^1 v\, d\tau \le 1. \tag{3.26}$$

Writing $v_d = v(1)$ *and* $y = \Lambda(\tau - 1)$*, the negative pulse multiplier obeys*

$$\frac{1}{2} \le \chi(y) := \frac{P(1 + y/\Lambda)}{P(1)} \le 1 + \frac{y}{\Lambda}, \qquad -4c_{\mathrm{p}} \le v(1 + y/\Lambda) \le v_d \le 1. \tag{3.27}$$

The endpoint map $\mu \mapsto v(\tau_b; \mu)$ *is smooth, strictly decreasing, positive at zero and negative at* c_{p}*. Its unique zero* μ_* *satisfies*

$$\frac{1}{2} - \frac{1}{4\Lambda} \le \mu_* \le 1, \qquad 0 \le v \le v_d \quad \text{on the selected negative pulse.} \tag{3.28}$$

For every trial, $0 \le \log P(\tau) \le 1 + \Lambda^{-1}$ *throughout the steering interval; at the selected endpoint,*

$$\log 2 \le \log P(\tau_b) = \int_0^{\tau_b} v\, d\tau \le 1 + \Lambda^{-1}. \tag{3.29}$$

Proof. The linear equation in (3.25) exists for every trial on the entire finite interval. On its first part $z \ge 0$. Apply the negative-quadrant assertion of Lemma 3.5 to $(-P, -P_\tau)$, with coefficients $a = 1$, $b = z$. This proves $P \ge 1$ and $P_\tau \ge 1$ before using their quotient. Now $v_\tau = 1 - \eta - v^2$. The constant 1 is a supersolution, and $v_\tau \ge -v^2$ gives $(1/v)_\tau \le 1$. This proves (3.26); in particular $v_d \in [1/2, 1]$ is independent of μ and $2 \le P(1) \le e$.

On the negative pulse the linear multiplier, defined without dividing by an evolving temperature, solves

$$\chi_{yy} + Q\chi = 0, \qquad Q = \frac{\mu h}{\Lambda} \le \frac{1}{2}, \qquad \chi(0) = 1, \quad \chi_y(0) = \frac{v_d}{\Lambda}. \tag{3.30}$$

Before a first possible zero, concavity gives $\chi \le 1 + y/\Lambda \le 2$. Hence $\chi_{yy} \ge -1$, and

$$\chi(y) \ge 1 + \frac{v_d}{\Lambda} y - \frac{y^2}{2} \ge \frac{1}{2}.$$

Continuity excludes a zero. This is the pointwise argument of Lemma 3.4 in the present normalization. The integral identity now gives

$$\chi_y(y) = \frac{v_d}{\Lambda} - \frac{\mu}{\Lambda} \int_0^y h(r)\chi(r)\, dr \ge -\frac{2c_{\mathrm{p}}}{\Lambda}.$$

Consequently $v = \Lambda\chi_y/\chi \ge -4c_{\mathrm{p}}$ is defined on the whole negative pulse. Its exact equation is

$$\frac{d}{dy} v(1 + y/\Lambda) = -\mu h(y) - \frac{v^2}{\Lambda} \le 0, \tag{3.31}$$

which proves the remaining upper bound in (3.27).

The coefficient in (3.30) is affine in μ, and $\chi \ge 1/2$ for every trial. Parameter dependence for the linear equation therefore gives smooth dependence of its quotient. Differentiating (3.31), with v_d independent of μ, yields

$$\partial_\mu v(\tau_b; \mu) = -\int_0^1 h(y) \exp\left(-\frac{2}{\Lambda} \int_y^1 v(1 + r/\Lambda; \mu)\, dr\right) dy < 0.$$

The profile is nonnegative and has unit mass, making the inequality strict. At $\mu = 0$ the endpoint equals $v_d/(1 + v_d/\Lambda) > 0$; at $\mu = c_{\mathrm{p}}$ it is at most $v_d - c_{\mathrm{p}} \le 1 - c_{\mathrm{p}} < 0$. The intermediate value theorem and strict monotonicity give exactly one root. At this root v decreases from v_d to zero, and integration gives

$$\mu_* = v_d - \frac{1}{\Lambda}\int_0^1 v(1+y/\Lambda)^2\,dy \in [v_d - v_d^2/\Lambda, v_d].$$

The function $x - x^2/\Lambda$ is increasing on $[1/2, 1]$ because $\Lambda > 2$. This proves (3.28).

On the first part $0 \le \log P(\tau) \le 1$. On every trial negative pulse,

$$\log P(1+y/\Lambda) = \int_0^1 v\,d\tau + \log\chi(y) \in [\log 2 - \log 2,\ 1 + \log(1+1/\Lambda)] \subset [0, 1+\Lambda^{-1}].$$

For the selected root, the negative pulse contribution is nonnegative and at most Λ^{-1}, proving (3.29). □

For the passage from the first model to physical time, choose $s \in (0, 1/8]$ with $c_{\mathrm{p}}\Lambda H \sin s \le 1/4$ and set $Z = \sin(s)z$ for (3.25). Define $\alpha = s - \arcsin Z$. For arbitrary $\lambda, \sigma > 0$ put $\Gamma = \sigma \sin s$, $a = \sigma^2 \sin s/\lambda$ and $b = \lambda Z$, where $\tau = \Gamma(t - t^1)$. Let the first-stage amplitude system $\dot\Theta = a\Omega$, $\dot\Omega = b\Theta$ have data

$$\Theta(t^1) = -P_{\mathrm{ent}} < 0, \qquad P_{\mathrm{ent}} > 0, \qquad \Omega(t^1) = \frac{\lambda}{\sigma}\Theta(t^1).$$

Its normalized multiplier is

$$P(\tau) = \frac{-\Theta(t^1 + \tau/\Gamma)}{P_{\mathrm{ent}}}. \tag{3.32}$$

Lemma 3.8 (From the first-stage model to the amplitude system)**.** *Under the preceding setup:*

(i) Angle bounds. *For every trial μ, the angle α is smooth, starts at zero and satisfies*

$$|Z| \le \tfrac14, \qquad \cos(s-\alpha) = \sqrt{1-Z^2} \ge \frac{\sqrt{15}}{4}, \qquad 0 \le \alpha < \tfrac12. \tag{3.33}$$

At the selected root μ_* of Lemma 3.7, Z, $\alpha - s$, v, and $P - P(\tau_b)$ are identically zero in a neighborhood of the endpoint after extension by the stationary continuation. Thus all time derivatives of positive order of the angle and the amplitude pair vanish at the endpoint.

(ii) Physical normalization. *For every trial the multiplier (3.32) is exactly the solution in (3.25), and $v = \sigma\Omega/(\lambda\Theta)$. If this steering is preceded by an exact transition on $[t^1 - \Gamma^{-1}, t^1]$ with $Z = \sin s$ and the same growing eigenline, that transition contributes exactly 1 to the logarithmic gain. The gain from its left endpoint is then in $[1 + \log 2,\ 2 + \Lambda^{-1}]$ for the selected trial, and in $[1,\ 2 + \Lambda^{-1}]$ for every trial. In the first-stage construction this left endpoint is the growth threshold crossing. If the preceding growth has length $L_{\mathrm{growth}}/\Gamma$, the duration through the end of steering is exactly*

$$\frac{L_{\mathrm{growth}} + 2 + \Lambda^{-1}}{\Gamma}. \tag{3.34}$$

Proof. The two pieces of z have disjoint interiors, and $c_{\mathrm{p}}\Lambda H \ge 1$. Hence $Z \le \sin s$ and $|Z| \le c_{\mathrm{p}}\Lambda H \sin s \le 1/4$. The principal arcsine is smooth there and gives (3.33). Indeed $\arcsin x \le x/\sqrt{1-x^2} \le 4x/3$ for $0 \le x \le 1/4$, so $0 \le \alpha \le s + \arcsin(1/4) \le 1/8 + 1/3 < 1/2$. In particular $\cos\alpha \ge 1 - \alpha^2/2 > 7/8$ as well. Differentiation gives $\alpha_\tau = -Z_\tau/\cos(s-\alpha)$, the feedback equation for this first-stage model. Its right-hand side is smooth where the cosine is positive, so uniqueness for this scalar equation identifies the displayed angle with the control starting at $\alpha(0) = 0$. The profile joins are smooth because the step is flat at its endpoints and h is supported in the interior. All derivatives of each fixed order are bounded by finite functions of the fixed design and profile norms.

Near the right endpoint, $Z = 0$ by the support of h. There $P_{\tau\tau} = 0$; at a selected root $P_\tau(\tau_b) = 0$, so $P_\tau = 0$ and P is constant on that neighborhood. The angle equals s there. These constants continue under the same equations with $Z = 0$, giving all the stated endpoint values and derivatives. Positivity of P, rather than positivity of the quotient on every trial negative pulse, is what permits this discussion.

In physical time $ab = \Gamma^2 z$ and a is constant. Hence $\Theta_{\tau\tau} = z\Theta$. The multiplier (3.32) has

$$P(0) = 1, \qquad P_\tau(0) = -\frac{a\Omega(t^1)}{\Gamma P_{\mathrm{ent}}} = \frac{a\lambda}{\Gamma\sigma} = 1.$$

Uniqueness identifies it with the normalized model before any division by the evolving temperature. Lemma 3.7 then gives $\Theta < 0$ on the full steering interval, and $\partial_\tau \log(-\Theta) = \sigma\Omega/(\lambda\Theta) = v$. The growing eigenline has rate Γ when $Z = \sin s$. The transition therefore multiplies its amplitude by e, independently of the steering normalization $P(0) = 1$. Adding that gain to Lemma 3.7 proves the two intervals. Adding the growth, transition, first-part and compressed pulse durations gives (3.34). In these variables the all-trial vorticity identity is $|\Omega| = (\lambda/\sigma)|v|(-\Theta)$; the absolute value on v is needed for trial pulses that carry the ratio below zero. □

3.6.3. *The reduced model for later stages.* The model for later stages includes the response of the preceding layer. Its three unknowns represent the length of the new phase gradient, the preceding vorticity amplitude divided by the growth scale, and a normalized ratio of the new amplitudes. Their precise identification in the controlled system is made in Subsection 3.7.6. The negative pulse changes the last component's endpoint sign; the middle component supplies the positive vorticity amplitude left behind. The geometric estimates must therefore precede estimates using the amplitude ratio.

In this subsection the negative pulse profile is again denoted by $h \in C_c^\infty((0,1))$, $h \geq 0$, $\int_0^1 h = 1$, independently of the profile used earlier in the construction. Set $m_2(y) = \int_0^y h(r)\,dr$ and keep the same smooth step η. The small parameter δ below measures coefficient errors. The length of the negative pulse in τ is L^{-1}.

Fix $c_{\mathrm{p}} > \bar{v} > 3/2$. The comparison will concern the following system, data and coefficient assumptions. Let $\delta > 0$, $L \geq 1$, $L^{-1} \leq \delta$, $0 \leq \mu \leq c_{\mathrm{p}}$, and $0 < \tau' \leq \tau_b = 1 + L^{-1}$. Suppose c_1, c_2, c_3, c_5 are continuous on $[0, \tau']$ with $|c_i - 1| \leq \delta$, and c_4 is a constant with $|c_4 - 1| \leq \delta$. Suppose d_1, d_2, d_4 are continuous on that interval with $|d_i| \leq \delta$, and d_3 is continuous on $[0, \min(1, \tau')]$ with $|d_3| \leq \delta$. Let $\chi_1 \in [1-\delta, 1+\delta]$ and let $\chi, W, v \in C^1([0,\tau'])$ solve the explicit system

$$\chi' = c_1 W + d_1, \tag{3.35}$$

$$W' = c_2 \mathsf{g}/\chi + d_2, \tag{3.36}$$

$$v' = \mathsf{f} - \widetilde{k} v^2, \qquad \widetilde{k} = c_5 \chi_1/\chi^2 + d_4, \tag{3.37}$$

where primes mean τ derivatives and

$$\begin{array}{c|c|c} & 0 \leq \tau \leq 1 & 1 \leq \tau \leq \tau_b \\ \hline \mathsf{g} & \eta(\tau) + d_3(\tau)(1-\eta(\tau)) & 1 + \mu c_3 L \chi_1 h(L(\tau-1)) \\ \mathsf{f} & (c_4/\chi_1)(1-\eta(\tau)) & -\mu L h(L(\tau-1)) \end{array} \tag{3.38}$$

on the portions present in $[0, \tau']$. The two definitions agree at $\tau = 1$. Assume the following initial data and bounds on χ and W:

$$\begin{gathered} \chi(0) = \chi_1, \quad |W(0)| \leq \delta, \quad v(0) \in [1-\delta, 1+\delta], \\ \frac{1}{2} \leq \frac{\chi}{\chi_1} \leq 2, \qquad -1 \leq W \leq 2 + c_{\mathrm{p}} \quad (0 \leq \tau \leq \tau'). \end{gathered} \tag{3.39}$$

The interval bounds involve only χ and W; the ratio v will be estimated from its equation. The constants in the comparison depend only on $c_{\mathrm{p}}, \bar{v}$, uniformly over the profiles with the stated sign, endpoint and unit-mass properties, the trial amplitude, the interval endpoint and coefficients satisfying these bounds.

Lemma 3.9 (Steering comparison for later stages)**.** *For the profiles, coefficients, system and data in Subsection 3.6.3, including the assumed bounds* (3.39)*, there are finite* $C = C(c_{\mathrm{p}}, \bar{v}) \geq 1$ *and* $\delta_0 = \delta_0(c_{\mathrm{p}}, \bar{v}) > 0$ *such that, whenever* $0 < \delta \leq \delta_0$*, the following conclusions hold on the supplied interval* $[0, \tau']$*.*

(i) First part. *On* $[0, \min(1, \tau')]$*,*

$$-C\delta \leq W \leq \tau + C\delta, \qquad 1 - C\delta \leq \chi/\chi_1 \leq (1 + C\delta)(1 + \tau^2/2),$$
$$\widetilde{k} \geq (1 - C\delta)(1 + \tau^2/2)^{-2}, \qquad \frac{1 - C\delta}{1 + \tau} \leq v < \bar{v}. \tag{3.40}$$

(ii) Whole interval. *On* $[0, \tau']$*,*

$$1 - C\delta \leq \chi/\chi_1 \leq \tfrac{3}{2} + C\delta, \qquad -C\delta \leq W \leq 1 + c_{\mathrm{p}} + C\delta,$$
$$0 < \widetilde{k} \leq 1 + C\delta, \qquad |v| \leq C. \tag{3.41}$$

The improved intervals for χ/χ_1 *and* W *in* (3.41) *are strictly contained in the intervals assumed in* (3.39)*.*

(iii) Negative pulse. *On the portion with* $\tau \geq 1$*, with* $y = L(\tau - 1)$ *and* $v_d = v(1)$*,*

$$W' > 0, \qquad W \leq 1 + \mu m_2(y) + C\delta, \qquad v' \leq 0,$$
$$v_d - \mu m_2(y) - C\delta \leq v \leq v_d,$$
$$\left| \int_1^{\tau} \widetilde{k} v \, dr \right| \leq C\delta, \qquad \int_1^{\tau} \widetilde{k} v^2 \, dr \leq C\delta. \tag{3.42}$$

(iv) Integrated bounds. *Including when* $\tau' < 1$*,*

$$\int_0^{\tau'} |W| \, d\tau \leq \tfrac{1}{2} + C\delta, \qquad \int_0^{\tau'} \widetilde{k} \, d\tau \leq 1 + C\delta, \qquad \int_0^{\tau'} |\mathsf{f}| \, d\tau \leq 1 + \mu + C\delta. \tag{3.43}$$

(v) Endpoint signs. *For* $\tau' = \tau_b$*, one has*

$$v(\tau_b) \in [v_d - \mu - C\delta, \, v_d - \mu]. \tag{3.44}$$

In particular the endpoint is positive for $\mu = 0$ *and negative for* $\mu = c_{\mathrm{p}}$*.*

(vi) Selected endpoint and gain. *If* $\tau' = \tau_b$ *and* $v(\tau_b) = 0$*, then* $v \geq 0$ *on the negative pulse,*

$$\tfrac{1}{2} - C\delta \leq \mu \leq \bar{v}, \qquad \tfrac{1}{3} - C\delta \leq W(\tau_b) \leq 1 + \bar{v} + C\delta, \tag{3.45}$$

and its total logarithmic model gain satisfies

$$(1 - C\delta) I_0 \leq \int_0^{\tau_b} \widetilde{k} v \, d\tau \leq (1 + C\delta)\bar{v}, \qquad I_0 = \int_0^1 \frac{d\tau}{(1 + \tau^2/2)^2 (1 + \tau)}, \quad \frac{2}{9} \leq I_0 \leq \log 2. \tag{3.46}$$

Proof. We first bound W and χ independently of v, then estimate v on the first part and negative pulse, and finally obtain the endpoint and integral bounds. All estimates are uniform in the endpoint of any existing solution satisfying (3.39). Throughout, $\delta \leq 1/10$; we enlarge $C = C(c_{\mathrm{p}}, \bar{v})$ and decrease δ_0 finitely many times, keeping each lower bound $1 - C\delta$ at least $1/2$.

First derive a lower bound for W, then one for χ. The assumptions $\chi \geq \chi_1/2$ and $\chi_1 \geq 1 - \delta$ give $\chi \geq (1 - \delta)/2$, and on the first part $\mathsf{g} \geq -\delta$. Thus (3.36) gives $W' \geq -C\delta$ and $W \geq -C\delta$. By (3.35), $\chi' \geq -C\delta$, hence $\chi \geq \chi_1 - C\delta\tau$ and $\chi/\chi_1 \geq 1 - C\delta$. In particular $\chi \geq 1 - C\delta$.

For the upper bounds, put $E(\tau) = \int_0^{\tau} \eta \leq \tau$. The improved lower bound for χ and $\mathsf{g} \leq \eta + \delta$ imply

$$W' \leq (1 + C\delta)\eta + C\delta, \qquad W \leq (1 + C\delta)E(\tau) + C\delta \leq \tau + C\delta.$$

Using this in (3.35) gives $\chi' \le \tau + C\delta$, so

$$\frac{\chi}{\chi_1} \le 1 + \frac{\tau^2/2 + C\delta\tau}{\chi_1} \le (1+C\delta)(1+\tau^2/2).$$

This proves the first geometric bounds without any information about v.

Using $\chi \le 2\chi_1$, $\chi_1 \le 1+\delta$, $c_5 \ge 1-\delta$ and $d_4 \ge -\delta$ in (3.37) shows that the quadratic coefficient is positive for small δ:

$$\widetilde{k} \ge \frac{1-\delta}{4(1+\delta)} - \delta > 0.$$

The improved lower bound for χ/χ_1 gives $\widetilde{k} \le 1 + C\delta$. On the first part its upper bound gives

$$\frac{c_5\chi_1}{\chi^2} \ge (1-C\delta)(1+\tau^2/2)^{-2}.$$

The term $d_4 \ge -\delta$ is absorbed in the same form because $(1+\tau^2/2)^{-2} \ge 4/9$. This proves the asserted lower bound for $\widetilde{k}$.

Now $v(0) \ge 1-\delta > 0$ and the first-part forcing is nonnegative. On each initial interval where $v > 0$,

$$(1/v)' = -\mathsf{f}/v^2 + \widetilde{k} \le 1 + C\delta, \qquad \frac{1}{v} \le (1-\delta)^{-1} + (1+C\delta)\tau \le (1+C\delta)(1+\tau).$$

The resulting positive lower bound excludes a first zero by continuity. For the upper barrier choose δ so $1+\delta < \bar{v}$. At a first contact with $\bar{v}$, (3.37) would give

$$v' \le 1 + C\delta - \tfrac{4}{9}(1-C\delta)\bar{v}^2 \le 1 - \tfrac{4}{9}\bar{v}^2 + C\delta(1+\bar{v}^2) < 0,$$

because $\bar{v} > 3/2$. This contradicts first contact from below. It also proves the first-part gain bounds

$$(1-C\delta)I_0 \le \int_0^1 \widetilde{k}v\,d\tau \le (1+C\delta)\bar{v} \quad \text{when } \tau' \ge 1. \tag{3.47}$$

The integrand defining I_0 is at least $2/9$ and at most $(1+\tau)^{-1}$, proving the stated bounds without evaluating the integral.

Suppose next that the interval reaches the negative pulse. The forcing of the preceding layer satisfies $\mathsf{g} \ge 1$. Using $\chi \le 2\chi_1 \le 2(1+\delta)$, $c_2 \ge 1-\delta$ and $d_2 \ge -\delta$ in (3.36) gives

$$W' \ge \frac{1-\delta}{2(1+\delta)} - \delta > 0.$$

Hence $W \ge W(1) \ge -C\delta$, then $\chi' \ge -C\delta$. The negative pulse has length at most δ, so $\chi \ge \chi(1) - C\delta^2$ and $\chi/\chi_1 \ge 1 - C\delta$. The previously established positivity of $\widetilde{k}$ and its upper bound $1 + C\delta$ therefore hold on the negative pulse as well. Splitting g into its constant and pulse terms in (3.36), then integrating over $[1,\tau]$, gives

$$\begin{aligned} W(\tau) - W(1) &= \int_1^\tau \left(\frac{c_2}{\chi} + d_2\right) dr + \mu \int_1^\tau c_2 c_3 \frac{\chi_1}{\chi} L h(L(r-1))\,dr \\ &\le C\delta + (1+C\delta)\mu m_2(y). \end{aligned} \tag{3.48}$$

This proves $W \le 1 + \mu m_2(y) + C\delta$. Reinserting it into (3.35) and integrating over a length at most δ gives $\chi/\chi_1 \le 3/2 + C\delta$. Notice the order of these estimates: the lower bound for the preceding vorticity amplitude gives the lower bound for χ. The resulting positive denominator then gives the upper vorticity-amplitude bound, followed by the upper bound for χ.

It remains to control v on the negative pulse, without assuming a bound for it. Write $V(y) = v(1 + y/L)$. Positivity of $\widetilde{k}$ gives $V_y \le 0$, and the exact integral equation is

$$V(y) = v_d - \mu m_2(y) - \frac{1}{L}\int_0^y \widetilde{k}(1 + r/L)V(r)^2\,dr. \tag{3.49}$$

Take the temporary bound $|V| \le 2B$, where $B = 1 + c_{\mathrm{p}} + \bar{v}$. It holds with a strict inequality initially. On any initial interval where it holds, $\widetilde{k} \le 2$ gives

$$-c_{\mathrm{p}} - 8B^2\delta \le V \le v_d < \bar{v} < B.$$

Choosing $8B^2\delta < B$ makes the lower bound greater than $-2B$. A first exit is therefore impossible. Thus $|V| \le 2B$ on every existing negative pulse interval, and substituting this back into (3.49) gives the two-sided negative pulse bound and $\int_1^\tau \widetilde{k}v^2 \le C\delta$. The same bound and the negative pulse length give $|\int_1^\tau \widetilde{k}v| \le C\delta$.

For a full interval, $m_2(1) = 1$ in (3.49) gives (3.44). At $\mu = 0$ it is at least $1/2 - C\delta > 0$, and at $\mu = c_{\mathrm{p}}$ it is at most $\bar{v} - c_{\mathrm{p}} < 0$. If its value is zero, the nonincreasing v is nonnegative on the negative pulse and

$$\mu = v_d - \int_1^{\tau_b} \widetilde{k}v^2\, d\tau \in [\tfrac{1}{2} - C\delta, \bar{v}].$$

To obtain the positive lower bound for $W(\tau_b)$, we use the integrated equation (3.48), not only the sign of v. For its pulse term, use $\chi/\chi_1 \le 3/2 + C\delta$, $c_2c_3 \ge (1-\delta)^2$. The first integral is at least $-\delta/L \ge -\delta^2$, so

$$W(\tau_b) \ge -C\delta - \delta^2 + \frac{(1-\delta)^2\mu}{3/2 + C\delta} \ge \tfrac{2}{3}\mu - C\delta \ge \tfrac{1}{3} - C\delta.$$

The upper bound for the preceding vorticity amplitude follows from (3.48) and $\mu \le \bar{v}$. For a parameter with $v(\tau_b) = 0$ the negative pulse gain lies between zero and $(1 + C\delta)\bar{v}/L$. Adding it to (3.47) and enlarging C proves (3.46).

The first-part bounds give $|W| \le \tau + C\delta$, and the negative pulse has bounded W on a length at most δ. Integration gives the first inequality of (3.43). The second follows from $\widetilde{k} \le 1 + C\delta$ and $\tau' \le 1 + \delta$. The first-part forcing has integral at most $(1+\delta)/(1-\delta) \le 1 + C\delta$, and the negative pulse forcing has absolute integral at most $\mu \int_0^1 h = \mu$. This proves the third.

Finally take C larger than the finitely many constants used above and than $2B$. Choose $\delta_0 > 0$ small enough to meet all preceding denominator bounds and to have $C\delta_0 < 1/4$, $1 + \delta_0 < \bar{v}$, $8B^2\delta_0 < B$, and

$$C\delta_0(1 + \bar{v}^2) < \tfrac{1}{2}(4\bar{v}^2/9 - 1).$$

At zero error, all these inequalities have positive gaps. Taking the minimum of the finitely many positive thresholds gives such a δ_0, depending only on the fixed pair $c_{\mathrm{p}}, \bar{v}$. The final condition $C\delta_0 < 1/4$ also places $[1 - C\delta, 3/2 + C\delta]$ strictly inside $(1/2, 2)$ and $[-C\delta, 1 + c_{\mathrm{p}} + C\delta]$ strictly inside $(-1, 2 + c_{\mathrm{p}})$. Thus the bounds for χ/χ_1 and W strictly improve those assumed in (3.39), completing all asserted a priori estimates. ☐

The preceding estimates have two distinct uses: they justify division by the temperature component on an existing interval, and they select a pulse amplitude from a fully constructed family. The proposition keeps the family-existence hypothesis explicit. The selection uses continuity and a compact zero set; no unique zero or smooth dependence of the selected zero is asserted.

Proposition 3.10 (Amplitude ratios and selection of a zero)**.** *Retain the profiles, coefficient bounds and parameter ranges of Subsection 3.6.3, with $0 < \delta \le \delta_0(c_{\mathrm{p}}, \bar{v})$ from Lemma 3.9. Suppose χ, W solve their equations (3.35)–(3.36) on $[0, \tau']$, with their initial data and bounds in (3.39). No amplitude quotient is assumed to exist on the whole interval at this point. Let (Θ, Ω) solve a linear system*

$$\Theta' = \mathfrak{a}\Omega, \qquad \Omega' = \mathfrak{b}\Theta$$

with continuous coefficients there, and $\Theta(0) < 0$. For a constant $c_0 > 0$ suppose $v = c_0\Omega/\Theta$ has the initial value $v(0) \in [1 - \delta, 1 + \delta]$ from (3.39) and obeys (3.37) wherever it is defined on an initial interval. Then $\Theta < 0$ on all of $[0, \tau']$, the quotient is C^1 there, and every conclusion of Lemma 3.9 holds. No sign assumptions on $\mathfrak{a}, \mathfrak{b}$ are needed for this implication.

If, in addition, a full family on $[0,\tau_b]$ *for* $\mu \in [0,c_{\mathrm{p}}]$ *is supplied satisfying these hypotheses and depending continuously on* μ*, its endpoint quotient map has a nonempty compact zero set in* $(0,c_{\mathrm{p}})$*. It has a least zero, and every family member with* $v(\tau_b)=0$ *has the endpoint and gain bounds of* (3.45)*–*(3.46)*. This statement asserts neither monotonicity of the endpoint map nor uniqueness of its root.*

Proof. On each initial interval preceding a possible first zero of Θ, Lemma 3.9 gives the same finite bound $|v| \le C$. Thus $|\Omega/\Theta| \le C/c_0$ independently of its endpoint. Lemma 3.5 excludes a zero on the whole supplied interval. The quotient and its equation extend to its endpoint by continuity, so the model estimates apply there. Equivalently, before that endpoint,

$$|\Theta(\tau)| \ge |\Theta(0)| \exp\left(-\frac{C}{c_0}\int_0^{\tau'} |\mathfrak{a}(r)|\, dr\right) > 0.$$

This lower bound is relative to the given initial amplitude; no stage-independent positive lower bound for its absolute size is claimed.

For the full continuous family the endpoint quotient is continuous because its denominator does not vanish. Its opposite signs at 0 and c_{p} give a zero by the intermediate value theorem. The zero set is closed in the compact parameter interval and excludes both endpoints, so it has a least element in its interior. The model lemma applies to each such family member. The proposition assumes existence on the supplied interval of the geometry, coefficients and every other variable entering their equations. For the controlled construction, that existence is established in Section 3.7 before this proposition is applied. ☐

Lemma 3.11 (Deformation from a transported phase gradient)**.** *Let* $0 < m_\zeta \le M_\zeta$ *be fixed bounds and let* $t \ge t_0$*. For real* 2×2 *matrices* D, M*, write* $D_{\mathrm{sym}} = (D+D^T)/2$*. Suppose* $\dot M = DM$*,* $M(t_0)=\mathrm{Id}$*,* $\operatorname{tr} D = 0$*, and* $\zeta = M^{-T}p$ *for* $|p|=1$ *on* $[t_0,t]$*. If* $m_\zeta \le |\zeta(s)| \le M_\zeta$ *throughout that interval, then*

$$\max\big(\|M(t)\|, \big\|M(t)^{-1}\big\|\big) \le M_\zeta \left(1 + 2m_\zeta^{-2}\int_{t_0}^{t} \|D_{\mathrm{sym}}(s)\|\, ds\right) + m_\zeta^{-1}.$$

Proof. For $|\bar p|=1$ let $\xi = M^{-T}\bar p$ and decompose $\xi = h\zeta + bJ\zeta/|\zeta|^2$. Since $\det M = 1$, $b = \det(\zeta,\xi) = \det(p,\bar p)$ is constant and $|b|\le 1$. Differentiating $h = \xi\cdot\zeta/|\zeta|^2$ gives

$$\dot h = -2b\frac{J\zeta\cdot D_{\mathrm{sym}}\zeta}{|\zeta|^4}.$$

Thus $|h(t)| \le 1 + 2m_\zeta^{-2}\int_{t_0}^{t}\|D_{\mathrm{sym}}\|$. The decomposition gives the displayed bound for $|\xi|$, uniformly in $\bar p$. Finally the two singular values of a two-dimensional determinant-one matrix are reciprocal, so $\|M\| = \big\|M^{-1}\big\|$. This proves the claimed polynomial bound. ☐

The comparison lemmas supply nonvanishing and steering gains; Lemma 3.11 converts their phase-length bounds into deformation bounds. These local estimates are used in the finite-stage argument that follows.

3.7. **Proof of the controlled-stage theorem.** We prove the theorem by induction on finite sequences of stages. The schedule estimates first supply bounds without using future growth scales. The closed system then gives the first stage and the estimates for older layers on a new stage. We end growth at a transversal crossing of the amplitude threshold, construct the steering evolution for every trial pulse amplitude, and select a zero of the new vorticity amplitude. Only after completing this finite-stage induction do we sum stage lengths and obtain lifetime deformation and propagator bounds. We check smoothness at the joins last, so no terminal regularity is used in constructing the stages. For reference, the older-layer and growth estimates, combined with the shooting gains, give parts (ii)–(iii) and the activation bound in part (v). The shooting argument also proves selection and (3.19); completion of the induction gives part (iv) and the lifetime assertions. The remaining parts of the proof establish the control-speed bound, the propagator estimate and smooth joining. Until the new endpoint is constructed, the only growth scales available are $\sigma_0, \dots, \sigma_{q-1}$.

3.7.1. *Scale inequalities for finite sequences of stages.* This first set of scale inequalities supports the low-order control argument. The later scale comparisons in the derivative section control differentiated equations, and the parameter section combines both sets with support and force estimates. Keeping these roles separate shows which quantities may already be used at each stage.

For later shear estimates set $h_2 = 0$ and $h_j = s_{j-1}\sigma_{j-2}$ for $j \geq 3$. The definitions of the growth rate give, for $j \geq 2$,

$$\Gamma_j = L_j \sigma_{j-2} \frac{\sin s_j}{s_j}, \qquad 1 - s_j^2/6 \leq \frac{\sin s_j}{s_j} \leq 1. \tag{3.50}$$

The following elementary estimate makes the thresholds uniform over the logarithms that occur in the schedule.

Lemma 3.12 (Polynomial-exponential absorption). *For any finite family of fixed numbers $C > 0$, $a \geq 0$, $b > 0$, there is a finite y_0 such that all its members satisfy*

$$C x^a e^{-bx} \leq e^{-bx/2} \qquad (x \geq y_0). \tag{3.51}$$

Proof. For each member, $\log C + a \log x - bx/2$ tends to $-\infty$ and is nonincreasing for $x \geq 2a/b$. Choose y_0 after its last positive value and take the maximum over the finite family. Fixed multipliers and positive tolerances may be included in C. □

In the next lemma use the frequencies, radii and scales $Q_j, \lambda_j, \ell_j, \Pi_j$ from (3.7), and the logarithmic gains L_j from (3.8) for arbitrary $0 \leq k_j \leq Q_j/120$. The insertion angles and growth rates are (3.10), with $s_1 = s_*$ and $\Gamma_1 = \sin s_*$. Recall $\varpi_j = L_j s_j$ for $j \geq 2$, $X_1 = \Lambda_1 \Gamma_1$ and $X_j = L_j \Gamma_j$ for $j \geq 2$. The fixed bounds, positive gaps and tolerance are those in (3.14)–(3.16).

Lemma 3.13 (Parameter estimates for finitely many stages). *For the fixed ϵ_{sch}, there is a finite threshold on λ_1, independent of the admissible derivative orders, with the following property. For any $q \geq 2$, let $\sigma_1, \dots, \sigma_{q-1}$ be arbitrary positive numbers satisfying the two-sided growth-scale bounds in (3.18), with $\sigma_0 = 1$. The following groups hold in their displayed index ranges. The logarithmic gains satisfy*

$$L_j^{-1} \leq \epsilon_{\mathrm{sch}} \quad (1 \leq j \leq q), \qquad 100 \leq L_j / L_{j-1} \leq Q_j^2 \quad (2 \leq j \leq q). \tag{3.52a}$$

The insertion angles and their finite tails satisfy

$$\begin{gathered} s_j \leq \epsilon_{\mathrm{sch}}, \quad L_j s_j \leq \epsilon_{\mathrm{sch}}, \quad L_j^4 \sigma_{j-2}^2 / \sigma_{j-1} \leq \epsilon_{\mathrm{sch}}, \quad \sigma_{j-2}/\sigma_{j-1} \leq \epsilon_{\mathrm{sch}} \quad (2 \leq j \leq q), \\ L_j s_j \leq \epsilon_{\mathrm{sch}} s_{j-1} \quad (3 \leq j \leq q), \qquad \sum_{j=k+1}^{q} s_j \leq \epsilon_{\mathrm{sch}} s_k \quad (1 \leq k < q). \end{gathered} \tag{3.52b}$$

The fixed weights required by the cone and transition estimates obey

$$\begin{gathered} 2 C_\vartheta s_2 / s_* \leq \epsilon_{\mathrm{sch}}, \\ 2(C_\varphi + C_\vartheta) \varpi_j / s_{j-1} \leq \epsilon_{\mathrm{sch}}, \qquad C_S s_{j-1} \leq \epsilon_{\mathrm{sch}} \quad (3 \leq j \leq q). \end{gathered} \tag{3.52c}$$

Growth-scale separation and summation give

$$\begin{gathered} \sigma_j \geq \max(\epsilon_{\mathrm{sch}}^{-1} \sigma_{j-1}, \sigma_{j-1}^3) \quad (1 \leq j \leq q-1), \\ \sum_{i=0}^{j} \sigma_i^\nu \leq (1 + 2\epsilon_{\mathrm{sch}}) \sigma_j^\nu \quad (0 \leq j \leq q-1, \ \nu = 1, 2). \end{gathered} \tag{3.52d}$$

The shear estimates use

$$h_jL_j/\Gamma_j \le 2\epsilon_{\rm sch} \quad (3 \le j \le q),$$

$$\sum_{i=1}^{j-2}\sigma_i s_{i+1} \le 2h_j, \quad \sum_{i=1}^{j-1}\sigma_i s_{i+1}^2 \le 2\epsilon_{\rm sch}L_2^{-2}, \quad \sum_{i=1}^{j-1}\sigma_i^2 s_{i+1} \le 2\sigma_{j-1}^2 s_j \quad (2 \le j \le q). \tag{3.52e}$$

Finally, the normalized base and older gradients satisfy

$$1 \le \epsilon_{\rm sch}\lambda_1^{1/8}, \qquad (\sigma_{j-1}^2 s_j)^{-1} \le \epsilon_{\rm sch}/L_j, \quad 2\sigma_{j-1}^2 \le \epsilon_{\rm sch}\lambda_j^{1/8} \quad (2 \le j \le q). \tag{3.52f}$$

Empty sums are zero; the shear sums begin at 1. *In addition, the threshold can ensure*

$$1,\ C_\Pi\sigma_{j-1}^2,\ \ell_j^{-1},\ \Gamma_j^{\pm 1} \le \Pi_j, \quad \Pi_j \le \lambda_j^{5/Q_j} \quad (1 \le j \le q). \tag{3.53a}$$

The control rates obey

$$\Gamma_j \le X_j \quad (1 \le j \le q), \qquad X_j \le \epsilon_{\rm sch}X_{j+1}, \quad X_{j+1} \le \epsilon_{\rm sch}\Pi_j \quad (1 \le j < q). \tag{3.53b}$$

The spatial radius and first lifetime have the majorants

$$\sigma_{j-1}\lambda_{j-1}^{-3} \le 1/4 \quad (2 \le j \le q), \qquad \frac{L_1 + 2 + \Lambda_1^{-1}}{\Gamma_1} + C_T \le \Pi_1. \tag{3.53c}$$

The first-layer statements use $\Pi_1 = (\log\lambda_1)^{10}$. *The comparison with* X_{j+1} *uses the bounds for* σ_j; *it is asserted only after those bounds are available.*

Proof. Write $y = \log\lambda_1$. We include the following two fixed-data conditions in the low-order threshold:

$$c_\sigma e^{y/8} \ge 16\epsilon_{\rm sch}^{-2}, \qquad 5y \ge C_K. \tag{3.54}$$

We also take $y \ge \max\{(Q_*+3)^3, \epsilon_{\rm sch}^{-1}\}$ and enlarge it using Lemma 3.12 for the finite families below. In particular $20\log x \le x$ for $x \ge y$. For the gain group (3.52a), recall from (3.8) that $L_j = (k_j + 41/8)\log\lambda_j$ with $0 \le k_j \le Q_j/120$. This gives $L_j \le Q_j\log\lambda_j$, $Q_j \le \log\lambda_{j-1}$ for $j \ge 2$, and $Q_j \le \log\lambda_{j-2}$ for $j \ge 3$. Indeed $\log\lambda_{j-2} \ge 201^{j-3}y \ge Q_* + j$; the first two indices are checked directly. The minimum and maximum log-gains give

$$\frac{L_j}{L_{j-1}} \ge \frac{(41/8)Q_j}{Q_{j-1}/120 + 41/8} \ge 100, \qquad \frac{L_j}{L_{j-1}} \le Q_j^2.$$

Both gain-ratio comparisons start at $j = 2$. The lower comparison is increasing as a function of $Q_{j-1} \ge 201$. All L_j^{-1} are at most $1/(5y)$, proving (3.52a).

The growth-scale bounds imply $\sqrt{c_\sigma}\lambda_j^{1/16} \le \sigma_j \le \sqrt{C_\sigma}\lambda_j^{1/16}$. Comparing consecutive powers proves the two growth-scale separations for $j \ge 2$; the case $j = 1$ follows from (3.54). We also require $\sigma_{j-1}/\sigma_j \le \epsilon_{\rm sch}/4$ for $1 \le j \le q-1$. Their geometric sums give the growth-scale sum bounds in (3.52d), using $(1-\epsilon_{\rm sch})^{-1} \le 1 + 2\epsilon_{\rm sch}$ for $\epsilon_{\rm sch} \le 1/2$.

For $j \ge 3$ put $x = \log\lambda_{j-2} \ge y$. Then $L_j \le x^4$, $\sigma_{j-2} \ge \sqrt{c_\sigma}e^{x/16}$ and $\sigma_{j-1} \ge \sigma_{j-2}^3$, so

$$s_j \le x^4e^{-x/8}/c_\sigma, \quad L_js_j \le x^8e^{-x/8}/c_\sigma, \quad \frac{L_j^4\sigma_{j-2}^2}{\sigma_{j-1}} \le x^{16}e^{-x/16}/\sqrt{c_\sigma}.$$

At $j = 2$ each of these is bounded by $C(Q_*+3)^8y^4e^{-y/16}$; here $L_2 \le Q_2^2y$ and $\sigma_0 = 1$. Equation (3.51) makes these bounds uniformly small once a single threshold is imposed, including their fixed weighted versions.

For the relative angles, with $2 \le j < q$,

$$\frac{s_{j+1}}{s_j} \le \frac{Q_{j+1}^2}{\sigma_{j-1}}, \qquad \frac{L_{j+1}s_{j+1}}{s_j} = \frac{L_{j+1}^2\sigma_{j-1}^2}{L_j\sigma_{j-2}\sigma_j} \le \frac{L_{j+1}^4\sigma_{j-1}^2}{\sigma_j}.$$

We require the first ratio at most $\epsilon_{\rm sch}/2$, treating $s_2/s_1=s_2/s_*$ separately. Summing gives the finite angle tails. The second gives $L_{j+1}s_{j+1}\le\epsilon_{\rm sch}s_j$. The same bounds, with fixed multipliers, ensure $2C_\vartheta s_2/s_*\le\epsilon_{\rm sch}$ and, for $3\le j\le q$, $2(C_\varphi+C_\vartheta)\varpi_j/s_{j-1}\le\epsilon_{\rm sch}$ and $C_Ss_{j-1}\le\epsilon_{\rm sch}$. This proves (3.52b) and (3.52c). The h_j identity and the sine comparison give $h_jL_j/\Gamma_j=s_{j-1}s_j/\sin s_j\le 2s_{j-1}$. The successive terms of σ_is_{i+1} and $\sigma_i^2s_{i+1}$ increase by a factor at least $1/\epsilon_{\rm sch}$: for example their reverse ratios are respectively $L_i\sigma_{i-2}/(L_{i+1}\sigma_{i-1})$ and $L_i\sigma_{i-2}/(L_{i+1}\sigma_i)$. For the opposite sum, the terms $\sigma_is_{i+1}^2$ decrease:

$$\frac{\sigma_{i+1}s_{i+2}^2}{\sigma_is_{i+1}^2}=\left(\frac{L_{i+2}}{L_{i+1}}\right)^2\frac{\sigma_i^3}{\sigma_{i-1}^2\sigma_{i+1}}.$$

The two-sided frequency-power bounds for σ_j above, together with (3.7), bound this ratio by $CQ_{i+2}^4\lambda_i^{-(Q_{i+1}-3)/16}$. Since $Q_{i+2}\le\log\lambda_i$ and $Q_{i+1}\ge 202$, (3.51) makes it at most $1/2$. Its first term is $\sigma_1s_2^2=L_2^2/\sigma_1$. The required bound $\sigma_1s_2^2\le\epsilon_{\rm sch}L_2^{-2}$ is precisely $L_2^4/\sigma_1\le\epsilon_{\rm sch}$, the $j=2$ estimate already proved. The decreasing sum is therefore at most $2\epsilon_{\rm sch}L_2^{-2}$, completing (3.52e).

The base-gradient ratio is exactly $(\sigma_{j-1}^2s_j)^{-1}=1/(L_j\sigma_{j-1}\sigma_{j-2})$. It is at most $\epsilon_{\rm sch}/L_j$ by the preceding growth-scale ratios, including $j=2$, where $\sigma_0=1$. We also impose $e^{-y/8}\le\epsilon_{\rm sch}$. For the old-gradient comparison, the stronger fixed-multiplier bound $4C_\sigma\lambda_{j-1}^{-(Q_j-1)/8}\le\epsilon_{\rm sch}$ follows from (3.51), uniformly for $2\le j\le q$. It also gives $2\sigma_{j-1}^2/\lambda_j^{1/8}\le\epsilon_{\rm sch}$, proving (3.52f).

Finally for $q\ge 2$, with $x=\log\lambda_{q-1}$, $(\log\lambda_q)^{10}\le x^{20}\le e^x$, so $\Pi_q\le\lambda_q^{5/Q_q}$. At $q=1$ we impose separately $y^{10}\le e^{5y/Q_1}$; it is not the preceding inequality with an index zero. The remaining size bounds in (3.53a) follow from $\Pi_q=\lambda_{q-1}^4(\log\lambda_q)^{10}$ and $\Gamma_q\in[(1-\epsilon_{\rm sch}^2/6)L_q\sigma_{q-2},L_q\sigma_{q-2}]$. For $q=1$ use $\Gamma_1=\sin s_*$ and $\ell_1^{-1}=C_\ell/s_0$, imposing the finitely many resulting polynomial conditions on y. In particular the majorant for the first lifetime length in (3.53) grows only linearly in y, whereas $\Pi_1=y^{10}$. These estimates prove (3.53a) and the first lifetime comparison in (3.53c); its radius comparison follows from $\sigma_{j-1}\lambda_{j-1}^{-3}\le\sqrt{C_\sigma}\lambda_{j-1}^{-47/16}$.

The exact definitions of X_j retain their sine factor: $X_j=L_j^2\sigma_{j-2}\sin s_j/s_j$ for $j\ge 2$. For $j\ge 2$ the ratios already proved imply

$$\frac{X_{j+1}}{X_j}\ge(1-\epsilon_{\rm sch}^2/6)\left(\frac{L_{j+1}}{L_j}\right)^2\frac{\sigma_{j-1}}{\sigma_{j-2}}>\epsilon_{\rm sch}^{-1}.$$

For $j=1$ include the separate polynomial threshold condition $X_2/X_1\ge(1-\epsilon_{\rm sch}^2/6)L_2^2/(\Lambda_1\sin s_*)\ge\epsilon_{\rm sch}^{-1}$. For $j\ge 2$, with $x=\log\lambda_j$,

$$X_{j+1}\le L_{j+1}^2\sigma_{j-1}\le\sqrt{C_\sigma}x^8\lambda_{j-1}^{1/16}\le\epsilon_{\rm sch}\lambda_{j-1}^4x^{10}=\epsilon_{\rm sch}\Pi_j.$$

Here $L_{j+1}\le Q_{j+1}^2x$ and $Q_{j+1}\le x$; the last comparison follows after the same fixed threshold. At $j=1$, $\sigma_0=1$ gives

$$X_2\le Q_2^4y^2\le\epsilon_{\rm sch}y^{10}=\epsilon_{\rm sch}\Pi_1\qquad\text{once } y^8\ge\epsilon_{\rm sch}^{-1}Q_2^4.$$

This proves (3.53b). Only finitely many fixed-multiplier bounds have been imposed, independently of the requested derivative order. Their actual ratios were estimated above before applying (3.51). The further first-layer polynomial, tail and nesting conditions are displayed in Section 8. Each calculation uses only its displayed growth scales, all of index at most $q-1$ when proving these assertions for finitely many stages. The threshold depends only on low-order data, independently of the constants in the correction estimates of Sections 6–7. ☐

3.7.2. *The closed system.* We first isolate the variables that determine the coefficients. Keeping them in a compact subset of their domain will allow the new linear pair to continue. For a stage $q\ge 2$ use the state

$$x=(\alpha;(M_j,\Theta_j,\Omega_j)_{j\le q})$$

on the open set

$$\det M_j > 1/2, \quad |M_j^{-T} p_j| > 1/4 \quad (j \le q), \qquad \zeta_{q-1,2} > z_*/2, \quad \zeta_{q,2} > z_*/2.$$

The coefficients are smooth there. In particular $A_i \mathsf{e}_i = -\lambda_i \Theta_i \zeta_i$, so no square-root regularity issue occurs at $\Theta_i = 0$. The denominators are powers of $\det M_j$, $|\zeta_j|^2$ and the two displayed second components. After the growth threshold crossing and transition have been located, set $z_1 = \zeta_{q,1}(t_q^1)$, $\chi_1 = |\zeta_q(t_q^1)|$. The single formula

$$\dot\alpha = \mathcal{F}_q + (1 - \eta(\Gamma_q(t - t_q^a)))(\mathcal{F}_{q-1} - \mathcal{F}_q) - \dot Z_q/\zeta_{q,2}$$

represents growth, transition, steering and the subsequent holding interval. In this formula Z_q is extended constantly before t_q^1 and by zero after its negative pulse. The resulting ODE vector field is smooth in $t, x, \mathfrak{h}$ and affine in $\mathfrak{h}$. Before $t_q^1 + \Gamma_q^{-1}$ it is independent of $\mathfrak{h}$. The variables

$$x' = (\alpha; (M_j)_{j\le q}; (\Theta_j, \Omega_j)_{j<q})$$

form a closed subsystem, whose coefficients depend only on these variables. The pair (Θ_q, Ω_q) solves a linear system on the whole existence interval of x'. Each matrix generator is trace free, hence $\det M_j = 1$.

We use the following elementary continuation principle. A solution of a smooth finite-dimensional ODE can cease to exist at a finite time only by leaving every compact subset of its domain. Consequently finitely many bootstrap inequalities hold up to a given finite time if assuming them together yields strict improvements and keeps the solution in one compact subset of the domain. To prove this, take the supremum of the times on which all inequalities hold. The strict improvements rule out a first exit before the prescribed final time; remaining in a fixed compact subset rules out loss of existence there. We first keep x' in such a compact subset and then solve for (Θ_q, Ω_q).

Let

$$\mathcal{G}_j = \sum_{i=0}^{j-1} w_i A_i \sin d_{ij}, \qquad \Sigma_j = \sum_{1\le i<j} w_i \Omega_i \sin^2 d_{ij}.$$

Besides (3.6), direct differentiation gives

$$\begin{aligned} a_j = \mathcal{G}_j/(\lambda_j r_j), \qquad \dot\Omega_j = -A_j \sin\varphi_j, \qquad (\log P_j)' = -\mathcal{G}_j\Omega_j/A_j, \\ \mathcal{F}_j = -\Sigma_j + \dot\rho_j \tan\varphi_j, \qquad \dot d_{ij} = \Sigma_i - \Sigma_j. \end{aligned} \tag{3.55}$$

The amplitude logarithm is used only after its positivity has been established. The determinant formed by adjacent phase gradients is also constant. Indeed the extra rank-one term in $D_{<j}^T - D_{<j-1}^T$ annihilates ζ_{j-1}, so both adjacent phase gradients solve the same transport equation. The constancy of this determinant gives

$$\mathfrak{d}_q := r_q \sin\vartheta_q = \sin s_q \, e^{-[\rho_{q-1}(t) - \rho_{q-1}(t_q^{\mathrm{in}})]}. \tag{3.56}$$

For $q \ge 2$ define the shear errors below. Their sums are empty and $E_3 = \dot\rho_1 = 0$ at $q = 2$:

$$E_2 = \sum_{i=1}^{q-2} \Omega_i \sin d_{iq} \cos d_{iq}, \quad E_3 = \dot\rho_{q-1}, \quad E_1 = \sin\vartheta_q \sum_{i=1}^{q-2} \Omega_i \sin(d_{iq} + d_{i,q-1}).$$

All older activation cutoffs equal one once their growth intervals have been constructed. Subtracting adjacent angle equations yields

$$\dot\vartheta_q = -E_1 - \Omega_{q-1}\sin^2\vartheta_q, \qquad \dot\rho_q = E_2 + \Omega_{q-1}\sin\vartheta_q\cos\vartheta_q. \tag{3.57}$$

The factor $\sin\vartheta_q$ in E_1 provides the small relative-angle factor used in the estimates for older layers.

3.7.3. *The first stage and the induction hypotheses.* The rigid rotation at the first stage supplies the initial endpoint data for the finite-stage induction, including the exceptional second transition. At layer 1, M_1 is a rotation and $r_1 = 1$ throughout. Growth and transition have $\alpha = 0$, constant positive coefficients and the exact growing-eigenline solution $P_1 = \lambda_1^{-k_1-6} e^{\Gamma_1(t-1)}$, $\Omega_1 = -\lambda_1 P_1$. Thus the threshold crossing occurs at $1 + L_1/\Gamma_1$ and the transition adds one unit of logarithmic gain. The steering is given by Lemmas 3.7 and 3.8, with the design (3.9). They prove existence of the entire first family before taking the amplitude ratio, $|Z_1| \le 1/4$, $\zeta_{1,2} \ge \sqrt{15}/4$, opposite endpoint signs and a unique first root (uniqueness will not be claimed later). At the selected root

$$(1 + \log 2) \le \log(P_1(t_1^b)/\lambda_1^{-7/8}) \le 2 + \Lambda_1^{-1}, \quad \alpha = s_*, \quad \zeta_1 = e_2, \quad \Omega_1 = 0. \tag{3.58}$$

The exact duration is $(L_1 + 2 + \Lambda_1^{-1})/\Gamma_1$. Since all old gradients make an acute angle with e_2, $A_1 \le \sigma_1^2 \le A_1 + 1$. The first gain gives $A_1(t_1^b) \ge 2e\lambda_1^{1/8}$, while $c_\sigma < 2e$. Thus (3.54) gives $1 \le \epsilon_{\mathrm{sch}}^2 c_\sigma \lambda_1^{1/8} \le \epsilon_{\mathrm{sch}}^2 A_1(t_1^b)$. The strict choices of G_-, G_+ give the first growth-scale bound and $A_1(t_1^b)/\sigma_1^2 \in [1 - C\epsilon_{\mathrm{sch}}, 1]$.

At $t_q^{\mathrm{in}} = t_{q-1}^b$, we assume the following induction hypotheses about the stages already constructed. Every time interval below ends at t_q^{in}; an interval reduced to one endpoint and an empty index range retain their literal meanings.

- (H1) Every stage $n < q$ has its prescribed ordered times and growth scale, with the two-sided growth-scale bounds of (3.18) through index $q-1$. For $2 \le n < q$, $t_n^b - t_n^{\mathrm{in}} < C_S/\sigma_{n-2}$; the first duration is $(L_1 + 2 + \Lambda_1^{-1})/\Gamma_1$.
- (H2) At each completed endpoint t_k^b, $1 \le k < q$, $\zeta_{k,1} = \Omega_k = 0$.
- (H3) For $2 \le k < q$ the endpoint geometry obeys

$$\rho_k(t_k^b) \in [-C\epsilon_{\mathrm{sch}}, \log(3/2) + C\epsilon_{\mathrm{sch}}], \qquad (2/3 - C\epsilon_{\mathrm{sch}})s_k \le \vartheta_k(t_k^b) \le (1 + C\epsilon_{\mathrm{sch}})s_k,$$

  and the logarithmic gain lies in $[1 + I_0 - C\epsilon_{\mathrm{sch}}, 1 + \bar{v} + C\epsilon_{\mathrm{sch}}]$. At $k = 1$ we use (3.58) and $\rho_1 = 0$.
- (H4) For $1 \le k < q$, $A_k(t_k^b)/\sigma_k^2 \in [1 - C\epsilon_{\mathrm{sch}}, 1]$.
- (H5) For $1 \le k \le q-2$, Ω_k is nondecreasing on $[t_{k+1}^b, t_q^{\mathrm{in}}]$, starting in $[1/3 - C\epsilon_{\mathrm{sch}}, 1 + \bar{v} + C\epsilon_{\mathrm{sch}}]\sigma_k$, with increment at most $Cs_{k+1}\sigma_k$. For $2 \le k < q$, ρ_k is nondecreasing on $[t_k^b, t_q^{\mathrm{in}}]$, with increment at most Cs_k, and $|\log(P_k/P_k(t_k^b))| \le C\epsilon_{\mathrm{sch}} s_k$ there. For $k = 1$, $\rho_1 = 0$ and the logarithmic-amplitude bound is $C\epsilon_{\mathrm{sch}}$.
- (H6) For $2 \le k < q$, the adjacent angle ϑ_k is nonincreasing on $[t_k^b, t_q^{\mathrm{in}}]$, and its cumulative logarithmic loss is bounded by integrating

$$|(\log \vartheta_k)'| \le C\big(\sigma_{k-1}s_k + \mathbf{1}_{k\ge 3} s_{k-1}\sigma_{k-2}\big).$$

The constants in these integral estimates are fixed finite low-order constants. The bootstrap (3.59) enlarges the instantaneous ranges needed to continue a stage. The endpoint data and integral hypotheses above are carried separately, and all cumulative estimates are recomputed from their original stage-end values.

In stage 2, growth and transition have a special fixed geometry: $\alpha = s_*$, $\Omega_1 = \varphi_1 = 0$ and $M_2 = \mathrm{Id}$ solve the closed subsystem uniquely. Its constant coefficients are

$$a_2 = [\sin(s_* + s_2) + A_1(t_1^b)\sin s_2]/\lambda_2, \qquad b_2 = \lambda_2 \sin s_2.$$

Their base term is $O(\epsilon_{\mathrm{sch}})$ relative to $\sigma_1^2 \sin s_2$, and all their logarithmic derivatives vanish. This exception includes the transition; outside this exception, the preceding layer's interval of zero vorticity amplitude ends at t_{k+1}^a.

3.7.4. *Bounds for the older layers on a new stage.* Fix $q \geq 2$, abbreviate $\sigma = \sigma_{q-1}$, $s = s_q$, $\Gamma = \Gamma_q$, $L = L_q$, and let $t_{\mathrm{hor}} = t_q^{\mathrm{in}} + C_S/\sigma_{q-2}$. On an initial interval ending before this horizon assume

$$\begin{aligned}
&|\varphi_{q-1}| \leq 2(C_\varphi + C_\vartheta)Ls, \\
&|\varphi_q| \leq 2(C_\varphi + C_\vartheta)Ls + 2C_\vartheta s, \quad 0 < \vartheta_q \leq 2C_\vartheta s, \\
&-\epsilon\sigma \leq \Omega_{q-1} \leq 2C_W^{\mathrm{all}}\sigma, \quad \zeta_-/2 \leq r_q \leq 2\zeta_+, \\
&(c_W/2)\sigma_k \leq \Omega_k \leq 2C_W\sigma_k, \quad 1-2\epsilon \leq A_k/\sigma_k^2 \leq 1+2\epsilon \quad (1 \leq k \leq q-2), \\
&(c_\theta/2)s_k \leq \vartheta_k \leq 2C_\vartheta s_k \quad (2 \leq k \leq q-1), \qquad 1-2\epsilon \leq A_{q-1}/\sigma^2 \leq 1+2\epsilon.
\end{aligned} \tag{3.59}$$

These inequalities are imposed simultaneously. The estimates for the older layers, the preceding layer and the new geometry must all improve before a continuation argument is invoked.

First, the signs $\Omega_i \geq 0$ for $i \leq q-2$ and (3.57) imply that every old ϑ_k, $k \leq q-1$, is nonincreasing: all angles in the factored sum belong to $[0, \pi/2]$ under the bootstrap and the schedule. Consequently the inherited upper bound for each such angle persists. Summing adjacent angles and using the insertion-angle tail bound in (3.52b) gives

$$d_{ik} \in [c_\theta/2, 1 + C\epsilon_{\mathrm{sch}}]s_{i+1} \quad (1 \leq i < k \leq q,\ (i,k) \neq (q-1,q)), \qquad d_{0k} \in [s_1, (1 + C\epsilon_{\mathrm{sch}})s_1].$$

All constants here are obtained with the coarse bounds just displayed; they do not depend on a later higher-order majorant. Since $2(C_\varphi + C_\vartheta)L_q s_q \leq \epsilon_{\mathrm{sch}} s_{q-1}$ for $q \geq 3$,

$$-(1 + C\epsilon_{\mathrm{sch}})s_{k+1} \leq \varphi_k = \varphi_{q-1} - d_{k,q-1} \leq -(c_\theta/4)s_{k+1} \quad (k \leq q-2).$$

There are no such indices when $q = 2$.

Define the older-gradient contribution by $R_q = \sum_{i=0}^{q-2} A_i \sin d_{iq}$. The factored identities and (3.52e)–(3.52f) give

$$\begin{gathered}
|E_2| + |E_3| \leq Ch_q, \qquad |E_1| \leq Csh_q, \\
|R_q| \leq C\epsilon_{\mathrm{sch}}\sigma^2 s.
\end{gathered} \tag{3.60a}$$

The vorticity estimate includes the first layer:

$$0 \leq \dot{\Omega}_k \leq C\sigma_k^2 s_{k+1} \quad (1 \leq k \leq q-2). \tag{3.60b}$$

The estimates for ρ_k, the temperature amplitude and the adjacent angle have the separate ranges

$$\begin{gathered}
0 \leq \dot{\rho}_k \leq C\sigma_{k-1}s_k, \quad |(\log P_k)'| \leq C\sigma_{k-1}^2 s_k/\sigma_k \quad (2 \leq k \leq q-2), \\
|(\log P_1)'| \leq C/\sigma_1, \qquad \dot{\rho}_1 = 0, \\
|(\log \vartheta_k)'| \leq C(\sigma_{k-1}s_k + s_{k-1}\sigma_{k-2}\mathbf{1}_{k\geq 3}) \quad (2 \leq k \leq q-1).
\end{gathered} \tag{3.60c}$$

For $q = 2$ the E_i, h_q terms are zero, but the old-gradient sum is $R_2 = \sin d_{02}$, not an empty shear sum. For $q \geq 3$ and $k = 1$, (3.55) gives $\dot{\Omega}_1 = -A_1 \sin \varphi_1$. The angle range just proved and $A_1 \leq (1+2\epsilon)\sigma_1^2$ give $0 \leq \dot{\Omega}_1 \leq C\sigma_1^2 s_2$, establishing the exceptional index in (3.60b). For $q \geq 3$ the largest nonbase contribution to R_q is $C\sigma_{q-2}^2 s_{q-1}$; dividing it by $\sigma^2 s$ gives $CL_{q-1}\sigma_{q-3}/(L_q\sigma_{q-1}) \leq C\epsilon_{\mathrm{sch}}$. The remaining terms sum geometrically. The base contribution uses $(\sigma^2 s)^{-1} \leq \epsilon_{\mathrm{sch}}/L$. For the ρ_k and angle rates use the shear sums $\sum_{i=1}^{k-1} \sigma_i s_{i+1}$ and the factored E_1. The temperature rate follows from $\mathcal{G}_k \leq C\sigma_{k-1}^2 s_k$ and $|\Omega_k|/A_k \leq C/\sigma_k$. All denominators are bounded below by the bootstrap. These computations give a constant independent of the derivative order, the frequency parameters or the duration of any later stage.

We now integrate these rates to establish (H5)–(H6), including the stage of the immediately following layer. Up to any present endpoint, past length bounds and the current horizon give

$$|[t_k^b, t']| \leq C_T/\sigma_{k-1} \quad (k \geq 2), \qquad |[t_{k+1}^b, t']| \leq C_T/\sigma_k.$$

During the growth of the next layer P_k is constant because $\Omega_k = \varphi_k = 0$. The rest of that next stage has length at most $3/\Gamma_{k+1}$ and $|\Omega_k| \le 2C_W^{\mathrm{all}}\sigma_k$. For $k \ge 2$ the logarithm of its temperature-amplitude magnitude changes by at most

$$C\frac{\sigma_{k-1}^2 s_k}{\sigma_k \Gamma_{k+1}} = C\frac{s_k}{L_{k+1}}\frac{\sigma_{k-1}}{\sigma_k}\frac{s_{k+1}}{\sin s_{k+1}} \le C\epsilon_{\mathrm{sch}} s_k. \tag{3.61}$$

For $k=1$ the corresponding bound is $C/(\sigma_1\Gamma_2) \le Cs_2/L_2^2$. On the subsequent stages, the old-layer rate estimate (3.60) adds at most $C\sigma_{k-1}^2 s_k/\sigma_k^2$, respectively C/σ_1^2. Therefore the total changes satisfy

$$\begin{gathered} 0 \le \Omega_k - \Omega_k(t_{k+1}^b) \le Cs_{k+1}\sigma_k, \\ 0 \le \rho_k - \rho_k(t_k^b) \le Cs_k, \qquad |\log(P_k/P_k(t_k^b))| \le C\epsilon_{\mathrm{sch}} s_k \quad (k\ge 2), \\ |\log(P_1/P_1(t_1^b))| \le C\epsilon_{\mathrm{sch}}, \qquad \vartheta_k \ge (2/3 - C\epsilon_{\mathrm{sch}})e^{-C\epsilon_{\mathrm{sch}}} s_k. \end{gathered} \tag{3.62}$$

For the preceding layer, use the current interval $[t_q^{\mathrm{in}}, t']$ of length at most C_S/σ_{q-2}. For $q \ge 3$, the same bound $\mathcal{G}_{q-1} \le C\sigma_{q-2}^2 s_{q-1}$ and the bootstrap bound $|\Omega_{q-1}|/A_{q-1} \le C/\sigma_{q-1}$ give

$$\left|\log\frac{P_{q-1}(t')}{P_{q-1}(t_q^{\mathrm{in}})}\right| \le CC_S\frac{\sigma_{q-2}s_{q-1}}{\sigma_{q-1}} \le C\epsilon_{\mathrm{sch}}^2$$

by (3.52c) and (3.52d). For $q=2$, $\mathcal{G}_1 = \sin s_*$ gives instead $|\log(P_1(t')/P_1(t_2^{\mathrm{in}}))| \le CC_S/\sigma_1 = O(\epsilon_{\mathrm{sch}})$, and $\rho_1 = 0$. The growth contribution is zero, also through the transition when $q=2$. Together with the phase-gradient length bound these estimates give

$$A_{q-1}/\sigma^2 = 1 + O(\epsilon_{\mathrm{sch}}), \qquad 0 \le \rho_{q-1} - \rho_{q-1}(t_q^{\mathrm{in}}) \le C\epsilon_{\mathrm{sch}}, \qquad \mathfrak{d}_q/\sin s = 1 + O(\epsilon_{\mathrm{sch}}).$$

The last upper bound is $\mathfrak{d}_q/\sin s \le 1$, by (3.56). Combining the growth-scale ratio at the end of the earlier stage with $A_k = \lambda_k r_k P_k$ proves $A_k/\sigma_k^2 = 1 + O(\epsilon_{\mathrm{sch}})$. By the fixed choice of K_{sch} and the strict gaps between the model ranges and bootstrap bounds, these estimates strictly improve (3.59). Each bound is obtained by integrating from its fixed earlier stage-end value, not by adding an old C-bound and a new full C-bound.

3.7.5. *The growth threshold and the transition interval.* First run only $\dot\alpha = \mathcal{F}_{q-1}$ to the finite horizon, before establishing the threshold crossing. The preceding layer's first phase-gradient component and vorticity amplitude remain zero. Equations (3.57), (3.60) and $h_qL/\Gamma \le 2\epsilon_{\mathrm{sch}}$ yield

$$\vartheta_q \in [(1-C\epsilon_{\mathrm{sch}})s, s], \qquad |\rho_q| \le C\epsilon_{\mathrm{sch}}, \qquad \mathcal{G}_q/(\sigma^2\sin s) = 1 + O(\epsilon_{\mathrm{sch}}).$$

Thus $a_q, b_q > 0$ and $\gamma_q = \sqrt{a_qb_q} = (1+O(\epsilon_{\mathrm{sch}}))\Gamma$. Their logarithmic derivatives satisfy

$$|(\log a_q)'| + |(\log b_q)'| \le Ch_q \le C\epsilon_{\mathrm{sch}}\Gamma/L. \tag{3.63}$$

For completeness, differentiate $\mathcal{G}_q = A_{q-1}\sin\vartheta_q + R_q$. There are four contributions to $(\log a_q)'$: $A_{q-1}'\sin\vartheta_q/\mathcal{G}_q$, $A_{q-1}\cos\vartheta_q\,\vartheta_q'/\mathcal{G}_q$, $-\rho_q'$ and $R_q'/\mathcal{G}_q$. The first two leading logarithmic rates are E_3 and $-\cot\vartheta_q E_1$, both $O(h_q)$. For an old summand, $(A_i\sin d_{iq})' = A_i'\sin d_{iq} + A_i\cos d_{iq}(\Sigma_i - \Sigma_q)$. The factored difference of the two Σ sums supplies one s_{i+1}; the remaining shear sum is bounded by $C\sigma_{q-2}s_{q-1} = Ch_q$. Also $|A_i'|/A_i \le Ch_q$ from (3.60). Summing the already geometrically dominated old gradient gives $|R_q'| \le Ch_q\sigma^2 s$. For the normalized base summand, when $q\ge 3$, the angle tails and (3.52e) give

$$|d_{0q}'| = \Sigma_q \le C\sum_{i=1}^{q-2}\sigma_i s_{i+1}^2 \le Cs_2\sum_{i=1}^{q-2}\sigma_i s_{i+1} \le Cs_2h_q.$$

Since $\mathcal{G}_q \ge \sigma^2 s/2$, the base ratio in (3.52f) yields

$$\frac{|(\sin d_{0q})'|}{\mathcal{G}_q} \le \frac{Cs_2h_q}{\sigma^2 s} \le C\frac{\epsilon_{\mathrm{sch}}s_2}{L}h_q \le Ch_q.$$

At $q = 2$ the coefficients and the base angle are constant on growth and transition, so this derivative vanishes along with h_2. Finally differentiate $\log a_q = \log \mathcal{G}_q - \rho_q - \log \lambda_q$ and $\log b_q = \log \lambda_q + \rho_q + \log \sin \vartheta_q$. On growth, $b_q = \lambda_q \mathfrak{d}_q$, so the determinant identity (3.56) gives $(\log b_q)' = -E_3$ exactly. This proves (3.63), with all four contributions bounded at scale $C\epsilon_{\mathrm{sch}}\Gamma/L$, not merely $C\epsilon_{\mathrm{sch}}\Gamma$. At layers $1, 2$ both derivatives are exactly zero.

To keep the subsystem in a compact subset of its domain, use the bounded older amplitudes and the lower bounds for r_j and the controlled second components. They bound $\dot{\alpha}$ and each deformation coefficient matrix on this finite horizon. For each $j \le q$, finite-interval Gronwall gives a number K_j^{hor} with $\|M_j^{\pm 1}\| \le K_j^{\mathrm{hor}}$, for example $K_j^{\mathrm{hor}} = 2\max\{\|M_j^{\pm 1}(t_q^{\mathrm{in}})\|\}\exp(D_{\mathrm{ap}} C_S/\sigma_{q-2})$, where D_{ap} bounds the coefficient matrices for the stages already constructed. The bound retains $\|M_j^{\pm 1}(t_q^{\mathrm{in}})\|$, which need not equal one for older layers. The state x' thus lies in a closed bounded subset with determinant one and the improved bounds $r_j \ge 1 - C\epsilon_{\mathrm{sch}} > 1/2$ and controlled second components at least z_*. These place the state strictly away from the domain boundaries $r_j = 1/4$ and from both controlled-component boundaries at $z_*/2$. Its older P_j, $j < q$, are bounded above and away from zero by the cumulative logarithmic estimates; (Θ_q, Ω_q) is not a variable of this subsystem. The continuation principle extends x' beyond the horizon. The linear pair (Θ_q, Ω_q) then extends with it and is nonzero as a vector.

On any interval before a first zero of the temperature amplitude, $u = \Omega_q/\Theta_q$ solves $u' = b_q - a_q u^2$. For $u^* = \sqrt{b_q/a_q}$ the ratio $z = u/u^*$ solves $z' = \gamma_q(1 - z^2) - z(\log u^*)'$. The vector field points inward at $1 \pm C\epsilon_{\mathrm{sch}}$ by (3.63), and $z = 1$ at activation. Lemma 3.3 gives $z = 1 + O(\epsilon_{\mathrm{sch}})$ and $(\log P_q)' = (1 + O(\epsilon_{\mathrm{sch}}))\Gamma > 0$. This bounded quotient precludes a first zero of the temperature amplitude, since the new amplitude pair (Θ_q, Ω_q) never vanishes as a vector. Thus the continued pair has positive P_q, bounded amplitude ratio and strictly positive logarithmic growth rate on the finite horizon. These are the amplitude bounds needed to reach its threshold.

The solution with growth feedback continued therefore exists long enough that the gain L_q is attained, at $t_q^a - t_q^{\mathrm{in}} = (1 + O(\epsilon_{\mathrm{sch}}))L/\Gamma$. The crossing is transversal and occurs before the horizon. The activation interval length Γ^{-1} is smaller than this duration. Both components retain the initial seed scale. Indeed,

$$u^* = \frac{b_q}{\gamma_q} = (1 + O(\epsilon_{\mathrm{sch}}))\frac{\lambda_q r_q}{\sigma}, \qquad \frac{u}{u^*} = 1 + O(\epsilon_{\mathrm{sch}}),$$

so on $[t_q^{\mathrm{in}}, t_q^{\mathrm{in}} + \Gamma^{-1}]$,

$$P_q \le e^{1 + C\epsilon_{\mathrm{sch}}}\lambda_q^{-k_q - 6}, \qquad \left|\frac{\sigma \Omega_q}{\lambda_q}\right| = \frac{\sigma |u|}{\lambda_q} P_q \le C P_q. \tag{3.64}$$

At layer 1 the growing eigenline gives exactly $u = \lambda_1$ and $P_1 \le e^1 \lambda_1^{-k_1 - 6}$ during its activation interval. This proves the two-component estimate (3.20). Including the forthcoming fixed-duration intervals gives

$$t_q^b - t_q^{\mathrm{in}} \le (1 + C\epsilon_{\mathrm{sch}})L/\Gamma + (2 + 1/L)/\Gamma \le (1 + C'\epsilon_{\mathrm{sch}})/\sigma_{q-2} < t_{\mathrm{hor}} - t_q^{\mathrm{in}}.$$

On the transition add the bootstrap assumptions $|\varphi_{q-1}| \le \epsilon_{\mathrm{sch}} s$ and $|\Omega_{q-1}| \le \epsilon_{\mathrm{sch}}\sigma$. Subtracting the feedbacks gives exactly

$$\dot{\varphi}_{q-1} = -\eta_b \Omega_{q-1}\frac{\sin \vartheta_q}{\cos \varphi_q}\sin \varphi_{q-1} + \mathcal{E}, \quad \mathcal{E} = \eta_b E_1 - \eta_b E_2 \tan \varphi_q - (1 - \eta_b)E_3 \tan \varphi_{q-1}.$$

Put $\Phi = \sup |\varphi_{q-1}|/s$ and $\widetilde{W} = \sup |\Omega_{q-1}|/\sigma$ on the elapsed transition. Integrating from the exact zero datum, using its length Γ^{-1}, gives

$$\Phi \le C\widetilde{W}\Phi + C\frac{h_q}{\sigma s}, \qquad \widetilde{W} \le C\Phi, \qquad \frac{h_q}{\sigma s} = \frac{s_{q-1}}{L}.$$

The fixed low-order smallness absorbs the first term under these additional assumptions, and the weighted-angle threshold condition gives $\Phi + \widetilde{W} \le C\epsilon_{\rm sch}/L < \epsilon_{\rm sch}$. At $q = 2$ both are zero. Inserting this improvement in the four logarithmic contributions above and in the factored equations for the angle and ρ_q gives

$$|(\log a_q)'| + |(\log b_q)'| \le C\epsilon_{\rm sch}\Gamma/L \quad \text{on the transition.}$$

The constant here dominates the products generated by substitution as well as the individual error constants. Thus the same comparison for $z = u/u^*$ remains valid, and

$$\begin{gathered} \log[P_q(t_q^1)/P_q(t_q^a)] = 1 + O(\epsilon_{\rm sch}), \quad \chi_1 = 1 + O(\epsilon_{\rm sch}), \quad W(0) = \Omega_{q-1}(t_q^1)/\sigma = O(\epsilon_{\rm sch}/L), \\ v(0) = \frac{\sigma\Omega_q(t_q^1)}{\lambda_q\chi_1\Theta_q(t_q^1)} = 1 + O(\epsilon_{\rm sch}), \quad z_1 = \mathfrak{d}_q(t_q^1)(1 + e_4), \quad e_4 = O(\epsilon_{\rm sch}/L). \end{gathered} \tag{3.65}$$

Indeed $z_1 = \chi_1 \sin(\vartheta_q + \varphi_{q-1})$ and $\mathfrak{d}_q = \chi_1 \sin\vartheta_q$, so $e_4 = \cos\varphi_{q-1} - 1 + \cot\vartheta_q \sin\varphi_{q-1}$. The geometric and older-layer bounds strictly improve the bootstrap assumptions. Together with the matrix bound obtained from the values at $t_q^{\rm in}$, they permit continuation of this subsystem through the transition. The pair (Θ_q, Ω_q) is then continued and the bounded quotient excludes a zero of the temperature amplitude. At layer 2, $W(0) = e_4 = 0$, $\chi_1 = 1$ exactly.

3.7.6. *The shooting family for later stages.* We verify the later-stage model's hypotheses for every pulse amplitude, then use its endpoint signs to select the stage. On the steering interval put $\chi = r_q$, $W = \Omega_{q-1}/\sigma$, $\tau = \Gamma(t - t_q^1)$, $\mu = \mathfrak{h}/(L\sin s) \in [0, c_{\rm p}]$. The geometric equations are independent of the quotient Ω_q/Θ_q. Since the angular control enforces $\zeta_{q,1} = Z$, one has $\sin\varphi_q = Z/\chi$ and $\sin\vartheta_q = \mathfrak{d}_q/\chi$. Substitution gives exactly the geometric equations (3.35)–(3.36), with forcing (3.38), and the following coefficients:

$$\begin{gathered} c_1 = (\mathfrak{d}_q/\sin s)\cos\vartheta_q, \qquad d_1 = \chi E_2/\Gamma, \\ c_2 = c_5 = (A_{q-1}/\sigma^2)(\mathfrak{d}_q/\sin s), \qquad c_3 = \sin s/\mathfrak{d}_q, \\ d_3 = 1 - (\mathfrak{d}_q(t_q^1)/\mathfrak{d}_q)(1 + e_4), \\ c_4 = (\mathfrak{d}_q(t_q^1)/\sin s)(1 + e_4), \qquad d_4 = \chi_1 R_q/(\chi\sigma^2 \sin s), \\ d_2 = \frac{A_{q-1}}{\sigma^2 \sin s}\,\frac{\mathfrak{d}_q(\cos\varphi_q - 1) - Z(\cos\vartheta_q - 1)}{\chi}. \end{gathered} \tag{3.66}$$

For example, the identity for the preceding layer

$$\dot{\Omega}_{q-1} = A_{q-1}\sin(\vartheta_q - \varphi_q)$$

gives c_2, d_2. The coefficients of the new linear pair satisfy

$$a_q\lambda_q\chi_1/(\sigma\Gamma) = c_5\chi_1/\chi^2 + d_4.$$

On an interval where the temperature amplitude is nonzero, set

$$v = \sigma\Omega_q/(\lambda_q\chi_1\Theta_q).$$

Division of the pair (Θ_q, Ω_q) then gives (3.37). In particular $c_4 = z_1/\sin s$ is constant, and the initial values $\chi_1, v(0)$ are the actual values in (3.65). All $c_i - 1, d_i$ are bounded by $C\epsilon_{\rm sch}$. The potentially large negative pulse causes no loss: $|Z| \le (c_{\rm p}H_2 + C\epsilon_{\rm sch})L\sin s$, $|\cos\vartheta_q - 1| \le Cs^2$, and $|\cos\varphi_q - 1| \le C(Ls)^2$, hence $|d_2| \le C(Ls)^2$. Choose the common dominating low-order constant so that $\delta = C\epsilon_{\rm sch}$ bounds every coefficient error and error in the initial data, and $L^{-1} \le \delta$. By (3.16) this is inside the hypotheses of Lemma 3.9, with $c_{\rm p} = 3$, $\bar{v} = 2$ and the fixed H_2.

We verify existence before using its quotient conclusion. We impose the coarse bounds $\chi_1/2 < \chi < 2\chi_1$, $-\epsilon < W < 2 + c_{\rm p}$, together with (3.59). The geometric proof of Lemma 3.9 applies on every initial existence interval. It gives $W \ge -C\epsilon_{\rm sch}$, $\chi/\chi_1 \ge 1 - C\epsilon_{\rm sch}$, $W \le \tau + C\epsilon_{\rm sch}$ and $\chi/\chi_1 \le 1 + \tau^2/2 + C\epsilon_{\rm sch}$ on $[0, 1]$; on the negative pulse it gives $W_\tau > 0$, $W \le 1 + c_{\rm p} + C\epsilon_{\rm sch}$ and

$\chi/\chi_1 \le 3/2 + C\epsilon_{\rm sch}$. The determinant identity (3.56), $\rho_q = \log\chi$, $\sin\vartheta_q = \mathfrak{d}_q/\chi$ and $\sin\varphi_q = Z/\chi$, with the angle ranges in (3.59), therefore imply

$$(2/3 - C\epsilon_{\rm sch})s \le \vartheta_q \le (1 + C\epsilon_{\rm sch})s, \qquad -C\epsilon_{\rm sch} \le \rho_q \le \log(3/2) + C\epsilon_{\rm sch},$$
$$|\varphi_q| \le (c_{\rm p}H_2 + C\epsilon_{\rm sch})\varpi_q, \qquad |\varphi_{q-1}| \le (c_{\rm p}H_2 + C\epsilon_{\rm sch})\varpi_q + (1 + C\epsilon_{\rm sch})s_q.$$

All are strictly within the fixed wider bounds. The integrals for older layers establish their amplitude, adjacent-angle and vorticity bounds; their signs give $\varphi_k \le 0$ for $k \le q-2$. The base angle is at most $s_* + C\sum_{j\ge 2} s_j$, while all nonbase relative angles are $O(\epsilon_{\rm sch})$, so the cone inequality and the lower bounds for both controlled second components also strictly improve the bootstrap assumptions. In particular every phase-gradient length satisfies $r_j \ge 1 - C\epsilon_{\rm sch} > 1/2$, uniformly over the whole trial interval, and the controlled second components are at least z_*. These inequalities keep the state strictly away from the same open-domain boundaries. Together with the older-amplitude bounds they confine the closed system for the geometry and older layers to a common compact subset of its smooth domain. For its matrices use the bound already obtained from their values at $t_q^{\rm in}$, with a finite bound for the profile derivative uniform over $0 \le \mu \le c_{\rm p}$. This matrix bound is used only to prove existence through finitely many stages. The continuation principle constructs the whole geometric family before (Θ_q, Ω_q).

The linear pair (Θ_q, Ω_q) now exists on the whole interval and never vanishes as a vector. On the first part of steering, $0 \le \tau \le 1$, preservation of $\Theta_q < 0$ and $\Omega_q < 0$ gives $P_q > 0$ and $v > 0$. On any interval preceding a putative zero of the temperature amplitude during the negative pulse, the interval-uniform bounds of Lemma 3.9 bound $|v|$, independently of that endpoint. Equivalently, Proposition 3.10 applies with the constant $c_0 = \sigma/(\lambda_q\chi_1) > 0$ and excludes the zero. This proves $P_q > 0$ for every trial; v need not be positive for an arbitrary negative pulse trial, and absolute vorticity-amplitude bounds use $|v|$. The same substitution gives $\partial_\tau \log P_q = \widetilde{k}v$. Its first-part integral is at most $(1 + C\epsilon_{\rm sch})\bar{v}$ and the absolute negative pulse integral is $O(\epsilon_{\rm sch})$. Including the transition, every trial therefore satisfies $\log(P_q/P_q(t_q^a)) \le 1 + \bar{v} + C\epsilon_{\rm sch}$ on the steering interval and $|\Omega_q| \le \widehat{C}_\Omega(\lambda_q/\sigma)P_q$. The first trial family has the separate upper gain $2 + \Lambda_1^{-1}$ proved in Lemma 3.8.

The ODE vector field and data before the pulse do not depend on μ. Smooth dependence on μ makes the endpoint map $\mu \mapsto v(1 + 1/L; \mu)$ continuous. Its value at $\mu = 0$ is positive, and at $\mu = c_{\rm p}$ it is negative, by (3.44); since $\Theta_q < 0$, the signs of Ω_q are opposite. As in Proposition 3.10, its zero set is nonempty and compact, and has a least member. No monotonicity of the shooting map or uniqueness of its root is used. At the selected root $v \ge 0$ on the negative pulse, and (3.45)–(3.46) give

$$W(t_q^b) \in [1/3 - C\epsilon_{\rm sch}, 1 + \bar{v} + C\epsilon_{\rm sch}], \qquad \int_0^{1+1/L} |W|\, d\tau \le \tfrac{1}{2} + C\epsilon_{\rm sch},$$
$$\log[P_q(t_q^b)/\lambda_q^{-7/8}] \in [1 + I_0 - C\epsilon_{\rm sch}, 1 + \bar{v} + C\epsilon_{\rm sch}]. \tag{3.67}$$

The transition contributes one additional unit of logarithmic gain. The first-part bound gives $W \le 1 + C\epsilon_{\rm sch}$ before the negative pulse, and on the pulse W increases to $W(t_q^b) \le 1 + \bar{v} + C\epsilon_{\rm sch} < C_W$ at the selected root. Together with $W \ge -C\epsilon_{\rm sch}$, the transition bound (3.65) and its zero growth value, this proves the claimed bound for Ω_{q-1} during stage q in part (iii). For arbitrary trial amplitudes use $W \le 1 + c_{\rm p} + C\epsilon_{\rm sch} < C_W^{\rm all}$ instead. The first-part comparison supplies $I_0 > 0$, and the selected negative pulse adds a nonnegative gain. Near the end of steering $Z = 0$ and $\Omega_q = 0$, so P_q is constant and the angular control is exactly the next growth feedback.

At this point the new growth scale has not been used. The selected gain gives $A_q(t_q^b) \ge (1 - C\epsilon_{\rm sch})e^{1+I_0-C\epsilon_{\rm sch}}\lambda_q^{1/8} > \lambda_q^{1/8}$. The old gradient sum satisfies

$$S_{\rm old} = \sum_{i=0}^{q-1} A_i(t_q^b) \le (1 + C\epsilon_{\rm sch})(1 + 2\epsilon_{\rm sch})\sigma_{q-1}^2 \le \epsilon_{\rm sch}\lambda_q^{1/8} \le \epsilon_{\rm sch}A_q(t_q^b). \tag{3.68}$$

Projection onto the new direction and the triangle inequality imply $A_q \le \sigma_q^2 \le (1+\epsilon_{\rm sch})A_q$, including the stage-end ratio $A_q/\sigma_q^2 \in [1-\epsilon_{\rm sch}, 1]$. Combining this with the gain and phase-gradient length bounds yields the strict common growth-scale bounds because

$$(1 - C\epsilon_{\rm sch})e^{1+I_0-C\epsilon_{\rm sch}} > c_\sigma, \quad (1+\epsilon_{\rm sch})(3/2 + C\epsilon_{\rm sch})e^{1+\bar{v}+C\epsilon_{\rm sch}} < C_\sigma;$$

the first layer uses (3.58) instead. These strict inequalities are among the finite choices preceding the frequencies. Once σ_q is defined, the parameter estimates involving it apply.

At the end of steering all older vorticity amplitudes are nonnegative; the preceding layer's vorticity amplitude is at least $(1/3 - C\epsilon_{\rm sch})\sigma$, and $0 \le \Sigma_q \le C\sigma s^2 + C\sum_{i\le q-2}\sigma_i s_{i+1}^2 \le C\epsilon_{\rm sch}\sigma$. Since $\dot\alpha = -\Sigma_q$ there, the origin vorticity is at least $(1/3 - C\epsilon_{\rm sch})\sigma > c_W\sigma$. More explicitly, the quantity estimated here is

$$\begin{aligned} 2\dot\alpha(t_q^b) + \sum_{j\le q} w_j\Omega_j(t_q^b) &= -2\Sigma_q(t_q^b) + \sum_{j<q}\Omega_j(t_q^b) \\ &\ge (1/3 - C\epsilon_{\rm sch})\sigma > c_W\sigma. \end{aligned} \tag{3.69}$$

This proves (3.19).

3.7.7. *Completion of the induction and lifetime bounds.* The new stage length and growth scale establish (H1); its zero and gain give (H2)–(H3); the gradient comparison gives (H4). The rates and cumulative integrals establish (H5)–(H6). Each induction bound for an older layer follows from the integrals (3.62), always taken from its fixed value at the end of steering; the next-stage calculation (3.61) treats the preceding layer. The same finite constants work at every stage. The actual stage length is at most $[1 + C\epsilon_{\rm sch} + (2 + L^{-1})/L]/\sigma_{q-2} < C_S/\sigma_{q-2}$, with C_S fixed before its threshold crossing. This completes the induction through finitely many stages without assuming a deformation bound over the full lifetime or using a growth scale not yet constructed.

After all stages have been constructed, their lengths and the growth scale ratio give

$$T_* - 1 \le \frac{L_1 + 2 + \Lambda_1^{-1}}{\Gamma_1} + C_S\sum_{n\ge 2}\sigma_{n-2}^{-1} < \infty, \qquad T_* - t_q^{\rm in} \le C_T/\sigma_{q-2} \quad (q \ge 2).$$

The geometric sum is at most $1 + 2\epsilon_{\rm sch}$, which is below 2. The first-layer polynomial comparison gives $|I_1| \le \Pi_1$; the other lifetime bounds give $|I_q| \le \Pi_q$. Letting the constructed endpoint approach the limiting time in the integral estimates, still measured from the fixed stage-end values, gives bounds over the full lifetime. In particular $r_j \le (3/2 + C\epsilon_{\rm sch})e^{Cs_j} < \zeta_+$ and $P_j \le \exp(1 + \bar{v} + C\epsilon_{\rm sch} + C\epsilon_{\rm sch}s_j)\lambda_j^{-7/8} < e^{G_+}\lambda_j^{-7/8}$ for $j \ge 2$. The first layer uses its different gain and $r_1 = 1$. Positivity follows from the bounded logarithmic losses. The adjacent-angle lower bound is $(2/3 - C\epsilon_{\rm sch})e^{-C\epsilon_{\rm sch}}s_j > c_\theta s_j$; each old vorticity amplitude is at most $(1+\bar{v}+C\epsilon_{\rm sch}+Cs_{j+1})\sigma_j < C_W\sigma_j$ and at least $c_W\sigma_j$. The selected steering has nonnegative quotient and cannot decrease P_j, while its later logarithmic loss is $O(\epsilon_{\rm sch})$. Together with $r_j \ge 1 - C\epsilon_{\rm sch}$, this gives the persistent gradient lower bound from every threshold crossing. For the named cone constant, $d_{01} = s_*$, while the angle tails and (3.52c) imply, for each active $k \ge 2$,

$$0 \le d_{1k} = \sum_{j=2}^{k}\vartheta_j \le C_\vartheta(1+\epsilon_{\rm sch})s_2 \le \tfrac{1}{2}\epsilon_{\rm sch}(1+\epsilon_{\rm sch})s_* < s_*.$$

Thus every direction projects onto e_1 by at least $c_{\rm cone} = \cos s_*$, including the base direction. During the first steering interval the component lower bound is $\sqrt{15}/4$; after it, the first direction and all subsequent directions have $|\varphi_j| = O(\epsilon_{\rm sch})$ in laboratory coordinates. Hence the fixed z_* holds on all lifetimes.

We now estimate the full-lifetime deformation. The preceding layer's shear is zero on growth, $O(\epsilon_{\rm sch}\sigma)$ on transition, and its selected steering integral is $(\sigma/\Gamma)(1/2 + C\epsilon_{\rm sch})$. After the new layer finishes steering it is bounded by $C_W\sigma$ for a time at most C_T/σ. Still older shears have total size at most $C_W(1+2\epsilon_{\rm sch})\sigma_{q-2}$ on a remaining time at most C_T/σ_{q-2}. Since $\|\operatorname{sym}(J\mathsf{e}\otimes\mathsf{e})\| = 1/2$ and the rotational term is antisymmetric,

$$\int_{I_q} \|(D_{<q})_{\rm sym}\| \le \frac{1/4 + C\epsilon_{\rm sch}}{\sin s_q} + \tfrac{3}{2}C_W C_T \le \frac{1/4 + C'\epsilon_{\rm sch}}{s_q} + C.$$

Lemma 3.11, with $m_\zeta = \zeta_-$ and $M_\zeta = \zeta_+$ on the actual lifetime, gives

$$K_q \le \zeta_+\left(1 + 2\zeta_-^{-2}\left[\frac{1/4 + C\epsilon_{\rm sch}}{s_q} + C\right]\right) + \zeta_-^{-1} \le C_K/s_q.$$

The last inequality uses the fixed positive gap between C_K and the leading coefficient in the bound, then $s_q \le \epsilon_{\rm sch}$; it does not follow from the finite-interval Gronwall bound. Since $C_K/s_q = C_K\sigma_{q-1}/(L_q\sigma_{q-2}) \le \sigma_{q-1}$ by $L_q \ge 5y \ge C_K$ and $\sigma_{q-2} \ge 1$ from (3.54), the frequency and Π_q comparisons follow. This fixed-data threshold is also collected in (8.1). These full-lifetime suprema supply the geometry hypothesis of Lemma 4.3; $K_1 = 1$ follows from the rigid rotation of the first layer. This proves part (iv).

3.7.8. *Control speed and relative amplitude propagation.* The elementary first-order bound is useful independently of the higher derivative induction. Since $\varphi_1 = s_1 - \alpha$ and the angle of the first phase gradient is uniformly bounded, $|\alpha| \le C$ on its whole preterminal domain. On a stage $q \ge 2$, the factored feedback formula gives

$$|\mathcal{F}_q| + |\mathcal{F}_{q-1}| \le C\sum_{i=1}^{q-2}\sigma_i s_{i+1}^2 + C\sigma_{q-1}s_q\varpi_q.$$

To see the factors, write $\mathcal{F}_j = \sum_{i=1}^{j-1}\Omega_i \sin\varphi_i \sin d_{ij}/\cos\varphi_j$. For a layer older than the preceding layer, both sine factors are bounded by Cs_{i+1}. The angle of the preceding layer's phase gradient is at most $C\varpi_q$, its relative angle is at most Cs_q, and its vorticity amplitude has size $C\sigma_{q-1}$. The second-component lower bound keeps the denominators away from zero. This argument covers growth, transition and steering, with the empty sum retained at $q = 2$.

On the steering interval the exact prescribed profile has derivative bounded by

$$|\dot Z_q| \le C\big(\|\eta'\|_\infty s_q\Gamma_q + \|\eta_2'\|_\infty L_q^2 s_q\Gamma_q\big), \qquad L_q^2 s_q\Gamma_q \le \frac{L_q^4\sigma_{q-2}^2}{\sigma_{q-1}} \le \epsilon_{\rm sch}.$$

Also $\sigma_{q-1}s_q\varpi_q = L_q s_q^2\sigma_{q-1} \le 2L_q^2 s_q\Gamma_q/L_q \le 2\epsilon_{\rm sch}$. The schedule bounds the old shear sum by $\epsilon_{\rm sch}$, and $\zeta_{q,2} \ge z_*$. The angular control thus yields

$$|\dot\alpha| \le C(1 + \|\eta'\|_\infty + \|\eta_2'\|_\infty)\epsilon_{\rm sch} \quad \text{on every stage } q \ge 2.$$

At the first stage growth and transition have zero angular speed. On its steering interval, differentiate $\alpha = s_1 - \arcsin Z_1$; $\sqrt{1 - Z_1^2} \ge \sqrt{15}/4$ and the fixed-duration profile imply $|\dot Z_1| \le \Gamma_1 \sin s_*(\|\eta'\|_\infty + c_{\rm p}\Lambda_1^2\|\eta_2'\|_\infty)$. The factor Λ_1^2 comes from both the height of the first negative pulse and the reciprocal of its compressed duration. On $[0,1]$ the control is zero. Altogether

$$\sup_{t<T_*}(|\alpha| + |\dot\alpha|) \le C_\alpha := C\big[1 + \sin^2 s_*(\|\eta'\|_\infty + c_{\rm p}\Lambda_1^2\|\eta_2'\|_\infty) + \|\eta'\|_\infty + \|\eta_2'\|_\infty\big]. \tag{3.70}$$

This constant is fixed before the frequency threshold; increasing that threshold makes $C_\alpha \le \Pi_1/2$, proving part (vi). Higher time derivatives are treated in Section 5.

The next lemma compares an arbitrary solution of the amplitude system with its growing reference solution. This relative estimate will be used both here and in the correction estimates of Section 7. Its hypotheses are verified immediately afterward from the control estimates; no higher-order lifetime bound is required.

Lemma 3.14 (A weighted propagator estimate)**.** *Let $\mathcal{B} = \begin{pmatrix} 0 & \widehat{a} \\ \widehat{b} & 0 \end{pmatrix}$ be continuous on a lifetime I partitioned, in order, into growth, a combined interval containing the transition and both steering parts, a holding interval, and a tail consisting of all later evolution. Let y^0 be a nonzero solution of $\dot{y} = \mathcal{B}y$, and let $\mathcal{U}(t,s)$ be the propagator of this system, normalized by $\mathcal{U}(s,s) = \mathrm{Id}$. On growth suppose $\widehat{a}, \widehat{b}$ are positive C^1 functions and put*

$$\gamma = \sqrt{\widehat{a}\widehat{b}}, \quad r = \sqrt{\widehat{a}/\widehat{b}}, \quad \epsilon_r = \dot{r}/(2r), \quad r_0 = \sup\max(r, r^{-1}).$$

The supremum defining r_0 is over growth. In the coordinates $z = (r^{-1/2}y_1, r^{1/2}y_2)$ and $u_\pm = z_1 \pm z_2$, assume $u_-^0 = 0$ initially and $2\gamma\rho_* > |\epsilon_r|(1+\rho_*^2)$ for some $0 < \rho_* < 1$. On the holding interval suppose $\widehat{b} = 0$ and $y_2^0 = 0$. Write

$$E = \int_{\text{growth}} |\epsilon_r|, \quad S = \int_{\text{transition and steering}} \|\mathcal{B}\|, \quad H = \int_{\text{hold}} |\widehat{a}|, \quad D = \int_{\text{tail}} (|\widehat{a}| + |\widehat{b}|).$$

Then, for every $s \le t$ with $s, t \in I$, the propagator satisfies

$$\|\mathcal{U}(t,s)\| \le r_0^2\sqrt{1+\rho_*^2}(1+H)e^{(1+\rho_*)E+2S+2D}\frac{|y^0(t)|}{|y^0(s)|}. \tag{3.71}$$

In particular any fixed finite majorants for r_0, ρ_*, E, S, H, D give one fixed finite constant. The symbols r, z, E, S, H, D in this lemma denote local coordinates and integrals.

Proof. On growth, differentiation gives $\dot{u}_\pm = \pm\gamma u_\pm - \epsilon_r u_\mp$. Until a possible first zero of u_+^0, the ratio $\rho = u_-^0/u_+^0$ solves $\dot{\rho} = -2\gamma\rho + \epsilon_r(\rho^2 - 1)$. The strict signs at $\pm\rho_*$ preserve $|\rho| \le \rho_*$. The equation $\dot{u}_+^0 = (\gamma - \epsilon_r\rho)u_+^0$ then excludes that zero. For any solution,

$$\frac{d}{dt}|u| \le (\gamma + |\epsilon_r|)|u|, \qquad |u^0(t)| \ge \frac{e^{\int_s^t \gamma - \rho_* \int_s^t |\epsilon_r|}}{\sqrt{1+\rho_*^2}}|u^0(s)|.$$

Their quotient bounds growth relative to y^0. The transformation from z to u is a scalar multiple of an orthogonal matrix; converting the two solutions at the two endpoints costs at most r_0^2. Thus the growth factor is $r_0^2\sqrt{1+\rho_*^2}e^{(1+\rho_*)E}$. On transition and steering, forward and backward Grönwall bounds give the relative factor e^{2S}. On the holding interval y^0 is constant and $\mathcal{U} = \begin{pmatrix} 1 & \int\widehat{a} \\ 0 & 1 \end{pmatrix}$ has norm at most $1 + H$. During the later evolution the same two Grönwall estimates give e^{2D}. Splitting $[s,t]$ at its internal junctions and multiplying makes the ratios $|y^0(t_i)|/|y^0(t_{i-1})|$ telescope. This proves (3.71) and its stated symbolic majorant. □

3.7.9. *Verification of the weighted propagator hypotheses.* We supply the four interval integrals in Lemma 3.14 and obtain C_R in (3.72), including the separate contribution of the first holding interval. On growth in later stages, $r = \sqrt{\widehat{a}_q/\widehat{b}_q} = 1 + O(\epsilon_{\text{sch}})$ and $|\dot{r}/(2r)| \le C\epsilon_{\text{sch}}\Gamma_q/L_q$. In Lemma 3.14 take $\rho_* = C\epsilon_{\text{sch}}/L_q$ with a fixed constant making $2\gamma_q\rho_* > |\dot{r}/(2r)|(1+\rho_*^2)$. Its growing-eigenline datum is exactly our activation datum. Therefore $E \le C\epsilon_{\text{sch}}$ and $r_0 = 1 + O(\epsilon_{\text{sch}})$, with exact $E = 0, r = 1$ at layer 1 and $E = 0$ at layer 2. This estimate uses the logarithmic drift at scale $\epsilon_{\text{sch}}\Gamma_q/L_q$ over the long growth time.

On the combined transition and steering intervals of a later stage, $\int|\widehat{a}_q| \le 2 + C\epsilon_{\text{sch}}$ and $\int|\widehat{b}_q| \le 2 + c_{\text{p}} + C\epsilon_{\text{sch}}$. The transition and the first part of steering each contribute at most $1 + C\epsilon_{\text{sch}}$ to both coefficient integrals. On the negative pulse, $\int|\widehat{a}_q| = O(L_q^{-1})$ and $\int|\widehat{b}_q|$ is exactly $\mu\chi_1\int\eta_2 \le c_{\text{p}}\chi_1$. At layer 1 the respective integral bounds are $2 + \Lambda_1^{-1}$ and at most $2 + c_{\text{p}}$: the exact negative-pulse integral is $\mathfrak{h}_1/(\Lambda_1\Gamma_1) \le c_{\text{p}}$. Fix $C_{\text{short}} > 4 + c_{\text{p}} + \Lambda_1^{-1}$ large enough to bound their sum, so $S \le C_{\text{short}}$ in the weighted lemma.

On the holding interval, ending at t^a_{q+1}, $\widehat{b}_q = \Omega_q = 0$ and the reference pair is constant. For $q \geq 2$, the coefficient $|\widehat{a}_q| \leq C\sigma_{q-1}s_q$ and the remaining time bound give $H_q \leq C\epsilon_{\mathrm{sch}}$. The first holding interval is different and is retained exactly:

$$H_1 = \Gamma_1 L_2/\gamma_2 = (1 + O(\epsilon_{\mathrm{sch}}))\sin s_*\, s_2/\sin s_2.$$

Choose a fixed $C_{\mathrm{hold}} > \sin s_*$, and then choose the error tolerance so $H_1 \leq C_{\mathrm{hold}}$. Its leading term does not tend to zero as λ_1 grows. The rigid second transition belongs to the tail and has small integral.

For $q \geq 2$ the remainder of the next stage has length at most $3/\Gamma_{q+1}$ and the later time is at most C_T/σ_q. The coefficient and angle estimates give

$$\int_{\mathrm{tail}} |\widehat{a}_q| \leq C\sigma_{q-1}s_q(\Gamma_{q+1}^{-1} + \sigma_q^{-1}) \leq C\epsilon_{\mathrm{sch}},$$
$$\int_{\mathrm{tail}} |\widehat{b}_q| \leq C\sigma_{q-1}\left(\frac{\varpi_{q+1}}{\Gamma_{q+1}} + \frac{s_{q+1}}{\sigma_q}\right) \leq C\epsilon_{\mathrm{sch}}.$$

The small but generally nonzero coefficient contributions during the next layer's transition are included in these integrals; no exact zero through its endpoint is claimed. At layer 1 the upper coefficient is exactly Γ_1, so its integral over the later evolution is bounded by $C\Gamma_1(\Gamma_2^{-1}+\sigma_1^{-1}) \leq C\epsilon_{\mathrm{sch}}$. The integral of $|\widehat{b}_1|$ uses $C(\varpi_2/\Gamma_2 + s_2/\sigma_1) \leq C\epsilon_{\mathrm{sch}}$.

Consequently there are fixed finite bounds $r_0 \leq C_r$, $\rho_* \leq \rho_0 < 1$, $E \leq C_E\epsilon_{\mathrm{sch}}$, $D \leq C_D\epsilon_{\mathrm{sch}}$ and $H_q \leq C_{\mathrm{hold}} + C_H\epsilon_{\mathrm{sch}}$. Define once

$$C_R = C_r^2\sqrt{1+\rho_0^2}(1 + C_{\mathrm{hold}} + C_H\epsilon_0)\exp\big((1+\rho_0)C_E\epsilon_0 + 2C_{\mathrm{short}} + 2C_D\epsilon_0\big). \tag{3.72}$$

The low-order constants defining it are fixed before the sequence of higher-derivative constants and the common correction majorant. The four estimates in Lemma 3.14 telescope to (3.21); the relative amplitude weights cancel at every junction, leaving no unweighted growth exponential.

On its own selected interval $|y^0_q| \leq (1+\widehat{C}_\Omega)P_q$, by the model and tracking quotient bounds, with $|v|$ where necessary. Afterward $|\sigma_{q-1}\Omega_q/\lambda_q| \leq C\sigma_{q-1}\sigma_q/\lambda_q \leq C\epsilon_{\mathrm{sch}}\lambda_q^{-7/8}$ by the squared growth-scale bounds and separation, including $\sigma_0 = 1$. The fixed choice of C_Θ therefore bounds Y_q on the full lifetime. Positivity of P_q makes the reference vector nonzero.

3.7.10. *Smooth matching and the order of construction.* The finite-dimensional ODE vector field is smooth through the transition endpoints, by flatness of η, and through the two steering pieces, by the interior support of η_2. Near the end of steering $Z = \dot{Z} = 0$; the chosen $\Omega_q = 0$ and hence $\dot{\Theta}_q = 0$ there. The angular control already equals the next growth feedback $\mathcal{F}_q$. The older components thus solve the same smooth ODE on both sides of activation. The new activation cutoff is flat at its left endpoint and every appearance in later background coefficients has that factor, so its zero extension introduces no jump. This proves smoothness on every finite preterminal slab, not bounded derivatives of every order up to T_*.

The activation uses only the already known growth scales, the predetermined seed and the current direction. The threshold crossing is obtained by continuing the growth feedback; the transition and the common state before the pulse are then fixed; the compact nonempty set of pulse amplitudes with zero endpoint vorticity selects its least member; the new growth scale is evaluated only at the resulting end of steering. This order determines each stage from the preceding ones. Lemma 3.13 is uniform over all admissible logarithmic gains, and its threshold is independent of any future correction constants. This completes the low-order construction claimed in Theorem 3.2.

Bounds for time derivatives over the full lifetime require the weighted argument of Section 5; smoothness on each finite sequence of stages alone is insufficient. Theorem 7.3 supplies the correction and force estimates. Theorem 8.2 supplies their common parameter choice used to construct the solution.

4. Compactly supported fields for the controlled evolution

We now represent one step of the controlled evolution by compactly supported fields. The streamfunction gives a compactly supported divergence-free velocity. A periodic profile that is linear near zero gives an affine temperature and velocity on a smaller region. Three sets must be distinguished: the full support, the plateau of the material envelope, and the intersection of that plateau with the linear phase interval. Only the last set is the affine region. The support of every later layer must remain in this region of each older layer for its whole lifetime.

4.1. **An oscillatory function with a linear interval.** Choose an even cutoff $\chi \in C_c^\infty((-\pi,\pi))$ equal to one on $[-s_0, s_0]$, with $0 < s_0 < 1$. On this period define

$$F(s) = \sin s + \chi(s)(s - \sin s) \tag{4.1}$$

and extend periodically. It is smooth and odd, has zero period mean, and equals s on the stated interval. Its mean-zero periodic primitive P is even and equals $P(0) + s^2/2$ there. There is no analyticity requirement. Oscillation away from the origin and exact linearity near the origin are therefore compatible.

Lemma 4.1 (Compact streamfunctions)**.** *If $\psi \in C_c^\infty(\mathbb{R}^2)$, then $K\Delta\psi = \nabla^\perp\psi$, $\int \Delta\psi = 0$, and $\nabla^\perp\psi$ is divergence free and has finite energy. If ψ vanishes on an open set, its induced velocity vanishes there.*

Proof. For the Newtonian kernel $\mathcal{N}(x) = (2\pi)^{-1}\log|x|$, $\Delta\mathcal{N}$ is the Dirac mass at the origin in distributions. Compact support allows differentiation of the convolution: $\mathcal{N} * \Delta\psi = (\Delta\mathcal{N}) * \psi = \psi$. Applying $\nabla^\perp$ proves the first identity. Integration of the Laplacian proves its zero mean, and the remaining claims follow from compact support and $\operatorname{div}\nabla^\perp = 0$. □

An arbitrary vorticity supported outside a ball can induce velocity inside it. Lemma 4.1 makes a more special statement: vorticity of the form $\Delta\psi$ with a compact streamfunction identically zero inside the ball has exactly zero velocity there. The corrections below are designed at the streamfunction level for this reason.

Take a radial $f \in C_c^\infty(B_1)$, $0 \le f \le 1$, with $f = 1$ on $B_{1/2}$. For layer q let

$$a = M_q^{-1}x, \qquad s = \lambda_q\zeta_q \cdot x = \lambda_q p_q \cdot a, \qquad g_q(a) = f(a/\ell_q).$$

The leading physical fields are evaluations of

$$V_0 = w_q\Theta_q F(s) g_q(a), \qquad \Psi_0 = \frac{w_q\Omega_q}{\lambda_q^2|\zeta_q|^2} P(s) g_q(a). \tag{4.2}$$

Define the support region, plateau, and affine region, respectively, by

$$\mathcal{T}_q = M_q\overline{B}_{\ell_q}, \qquad \mathcal{P}_q = M_q B_{\ell_q/2}, \qquad \mathcal{A}_q = \mathcal{P}_q \cap \{|\lambda_q\zeta_q \cdot x| < s_0\}. \tag{4.3}$$

The plateau alone is not affine: F can still oscillate there. On $\mathcal{A}_q$ the leading fields satisfy

$$\theta_q = w_q\Theta_q\lambda_q\zeta_q \cdot x, \quad u_q = w_q\Omega_q(J\mathsf{e}_q \otimes \mathsf{e}_q)x, \quad \omega_q = w_q\Omega_q. \tag{4.4}$$

The corrections vanish on the whole plateau, for every phase s, by Proposition 6.1. In particular the affine region contains the ball $B_{r_q^{\mathrm{aff}}(t)}$, where

$$r_q^{\mathrm{aff}}(t) = \min\left\{\frac{\ell_q}{2\|M_q(t)^{-1}\|}, \frac{s_0}{\lambda_q|\zeta_q(t)|}\right\}. \tag{4.5}$$

Thus the central affine set contains a ball, while the corrections are supported in a surrounding material annulus. The full plateau may contain many oscillations. These three regions serve different purposes.

Lemma 4.2 (A lifetime nesting criterion)**.** *Suppose K_j bounds both M_j and M_j^{-1} throughout the remaining lifetime of layer j, and $|\zeta_j| \le C_\zeta$ there. If, for all $j < q$,*

$$K_j K_q \ell_q < \ell_j/2, \qquad C_\zeta \lambda_j K_q \ell_q < s_0, \tag{4.6}$$

then $\mathcal{T}_q(t) \subset \mathcal{A}_j(t)$ throughout the lifetime of q. If additionally $K_q\ell_q < s_0$, the base defined in (4.9) is affine on $\mathcal{T}_q$ for $t \ge 1$, after the initial interpolation.

Proof. For $x \in \mathcal{T}_q$, $|x| \le K_q\ell_q$. Hence $|M_j^{-1}x| \le K_jK_q\ell_q$ and $|\lambda_j\zeta_j \cdot x| \le C_\zeta\lambda_jK_q\ell_q$. The two conditions in (4.6) place x in the older material plateau and its linear phase cell, respectively. If $K_q\ell_q < s_0 < 1$, the support also lies in the base cutoff plateau B_2 and its unit-frequency linear phase cell. The base fields are $\theta^b = -\mathsf{e}_0 \cdot x$ and $u^b = \dot\alpha Jx$ there after the interpolation, by (4.9). □

The parameter estimates give sufficient separation conditions that can be imposed once before the frequencies are fixed.

Lemma 4.3 (Lifetime nesting under the schedule)**.** *For the schedule (3.7), suppose that K_j bounds M_j and M_j^{-1} on every lifetime $I_j = [t_j^{\mathrm{in}}, T_*)$, and $|\zeta_j| \le C_\zeta$ there. Suppose also $K_1 = 1$ and $K_q \le B\lambda_{q-1}^{1/16}$ for $q \ge 2$, with fixed B. Writing $y = \log\lambda_1 \ge 1$, sufficient conditions are*

$$Be^{-47y/16} \le \ell_1/4, \qquad B^2e^{-y} \le 1/4, \qquad C_\zeta Be^{-31y/16} \le s_0/2. \tag{4.7}$$

They give $K_jK_q\ell_q \le \ell_j/4$ and $C_\zeta\lambda_jK_q\ell_q \le s_0/2$ for every $j < q$, so $\mathcal{T}_q(t) \subset \mathcal{A}_j(t)$ for every $j < q$ and every $t \in I_q$. For $q \ge 2$ one also has the radius estimate

$$K_q\ell_q \le B\lambda_{q-1}^{-47/16} \le Be^{-47y/16} \le \ell_1/4 < 1/4. \tag{4.8}$$

For $q = 1$, $K_1\ell_1 = \ell_1 = s_0/C_\ell < \min(1/4, s_0)$. Thus the base (4.9) is affine on every support for $t \ge 1$, after the initial interpolation, and all supports lie in one ball of radius strictly below $1/4$.

Proof. For $q \ge 2$, $K_q\ell_q \le B\lambda_{q-1}^{-47/16}$; the first condition in (4.7) proves (4.8). For $2 \le j < q$, use $\lambda_{q-1} \ge \lambda_j = \lambda_{j-1}^{Q_j}$ to obtain

$$\frac{K_jK_q\ell_q}{\ell_j} \le B^2\lambda_{j-1}^{-(47Q_j-49)/16} \le B^2e^{-y} \le \tfrac14.$$

Only the strict gap $47Q_j - 49 > 16$ for $Q_j \ge 202$ is needed: the frequency growth makes the new support small enough despite the older inverse deformation and shrinking material radius. For $j = 1$, the same conclusion follows from $K_1 = 1$ and (4.8). For every $1 \le j < q$, $\lambda_j \le \lambda_{q-1}$ gives $C_\zeta\lambda_jK_q\ell_q \le C_\zeta B\lambda_{q-1}^{-31/16} \le s_0/2$.

For the base and $q \ge 2$, the radius bound gives $K_q\ell_q \le \ell_1/4 = s_0/(4C_\ell) < s_0$. For $q = 1$, $K_1\ell_1 = \ell_1 = s_0/C_\ell < \min(1/4, s_0)$ directly. The base envelope condition follows from these radius bounds. Lemma 4.2 now applies on the full common lifetime, since $I_q \subset I_j$ for $j < q$. □

4.2. **The exact initial datum and the rotating base.** The periodic profile used for the layers need not equal the prescribed unit sine in (1.8). We connect the two by a smooth initial interpolation before any layer is activated. Both profiles have the same derivative at the origin, so this interpolation preserves the temperature gradient at that point. At its completion, the profile is linear near the origin and gives the affine background that starts the controlled evolution.

Choose $R_H > R_0 + 1$ and take radial $H \in C_c^\infty(B_{R_H})$ equal to $|x|^2/2$ on B_{R_0+1}. Require $\alpha = 0$ on $[0, 1]$. Define

$$\begin{aligned} \theta^0(x,t) &= -F(R_\alpha e_2 \cdot x)\chi_0(x), \qquad & \theta^{\mathrm{ref}}(x) &= \theta^0(x,0), \\ \theta^b &= \theta^0 + (1-\eta(t))(\theta_{\mathrm{in}} - \theta^{\mathrm{ref}}), \qquad & \psi^b &= \dot\alpha H, \quad u^b = \nabla^\perp\psi^b. \end{aligned} \tag{4.9}$$

On $[0,1]$, $u^b = 0$ and $\partial_t\theta^b = \eta'(t)(\theta^{\mathrm{ref}} - \theta_{\mathrm{in}})$. Thus the initial datum remains the unit sine in (1.8). After $t = 1$ the interpolation term vanishes. On the support of θ^0, $u^b = \dot\alpha Jx$, and direct differentiation gives $(\partial_t + u^b\cdot\nabla)\theta^0 = 0$; radiality of χ_0 is used here. Since ΔH is radial, $u^b\cdot\nabla\Delta H = 0$. Consequently the complete base forces in scalar and curl form are

$$f^b_\theta = \eta'(t)(\theta^{\mathrm{ref}} - \theta_{\mathrm{in}}), \qquad E^b = \ddot\alpha\,\Delta H - \partial_1\theta^b. \tag{4.10}$$

Here E^b is the curl-form force; its buoyancy derivative is the fixed laboratory ∂_1. These forces have the required parity and compact support. Their all-order bounds through T_* require the all-order control of α in Theorem 5.5. The compact core fields (4.4), the full-lifetime nesting of Lemma 4.3, and the explicit base residuals (4.10) are the conclusions of this section. The next section controls their time derivatives; the correction and compact inverse-curl arguments then give the full scalar and vector forces.

5. Derivative bounds over the full lifetime

Theorem 3.2 constructs smooth controls on every finite sequence of stages. We now need derivative bounds that persist as the stages accumulate. Later controls vary on shorter time scales, but their effects on an older layer carry small angular and amplitude weights. Replacing all these contributions by the fastest rate would lose this smallness. We keep each weight with its rate, differentiate on every finite stage, and then sum over the lifetime of a fixed layer. The resulting bounds are collected in Theorem 5.5. Throughout, the selected shooting parameters of Theorem 3.2 are held fixed while we differentiate the resulting trajectories in time.

The scales already fixed are the frequencies, radii and lifetime scales in (3.7), the logarithmic gains in (3.8), and the insertion angles and growth rates in (3.10). The squared growth scale is $\sigma_j^2 = |G_{<j+1}(t_j^b)|$, and K_j bounds the deformation and its inverse over the lifetime, as in (3.17).

Write

$$\mathcal{I}_l = [t_l^{\mathrm{in}}, t_l^b], \quad \mathcal{I}_l^g = [t_l^{\mathrm{in}}, t_l^a], \quad \mathcal{I}_l^b = [t_l^a, t_l^1], \quad \mathcal{I}_l^s = [t_l^1, t_l^b]. \tag{5.1}$$

These are finite stage intervals, whereas $I_q = [t_q^{\mathrm{in}}, T_*)$ is the entire lifetime. The controlled index is $c = l-1$ on growth and transition, and $c = l$ on steering. At stage 1 the controlled base index is 0 on growth and transition. The controlled angle vanishes on growth and on transitions $1, 2$. On a transition after stage 2 it is small but generally nonzero; the choice $c = l-1$ specifies the feedback index, not a zero-angle constraint. Put $P_q = -\Theta_q > 0$, $\rho_q = \log|\zeta_q|$ and $\vartheta_q = d_{q-1,q}$ for $q \ge 2$, where $d_{ij} = \varphi_j - \varphi_i$ is the difference of phase directions in (3.6). Write $\widehat\Omega_q = \sigma_{q-1}\Omega_q/\lambda_q$ for the normalized vorticity-amplitude component of y_q^0. The constant base difference is $d_{01} = s_1 = s_*$. The profile norms used throughout are

$$\nu_n = 1 + \|\eta\|_{C^{n+1}} + \|\eta_2\|_{C^{n+1}} \qquad (n \ge 0). \tag{5.2}$$

5.1. **Rates and auxiliary scale comparisons.** The weighted estimates use separated rates, summable amplitude weights, and domination by the lifetime scale Π_q. We collect these comparisons before applying the differentiation rules. The two kinds of smallness are kept distinct: ϵ_{sch} is the schedule tolerance of Theorem 3.2, and

$$\varepsilon_l = L_l s_l, \quad X_l = L_l\Gamma_l \quad (l \ge 2), \qquad \varepsilon_1 = c_{\mathrm{p}}\Lambda_1\sin s_*, \quad X_1 = \Lambda_1\Gamma_1. \tag{5.3}$$

For $l \ge 2$, ε_l is the quantity ϖ_l in Theorem 3.2. The positive scale $U_q = \sigma_{q-1}\Gamma_q$ measures the scalar coupling $\mathcal{G}_q$ defined in (5.14) below. For $l \ge 3$ put $\mathsf{H}_l = s_{l-1}\sigma_{l-2}$, the old-strain scale h_l of Section 3.7. Define

$$\begin{aligned} \upsilon_{p,j} &= \frac{\sigma_p}{\sigma_{j-1}} \quad (j \ge p+1), \qquad \upsilon^+_{p,p} = 1, \quad \upsilon^+_{p,j} = \upsilon_{p,j} \quad (j > p), \\ h_k(j) &= s_{\min(j,k)}\frac{\sigma_{\min(j,k)-1}}{\sigma_{j-1}} \quad (j \ge 2). \end{aligned} \tag{5.4}$$

Thus $h_k(j) = s_j$ for $j \le k$. The weight $\upsilon_{p,j}$ records the amplitude of an older layer p at the rate of stage j; $h_k(j)$ is its angular counterpart. Constants C depend on the fixed normalized data, the design constants, and the admissible coefficient, angle and geometry bounds in Theorem 3.2. They may be enlarged finitely many times. Derivative constants may additionally depend on the differentiation order and the indicated profile norms, but are uniform in the stage indices, frequencies and admissible choices of k_j in (3.8).

Lemma 5.1 (Auxiliary scale comparisons)**.** *There is one enlargement of λ_{lo} in Theorem 3.2, independent of the correction constants and uniform over $0 \le k_l \le Q_l/120$, for which the following comparisons hold. It includes the fixed requirements*

$$y = \log \lambda_1 \ge \max\{\epsilon_{\mathrm{sch}}^{-1}, (Q_* + 3)^3, 1, C_\sigma^4\}, \qquad \frac{y}{\log y} \ge 160(Q_* + 2), \tag{5.5}$$

together with the finitely many low-order comparisons in the proof. The growth-scale constants c_σ, C_σ are fixed in (3.15); the scales obey the two-sided bounds in (3.18).

(a) Rate identities and separation. *For $l \ge 2$,*

$$X_l = \frac{\sin s_l}{s_l} L_l^2 \sigma_{l-2}, \quad \varepsilon_l \sigma_{l-1} = L_l^2 \sigma_{l-2}, \quad \frac{\Gamma_l}{X_l} = L_l^{-1} \le \epsilon_{\mathrm{sch}}, \quad \frac{\sigma_{l-2}}{\sigma_{l-1}} = \frac{s_l}{L_l}, \tag{5.6}$$

$$\Gamma_{l-1} \le \epsilon_{\mathrm{sch}} \Gamma_l, \quad X_l \ge \epsilon_{\mathrm{sch}}^{-1} X_{l-1}, \quad \varepsilon_l \le \epsilon_{\mathrm{sch}} \varepsilon_{l-1}, \quad \varepsilon_l X_l \le 10^{-l}, \quad \sum_{j \ge 1} \varepsilon_j \le 1, \tag{5.7}$$

(b) Weights for later stages and their sums. *For $l \ge 2$,*

$$\upsilon_{p,l} X_l \le \epsilon_{\mathrm{sch}}^{l-p-1} X_{p+1} \quad (0 \le p \le l-2), \qquad \upsilon_{p,l} \le \epsilon_{\mathrm{sch}} \varepsilon_l \quad (0 \le p \le l-2). \tag{5.8}$$

The consecutive growth-scale ratio is at most ϵ_{sch} with $\epsilon_{\mathrm{sch}} < 1/4$. Consequently

$$\sum_{j \ge p+1} \upsilon_{p,j} \le 1 + C\epsilon_{\mathrm{sch}}, \quad \sum_{i \le p} \sigma_i^a \le C\sigma_p^a \ (a = 1, 2), \qquad \sum_{j \le q} \Gamma_j \le C\Gamma_q, \quad \sum_{j > q} s_j \le \epsilon_{\mathrm{sch}} s_q. \tag{5.9}$$

(c) Older strains. *For $l \ge 3$,*

$$\mathsf{H}_l \asymp \Gamma_{l-1}, \quad \frac{\mathsf{H}_l}{\Gamma_l} \le C\frac{\epsilon_{\mathrm{sch}}}{L_l}, \quad \frac{\mathsf{H}_l}{X_l} \le C\frac{\epsilon_{\mathrm{sch}}}{L_l^2}, \quad \frac{\Gamma_l s_l}{L_l} \le \epsilon_{\mathrm{sch}} \mathsf{H}_l, \tag{5.10}$$

$$\sum_{i \le l-2} \sigma_i s_{i+1} \le C\mathsf{H}_l, \qquad \sum_{i \le k} \sigma_i^2 s_{i+1} \le C\sigma_k^2 s_{k+1}, \qquad 1 \le \epsilon_{\mathrm{sch}} U_l. \tag{5.11}$$

The sums over shears begin at 1; the last assertion holds also for $l = 2$.

(d) Lifetime coefficient scales. *For every $q \ge 1$,*

$$X_{q+1} \le \epsilon_{\mathrm{sch}} \Pi_q, \quad \sum_{j \le q+1} X_j^n \le \Pi_q^n \quad (n \ge 1), \quad C_\Pi \sigma_{q-1}^2 \le \Pi_q, \quad \Pi_1 \le \Pi_q, \tag{5.12}$$

$$K_q^2 \le \Pi_q, \quad \ell_q^{-1} \le \Pi_q, \quad 1 \le \sigma_{q-1} \le \Pi_q, \quad \Gamma_q^{\pm 1} \le \Pi_q, \quad |I_q| \le \Pi_q, \quad \|\alpha'\|_\infty \le C_\alpha \le \Pi_1/2. \tag{5.13}$$

This single choice of λ_{lo} is valid for every derivative order.

Proof. We verify (a)–(d), keeping the exceptional layers $l = 1, 2$ explicit. Lemma 3.13 supplies the gain comparisons and the bounds for s_l, $\varepsilon_l = L_l s_l$ and $L_l^4 \sigma_{l-2}^2 / \sigma_{l-1}$. The frequency recursion gives

$\log\lambda_l \ge 201^{l-1}y \ge l^2$ and $Q_l \le \log\lambda_{l-1}$ for $l \ge 2$. For $l \ge 3$ put $x = \log\lambda_{l-2} \ge y$. Besides the bounds for s_l, ε_l and $L_l^4\sigma_{l-2}^2/\sigma_{l-1}$ just recalled, the rate and future-scale ratios use

$$\frac{Q_{l+1}^4}{\sigma_{l-1}} \le Cx^{12}e^{-x/16}.$$

It follows from the same growth-scale bounds and $\sigma_{l-1} \ge \sigma_{l-2}^3$. All four bounds are of the form $Cx^a e^{-bx}$ with $b \ge 1/16$. Equation (3.51) therefore makes them at most $e^{-x/32}$ for every $x \ge y$, retaining decay as the logarithmic frequency increases. At $l = 2$ the corresponding small quantities are bounded by $C(Q_*+3)^8y^4e^{-y/16}$ and the same argument applies at y. Here $\sigma_0 = 1$; no frequency law at index zero is used. The first-layer separation $\sigma_1 \ge \epsilon_{\mathrm{sch}}^{-1}$ was already established in Lemma 3.13.

The identities (5.6) and $1 - s^2/6 \le \sin s/s \le 1$ give the stated comparisons between exact and comparison rates. For $l \ge 3$, $\Gamma_{l-1}/\Gamma_l \le C(L_{l-1}/L_l)(\sigma_{l-3}/\sigma_{l-2})$; at $l = 2$ the ratio is at most Cs_1/L_2. This proves the rate separation. The identities

$$\frac{s_{l+1}}{s_l} = \frac{L_{l+1}}{L_l}\frac{\sigma_{l-1}^2}{\sigma_l\sigma_{l-2}}, \qquad \frac{\varepsilon_{l+1}}{\varepsilon_l} = \left(\frac{L_{l+1}}{L_l}\right)^2\frac{\sigma_{l-1}^2}{\sigma_l\sigma_{l-2}}$$

are bounded by CQ_{l+1}^4/σ_{l-1}; their first ratios use s_1 and ε_1 separately. They give geometric summability. At the first layer $\varepsilon_1 H_2 \le 1/4$ and $H_2 \ge 1$; make the tail over later layers at most $1/2$ by one finite threshold, so $\sum\varepsilon_l \le 1$. Moreover $\varepsilon_l X_l \le L_l^4\sigma_{l-2}^2/\sigma_{l-1}$. For $l \ge 3$ its bound is $Ce^{-201^{l-3}y/32} \le 10^{-l}$; the second-layer term is made at most 10^{-2} directly by the bound $C(Q_*+3)^8y^4e^{-y/16}$ above. This proves $\varepsilon_l X_l \le 10^{-l}$ in (5.7).

For (5.8), at $l = p+2$ and $p \ge 1$ the ratio to X_{p+1} is at most CQ_{p+2}^4/σ_p. For $p = 0$ it is at most $CL_2^2/\sigma_1 = C\varepsilon_2$, since $X_1 = \Lambda_1\sin s_*$. Advancing l multiplies this ratio by at most CQ_{l+1}^4/σ_{l-1}. The other assertion follows from $\varepsilon_l = L_l^2\sigma_{l-2}/\sigma_{l-1}$ and $L_l \ge \epsilon_{\mathrm{sch}}^{-1}$.

The identities for the earlier-stage strain scales follow from

$$\frac{\mathsf{H}_l}{\Gamma_l} = \frac{s_{l-1}s_l}{L_l\sin s_l}, \qquad \frac{\Gamma_l s_l}{L_l\mathsf{H}_l} \le \frac{s_l}{s_{l-1}}.$$

Successive summands in the two sums over older shears have ratios at most ϵ_{sch}: use $\sigma_i s_{i+1} = L_{i+1}\sigma_{i-1}$ and $\sigma_i^2 s_{i+1} = L_{i+1}\sigma_{i-1}\sigma_i$. For the base term, $U_2^{-1} \le C/(L_2\sigma_1) \le \epsilon_{\mathrm{sch}}$; the later U_l increase.

For $q \ge 2$,

$$X_{q+1} \le Q_{q+1}^4(\log\lambda_q)^2\sigma_{q-1} \le C(\log\lambda_q)^6\lambda_{q-1}^{1/16} \le \epsilon_{\mathrm{sch}}\Pi_q.$$

For $q = 1$, $X_2 \le Q_2^4y^2 \le \epsilon_{\mathrm{sch}}y^{10} = \epsilon_{\mathrm{sch}}\Pi_1$ by the first-layer polynomial condition. Rate separation now gives the sum in (5.12). The growth-scale bound gives $C_\Pi\sigma_{q-1}^2 \le \Pi_q$ for $q \ge 2$. For $q = 1$ it suffices to require $C_\Pi \le y^{10} = \Pi_1$. The already proved $K_q \le C_K/s_q \le \sigma_{q-1}$ for $q \ge 2$, and $K_1 = 1$, prove $K_q^2 \le \Pi_q$ in (5.13). The first material radius is a fixed number and $\ell_q^{-1} = \lambda_{q-1}^3$ otherwise. For $q \ge 2$, $\Gamma_q \ge (1 - C\epsilon_{\mathrm{sch}})L_q\sigma_{q-2} \ge 1$; $\Gamma_1^{-1} \le (1/\sin s_1)$ is a fixed-data bound. The lifetime estimate gives $|I_q| \le C/\sigma_{q-2}$ for $q \ge 2$, while $|I_1| \le (L_1 + 2 + \Lambda_1^{-1})/\Gamma_1 + C$. All are absorbed by the displayed Π_q, including the logarithmic Π_1 after increasing y.

The first-order control bound is already proved in (3.70), independently of higher derivatives. We retain that same C_α, including its actual first-pulse factor $c_{\mathrm{p}}\Lambda_1^2$. Thus $\|\alpha'\|_\infty \le C_\alpha$, and $2C_\alpha \le \Pi_1$ is the fixed-data polynomial condition already included in λ_{lo}. Thus the separated-rate and weight estimates in (a)–(c), and the coefficient scales in (d), are available at every derivative order with this one threshold. □

5.2. **Weighted differentiation.** A weighted derivative bound is a sum of nonnegative contributions, each with its own amplitude and rate. Differentiation raises the rate to the derivative order but leaves its small amplitude attached. The finite lists below record this information. Their mass is the sum of their amplitudes, while effective size also includes an independent bound for the undifferentiated function. The lists do not change when the derivative order increases.

A finite list $\mathsf{L} = (a_j@r_j)$ consists of nonnegative amplitudes and positive rates. Write $\mathsf{L}[n] = \sum_j a_j r_j^n$, its mass $\Sigma_{\mathsf{L}} = \mathsf{L}[0]$, and $f \preccurlyeq [S;\mathsf{L}]$ through order n when

$$|f| \le S, \qquad |f^{(b)}| \le C_b \mathsf{L}[b] \quad (1 \le b \le n).$$

The effective size is $\widehat{S}_f = \max(S, \Sigma_{\mathsf{L}_f})$. Absolute factors may be put into C_b, but the lists themselves and their effective-size bounds are fixed, independently of the order. The symbol $\sqcup$ denotes concatenation of lists, and $r \vee R = \max(r, R)$.

For example, the product of two functions has the first function's list multiplied by the second's effective size, together with the second list multiplied by the first's effective size. Terms with time derivatives on both factors are included by the rate inequality in the proof below. The minimum rate used for an antiderivative records its undifferentiated drift without discarding contributions at faster rates.

Lemma 5.2 (Calculus of weighted derivative bounds)**.** *The following rules preserve order-dependent constants, independent of the number of stages.*

(i) Products. *Products of r factors have the list*

$$\mathsf{L}_{\prod f_i} = \bigsqcup_{i=1}^{r} \left(\prod_{h \ne i} \widehat{S}_{f_h} \right) \mathsf{L}_{f_i}.$$

(ii) Composition. *For F smooth on a fixed neighborhood of the range of f, composition $F(f)$ has the same list with a new constant depending on the order, the derivatives of F, and an upper bound for $\widehat{S}_f$.*

(iii) Antiderivatives. *If $X' = V$, then through the next order the list of X, with any minimum rate $R > 0$, is*

$$(S_V/R)@R \quad \sqcup \quad \left(\frac{a_j}{r_j \vee R} @ (r_j \vee R) \right)_j \qquad \text{when } \mathsf{L}_V = (a_j@r_j).$$

(iv) Sums. *For sums, concatenate lists and use the maximum of the constants, not their sum. Merge equal rates by adding their amplitudes.*

(v) Linear systems. *If $Y' = BY$ and $\|B^{(i)}\| \le c_i R^{i+1}$ for $0 \le i < n$, then*

$$Y^{(n)} = \mathcal{P}_n Y, \quad \mathcal{P}_1 = B, \quad \mathcal{P}_{n+1} = \mathcal{P}_n' + \mathcal{P}_n B, \qquad \|\mathcal{P}_n\| \le n! \max_{i<n}(1, c_i)^n R^n.$$

Proof. For positive rates and positive integers b_i with sum b, $\prod_i r_i^{b_i} \le \sum_i r_i^b$. Expanding the finite sums yields

$$\prod_i \mathsf{L}_i[b_i] \le \sum_i \left(\prod_{h \ne i} \Sigma_{\mathsf{L}_h} \right) \mathsf{L}_i[b].$$

In the multinomial Leibniz formula, undifferentiated factors use their size bounds; products of differentiated factors use the displayed mass inequality. The resulting constant is determined by the multinomial coefficients and the derivative constants of the factors. For composition, the set-partition form of the chain rule gives $F^{(|\pi|)}(f) \prod_{B \in \pi} f^{(|B|)}$. The preceding inequality bounds the product of lists by $|\pi| \widehat{S}_f^{|\pi|-1} \mathsf{L}_f[b]$. There are finitely many partitions depending only on b. The minimum-rate rule is the identity $X^{(b)} = V^{(b-1)}$: for $b = 1$ use the drift term, and for $b > 1$ use $r_j^{b-1} \le (r_j \vee R)^{b-1}$. The sum and merge rules are the triangle inequality. For the last rule, $\mathcal{P}_n$ is a sum of at most $n!$ ordered monomials $\prod B^{(i_h)}$ with $\sum_h (i_h + 1) = n$, as follows by differentiating and multiplying once more by B. This proves both the recurrence and its bound. $\square$

Only $\sin, \cos, \tan, \sec, \exp(\pm\cdot)$ and arcsin occur as outer functions. Their arguments remain in fixed compact subsets of their smooth domains by the low-order cone and phase-gradient length estimates. Thus the composition constants never depend on a frequency parameter.

5.3. **Weighted bounds for the coupled equations.** On $\mathcal{I}_l$, every older activation cutoff is 1. Define the scalar coefficient

$$\mathcal{G}_q = \sum_{0\le i<q} A_i \sin d_{iq}, \qquad A_0 = 1. \tag{5.14}$$

Its sign is fixed by $J\zeta_q \cdot G_{<q} = -|\zeta_q|\mathcal{G}_q$. The exact angle and amplitude equations are

$$\begin{aligned} \vartheta_q' &= -\Omega_{q-1}\sin^2\vartheta_q - \sin\vartheta_q \sum_{i\le q-2} \Omega_i \sin(d_{iq} + d_{i,q-1}), \\ \rho_q' &= \tfrac{1}{2}\sum_{i<q} \Omega_i \sin(2d_{iq}), \\ P_q' &= -\frac{\mathcal{G}_q}{\lambda_q e^{\rho_q}}\Omega_q, \qquad \Omega_q' = -A_q \sin\varphi_q. \end{aligned} \tag{5.15}$$

The first identity follows from $\sin^2 a - \sin^2 b = \sin(a+b)\sin(a-b)$. Moreover

$$d_{0k} = s_1 + \sum_{2\le q\le k} \vartheta_q, \quad d_{pk} = \sum_{p<q\le k} \vartheta_q \ (p \ge 1), \quad \varphi_p = \varphi_c - d_{pc} \ (p<c), \quad \alpha = d_{0c} - \varphi_c.$$

In particular, no derivative of α is an input to (5.15).

Theorem 3.2 supplies $|\rho_q| \le C_\rho$, $e^{\rho_q} \in [1 - C\epsilon_{\mathrm{sch}}, \zeta_+]$, adjacent angles comparable to s_q, $A_q \asymp \sigma_q^2$ after the end of steering and $|\Omega_q| \le C\sigma_q$ after its holding interval. Write $S_q = C_P P_q(t_q^b)$ for a fixed design-dependent C_P large enough that $P_q \le S_q$ thereafter. The value at the end of steering gives

$$\lambda_q S_q \le C\sigma_q^2, \qquad \lambda_q^{-1} \le C S_q/\sigma_q^2. \tag{5.16}$$

The explicit low-order estimates of Theorem 3.2 continue to hold wherever we use a coarse bound C. In particular, $\mathcal{G}_q > 0$ and $\mathcal{G}_q \le CU_q$: the preceding layer term is at most $C\sigma_{q-1}^2 s_q \le CU_q$, the earlier terms sum by (5.11), and the base term is at most $\epsilon_{\mathrm{sch}} U_q$. At $q = 2$ use $(\sigma_1^2 s_2)^{-1} = 1/(L_2\sigma_1)$. For $q = 1$, $\rho_1 = 0$ and $\mathcal{G}_1 = U_1$ exactly.

For the amplitude derivatives put

$$\beta_q(j) = \frac{\Gamma_q}{X_j}\upsilon_{q-1,j}\,\mathbf{1}_{q\ge 2,\ j\ge q} + \frac{\sigma_{q-1}}{\sigma_q}\upsilon_{q,j}^+\,\mathbf{1}_{j\ge q}. \tag{5.17}$$

A table row with size S and weights b_j means $f \preccurlyeq [S; (b_j @ X_j)_j]$ through the indicated order; unstated entries are zero. The quantities $e_q(j)$ in the derived table are nonnegative scale weights constructed in (5.24). The next proposition treats the primary and derived estimates together: the derived estimates at the previous order are inputs to the next primary estimates, and are established again before closing each order.

The stage intervals and their controlled index c are specified at (5.1); the rates and angular/amplitude weights are (5.3)–(5.4). In the table below, *generic transition* means $l \ge 3$, whereas *generic steering* means $l \ge 2$.

Proposition 5.3 (Weighted derivative bounds on each stage)**.** *There are nondecreasing constants C_n, depending only on n, ν_n from (5.2) and the fixed design and low-order bounds, such that the following estimates hold through order n, simultaneously on every $\mathcal{I}_l$.*

Primary quantities. *The angle, phase-gradient length and older-amplitude estimates have the following weights at rate X_j:*

$$\begin{array}{c|c|c} \textit{quantity} & \textit{size} & \textit{derivative weights and index range} \\ \hline \vartheta_q,\ 2\le q\le l & Cs_q & s_q\upsilon_{q-1,j},\ q\le j\le l \\ \rho_q,\ e^{\pm\rho_q},\ 2\le q\le l & C & (\Gamma_q/X_j)\upsilon_{q-1,j},\ q\le j\le l \\ P_q,\ 1\le q<l & S_q & S_q(\sigma_{q-1}/\sigma_q)\upsilon^+_{q,j},\ q\le j\le l \\ \Omega_p,\ 1\le p<l & C\sigma_p & \sigma_p\upsilon_{p,j},\ p+1\le j\le l. \end{array} \tag{5.18}$$

Controlled angle and normalized amplitude pair. *The controlled-angle estimates are*

$$\begin{array}{l|c|c} \textit{interval} & \textit{size of } \varphi_c & \textit{list} \\ \hline \textit{growth; transitions } 1,2 & 0 & \varnothing \\ \textit{generic transition} & C\epsilon_{\rm sch}s_l & s_l@X_l \\ \textit{generic steering} & C\varepsilon_l & \varepsilon_l@X_l \\ \textit{first steering} & C & \sin s_*@X_1 \end{array} \tag{5.19}$$

Here the first-steering constant is fixed by the design. The normalized pair $y_l^0=(\Theta_l,\sigma_{l-1}\Omega_l/\lambda_l)$ has a pointwise relative derivative estimate on its own interval:

$$(y_l^0)^{(n)}=\mathcal{P}_{l,n}y_l^0,\qquad \|\mathcal{P}_{l,n}\|\le C_nX_l^n.$$

Derived quantities. *The following estimates bound pairwise angles, gradient amplitudes and the scalar coupling for use at the next order:*

$$\begin{array}{c|c|c} d_{pk},\ 0\le p<k\le l & Cs_{p+1}\ (p\ge 1),\ Cs_1\ (p=0) & Ch_k(j),\ \max(p+1,2)\le j\le l \\ A_q,\ 1\le q<l & C\sigma_q^2 & C\sigma_q^2\beta_q(j),\ q\le j\le l \\ \mathcal{G}_q,\ 1\le q\le l & CU_q & U_qe_q(j),\ j\le l. \end{array} \tag{5.20}$$

Here $e_1=0$. For $q\ge 2$ there exist nonnegative scale weights $e_q(j)$, independent of n,l, constructed in the proof, such that

$$e_q(j)\le C\upsilon_{q-1,j}\ (j\ge q),\qquad e_q(q-1)\le C\epsilon_{\rm sch},\qquad \sum_{j\le q-2}e_q(j)\le C\epsilon_{\rm sch}.$$

Rotation. *Recovering α from the controlled angle gives*

$$|\alpha^{(n)}|\le C_n\sum_{j\le l}\varepsilon_jX_j^n\quad (n\ge 1). \tag{5.21}$$

All effective sizes are at most a fixed multiple of their indicated sizes, except on transitions after stage 2: the effective sizes of φ_{l-1} and Ω_{l-1} are s_l and σ_{l-1}, although their actual sizes are $C\epsilon_{\rm sch}s_l$ and $C\epsilon_{\rm sch}\sigma_{l-1}$.

Proof. The order-zero statements follow from Theorem 3.2 and (5.16). Here we use the pairwise-angle bounds following (3.59) and the transition estimates leading to (3.65). The list masses follow by summing $\upsilon_{p,j}$ and s_j. In particular the mass of the list for $\rho_q,e^{\pm\rho_q}$ is $O(L_q^{-1})$, the P_q mass is $O(S_q\sigma_{q-1}/\sigma_q)$ and the A_q mass is $O(\epsilon_{\rm sch}\sigma_q^2)$. The two transition exceptions have precisely the masses stated.

The induction is on the derivative order, simultaneously for every finite stage $\mathcal{I}_l$, and includes all primary and derived estimates. Assume all estimates through order n, simultaneously for every l. We produce the next order in the following order of dependence:

$$\vartheta,\rho,\ e^{\pm\rho};\quad \Omega,P;\quad \varphi_c;\quad y_l^0;\quad d,A,\mathcal{G};\quad \alpha.$$

The first two differential equations use only estimates at the previous order. The steering angle then uses the new derivatives of ρ_l; the transition angle uses only estimates from previous orders. The normalized pair y_l^0 uses coefficients through order n. The remaining estimates are algebraic consequences of rows already obtained at the new order. Thus each new estimate uses old-order estimates and only those new-order estimates earlier in the displayed sequence. In particular the derived estimates are restored at every order.

We give the list computations, including the terms at rates below the relevant minimum rate. Constants arising from the rules in Lemma 5.2 are absorbed into an order-dependent number common to each finite family of rows. A target ratio compares a collected amplitude with the claimed amplitude at the same rate. The antiderivative rule, Lemma 5.2(iii), raises entries below its minimum rate R to R; the contribution $(S_V/R)@R$ records the undifferentiated drift.

Adjacent angles. The right side of the first equation in (5.15) has size $C\Gamma_q s_q$. For its squared-sine term, apply the product rule of Lemma 5.2(i) to $\Omega_{q-1}\sin\vartheta_q\sin\vartheta_q$. Each sine factor has effective size Cs_q and weight $Cs_q\upsilon_{q-1,j}$ at X_j. Their product therefore has weight $Cs_q^2\upsilon_{q-1,j}$, retaining the full square. Multiplication by Ω_{q-1} yields the first of the following four families:

$$\begin{array}{ll}\Gamma_q s_q\upsilon_{q-1,j}, & j\ge q,\\ s_q s_{i+1}\sigma_i\upsilon_{i,j}, & j\ge i+1,\ i\le q-2,\\ s_q\sigma_i(h_q(j)+h_{q-1}(j)), & j\ge i+1,\ i\le q-2,\\ s_q\sigma_i s_{i+1}\upsilon_{q-1,j}, & j\ge q,\ i\le q-2.\end{array}$$

These are respectively the preceding layer term and the three choices of differentiated factor in the older product. Apply the minimum rate X_q. The first and fourth families divided by X_j are bounded by $Cs_q\upsilon_{q-1,j}\Gamma_q/X_j$. For the second family at $j\ge q$ use $\upsilon_{i,j}/\upsilon_{q-1,j}=\sigma_i/\sigma_{q-1}$. Its sum of ratios to the target is at most $CX_j^{-1}\sum_{i\le q-2}s_{i+1}\sigma_i(\sigma_i/\sigma_{q-1})\le C\Gamma_{q-1}/X_j$. For the third family use

$$h_q(j)=s_q\upsilon_{q-1,j},\qquad h_{q-1}(j)=s_{q-1}\upsilon_{q-2,j}\quad (j\ge q).$$

Its summed target ratio is bounded by $C\Gamma_q/X_j$. For the contributions with $j<q$, which the antiderivative rule estimates at rate X_q, use $\sum_{j\ge i+1}\upsilon_{i,j}\le C$ from (5.9), and $\sum_{i+1\le j<q}s_j\le Cs_{i+1}$. Their total amplitude is at most $Cs_q\Gamma_{q-1}/X_q\le Cs_q$. The drift has amplitude $C\Gamma_q s_q/X_q\le Cs_q$. This proves the first estimate at order $n+1$.

Phase-gradient lengths. The equation for ρ_q has size $C\Gamma_q$ and list families

$$\Gamma_q\upsilon_{q-1,j}\ (j\ge q),\qquad s_{i+1}\sigma_i\upsilon_{i,j},\sigma_i h_q(j)\ (j\ge i+1,\ i\le q-2).$$

Use the minimum rate X_q and target $(\Gamma_q/X_j)\upsilon_{q-1,j}$. The preceding layer ratio is C. The older ratios at $j\ge q$ are bounded respectively by $C\Gamma_{q-1}/\Gamma_q$ and $C(s_q/\Gamma_q)\sum_{i\le q-2}\sigma_i$. The raised total and the drift have ratios at most $C\Gamma_{q-1}/\Gamma_q+C$. This proves the estimate for ρ_q. Composition with $e^{\pm\rho_q}$ on $|\rho_q|\le C_\rho$ gives the stated lists for both exponentials at the same new order.

Earlier vorticity amplitudes. For $p<l$, the holding interval has $\Omega_p=\varphi_p=0$ through t^a_{p+1}; $\Omega_1=0$ also on transition 2. Elsewhere put $\varphi_p=\varphi_c-d_{pc}$, with the angle term absent if $p=c$. Write $\mathfrak{a}_c$ for the controlled-angle amplitude at X_l, so $\mathfrak{a}_c=0,s_l,\varepsilon_l$ on the nonexceptional growth, transition and steering intervals, respectively. The effective size $\widehat{\Phi}_p$ of $\sin\varphi_p$ satisfies

$$\widehat{\Phi}_p\le C(\mathfrak{a}_c+s_{p+1}\mathbf{1}_{p<c}),\qquad \sigma_p\widehat{\Phi}_p\le CX_{p+1}.$$

For the first inequality include the controlled-angle size $C\varepsilon_l$ on steering; it is comparable to $\mathfrak{a}_c$. For the second, use $\sigma_p\varepsilon_{p+1}\le CX_{p+1}$ and $\sigma_p\varepsilon_l\le C\upsilon_{p,l}X_l\le CX_{p+1}$ when $l\ge p+2$. On the immediate transition the effective size is s_{p+1}, not its smaller actual size.

The equation $\Omega'_p=-A_p\sin\varphi_p$ therefore gives list

$$C\sigma_p^2\widehat{\Phi}_p\beta_p(j)\ (j\ge p),\qquad C\sigma_p^2\mathfrak{a}_c\ (j=l),\qquad C\sigma_p^2 h_c(j)\ (j\ge p+1,\ p<c).$$

Use the minimum rate X_{p+1} and target $\sigma_p\upsilon_{p,j}=\sigma_p^2/\sigma_{j-1}$. For the controlled entry its ratio after integration is $C\sigma_{l-1}\mathfrak{a}_c/X_l\le C$. For the angle entries it is $C\sigma_{j-1}h_c(j)/X_j\le C/L_j$ if $j\le c$. The only other case is $j=l=c+1$, whose ratio is $C\mathsf{H}_l/X_l$. For $j>p$,

$$\beta_p(j)\sigma_{j-1}=\sigma_{p-1}\left(1+\frac{\Gamma_p}{X_j}\mathbf{1}_{p\ge2}\right);$$

hence those ratios are at most $C(\sigma_{p-1}/\sigma_p)(\sigma_p\widehat{\Phi}_p/X_{p+1}) \le C\epsilon_{\mathrm{sch}}$. The raised $j = p$ entry has the same bound because $\beta_p(p) \le C\epsilon_{\mathrm{sch}}$. The drift ratio is bounded by $C\sigma_p|\sin\varphi_p|/X_{p+1} \le C$. This proves the vorticity-amplitude estimate without assuming a sign for the preceding layer's vorticity amplitude during the current stage's transition interval.

Earlier temperature amplitudes. The equation for P_q and (5.16) give size $C\overline{S}_q$, where $\overline{S}_q = S_q\Gamma_q\sigma_{q-1}/\sigma_q$, and list

$$C\overline{S}_q\left[e_q(j) + \frac{\Gamma_q}{X_j}\upsilon_{q-1,j}\mathbf{1}_{q\ge 2,\ j\ge q} + \upsilon_{q,j}\mathbf{1}_{j\ge q+1}\right].$$

Use the minimum rate X_q and target $S_q(\sigma_{q-1}/\sigma_q)\upsilon^+_{q,j}$. The drift ratio is Γ_q/X_q, equal to L_q^{-1} for $q \ge 2$ and exactly Λ_1^{-1} for $q = 1$. The raised entries below q have total ratio $C(\Gamma_q/X_q)\sum_{j<q} e_q(j) \le C$. For $j \ge q+1$, $e_q(j) \le C\upsilon_{q-1,j} \le C\upsilon_{q,j}$, so the remaining ratios are at most $C\Gamma_q/X_j \le C$. The $j = q$ term obeys the same estimate. For $q = 1$, the lists for ρ_1 and $\mathcal{G}_1$ are empty and the drift ratio is the fixed design quantity $1/\Lambda_1$. This proves the temperature estimate at the next order.

Controlled angle on steering. The prescribed profile has derivatives $|Z_l^{(b)}| \le C\nu_b\varepsilon_l X_l^b$ on steering for $l \ge 2$. It has size $C\varepsilon_l$, since its two profile pieces are disjoint and its fixed height is at most $3L_l\sin s_l$. At $q = l$ the list for ρ_l has the single amplitude $\Gamma_l/X_l = 1/L_l$. Thus the product $Z_l e^{-\rho_l}$ has list $C\varepsilon_l@X_l$ at the new order. The positive cosine in Theorem 3.2 fixes the branch $\varphi_l = \arcsin(Z_l e^{-\rho_l})$, and the composition rule proves the required row. At stage 1 use $\rho_1 = 0$, $|Z_1| \le 1/4$ and $|Z_1^{(b)}| \le \nu_b(\Lambda_1^{-b} + c_{\mathrm{p}}\Lambda_1)\sin s_* X_1^b$ instead. The first controlled-angle list can be $\sin s_*@X_1$, with a fixed design factor in its constants; arcsin is composed only on the fixed compact interval $[-1/4, 1/4]$. Growth and transitions $1, 2$ have their exact zero controlled angles.

Controlled angle on transitions after stage 2. Here the factored feedback identity is

$$\begin{aligned}
\varphi'_{l-1} = {} & -\eta_b\Omega_{l-1}\sin\vartheta_l\sec\varphi_l\sin\varphi_{l-1} \\
& + \eta_b\sin\vartheta_l\,\mathfrak{H}_1 - \tfrac12\eta_b\tan\varphi_l\,\mathfrak{H}_2 - \tfrac12(1-\eta_b)\tan\varphi_{l-1}\,\mathfrak{H}_3, \\
\mathfrak{H}_1 = {} & \sum_{i\le l-2}\Omega_i\sin(d_{il} + d_{i,l-1}), \\
\mathfrak{H}_2 = {} & \sum_{i\le l-2}\Omega_i\sin(2d_{il}), \qquad \mathfrak{H}_3 = \sum_{i\le l-2}\Omega_i\sin(2d_{i,l-1}).
\end{aligned}\tag{5.22}$$

It follows by inserting (3.4) and (3.5) into $\varphi'_{l-1} = -\alpha' - \Sigma_{l-1}$. Using the angular and vorticity estimates through order n, each $\mathfrak{H}_r$ has size $C\mathsf{H}_l$ and the list

$$(C\Gamma_j@X_j)_{2\le j\le l-1} \quad \sqcup \quad (C\Gamma_l s_l/L_l)@X_l, \qquad \widehat{S}_{\mathfrak{H}_r} \le C\mathsf{H}_l.\tag{5.23}$$

Indeed the product rule at X_j, $j < l$, gives

$$\frac{C}{\sigma_{j-1}}\sum_{i\le j-1}\sigma_i^2 s_{i+1} + Cs_j\sum_{i\le j-1}\sigma_i \le C\Gamma_j.$$

At X_l, the two angular lists contribute $C(s_l + s_{l-1}\sigma_{l-2}/\sigma_{l-1})\sum_{i\le l-2}\sigma_i$; the vorticity list contributes $C\sigma_{l-1}^{-1}\sum_{i\le l-2}\sigma_i^2 s_{i+1}$. Their sum is at most $C(\sigma_{l-2}/\sigma_{l-1})(\Gamma_l + \mathsf{H}_l) \le C\Gamma_l s_l/L_l$. The mass bound follows from the separated Γ_j and $\Gamma_l s_l/L_l \le \epsilon_{\mathrm{sch}}\mathsf{H}_l$.

On this transition, use effective sizes $\widehat{S}_{\Omega_{l-1}} = \sigma_{l-1}$ and $\widehat{S}_{\sin\varphi_{l-1}} \le Cs_l$. The first product in (5.22) has actual size $C\epsilon^2_{\mathrm{sch}}\Gamma_l s_l$, but list amplitude $C\Gamma_l s_l$ at X_l. The remaining three products have amplitudes $Cs_l\Gamma_j$ at $j < l$ and $C(s_l\mathsf{H}_l + \Gamma_l s_l^2/L_l)$ at X_l. This accounts for derivatives on every factor, including the profile and the secant or tangent. The size of the whole right side is at most

$C(\epsilon_{\rm sch}^2\Gamma_l s_l + s_l\mathsf{H}_l)$. Using the minimum rate X_l, the resulting single amplitude is at most

$$Cs_l\left[\frac{\Gamma_l}{X_l}+\frac{\mathsf{H}_l}{X_l}+\frac{s_l}{L_l^2}+\frac{1}{X_l}\sum_{j<l}\Gamma_j\right]\le C\epsilon_{\rm sch}s_l.$$

The drift obeys the same bound. This proves the transition estimate in (5.19). The product estimate uses the effective sizes s_l and σ_{l-1}; the decisive ratios are $\sigma_{l-1}s_l/X_l \asymp L_l^{-1}$ and $\mathsf{H}_l/X_l \le C\epsilon_{\rm sch}/L_l^2$.

The new amplitude pair on its own stage. The normalized pair of (3.17) has matrix

$$\mathcal{B}_l=\begin{pmatrix}0 & \mathcal{G}_l/(\sigma_{l-1}e^{\rho_l})\\ \sigma_{l-1}e^{\rho_l}\sin\varphi_l & 0\end{pmatrix}.$$

All list rates are at most X_l. The list for $\mathcal{G}_l$ has mass CU_l and the list for ρ_l has mass C/L_l, so the upper entry satisfies $|\widehat{a}_l^{(i)}|\le C_i\Gamma_l X_l^i$. On growth and transition, $\varphi_l=\varphi_{l-1}+\vartheta_l$ has size and mass Cs_l, giving the same bound for $\widehat{b}_l$. On steering the exact equality $\widehat{b}_l=\sigma_{l-1}Z_l$ gives $|\widehat{b}_l^{(i)}|\le C_iX_l^{i+1}$. The first upper entry is exactly Γ_1; the first lower entry is Γ_1 on growth and transition and Z_1 on steering. Here $\Gamma_1/X_1=\Lambda_1^{-1}$, and $\varepsilon_1/X_1=c_{\rm p}$ exactly. Thus $|\widehat{b}_1^{(i)}|\le C_iX_1^{i+1}$, with constants depending on the fixed design and profile norms, as required. Lemma 5.2(v) gives the relative derivative estimate for y_l^0 at order $n+1$ from coefficient estimates through order n only.

Derived estimates. Summing adjacent-angle lists gives, at X_j,

$$\sum_{\max(p+1,2)\le q\le\min(k,j)} s_q\frac{\sigma_{q-1}}{\sigma_{j-1}}\le C\frac{\Gamma_{\min(k,j)}}{\sigma_{j-1}}\le Ch_k(j).$$

The constant base angle has zero derivatives of every positive order. Next $A_q=\lambda_q e^{\rho_q}P_q$, the product rule and (5.16) give exactly the $\beta_q(j)$ row.

We construct the gradient weights by collecting the following nonnegative amplitudes and dividing their sum at each rate by U_q:

$$\begin{array}{ll} C\sigma_i^2 s_{i+1}\beta_i(j), & 1\le i<q,\ j\ge i,\\ C\sigma_i^2 h_q(j), & 1\le i<q,\ j\ge i+1,\\ Ch_q(j), & j\ge 2.\end{array}\tag{5.24}$$

At the direct preceding layer term use the adjacent-angle list itself, so its angular amplitude is $C\sigma_{q-1}^2 s_q\upsilon_{q-1,j}$ for $j\ge q$. Divide the collected amplitudes by U_q to define $e_q(j)$. These weights depend only on the scales, not on the derivative order. To verify their bounds, note

$$\beta_i(i)\le C\epsilon_{\rm sch},\quad \beta_i(j)\le C\epsilon_{\rm sch}\upsilon_{i,j}\ (j>i),\quad U_{i+1}/U_q\le C\epsilon_{\rm sch}\,\varrho^{q-2-i}\quad (i\le q-2)$$

for a fixed $\varrho<1/2$; the last assertion uses the product of the growth scale and Γ ratios. For $j\ge q$, $h_q(j)=s_q\upsilon_{q-1,j}$, $\sigma_i^2 s_q/U_q\le C(\sigma_i/\sigma_{q-1})^2$, and $\upsilon_{i,j}=(\sigma_i/\sigma_{q-1})\upsilon_{q-1,j}$. The preceding layer angular term is $C\upsilon_{q-1,j}$, its amplitude term is $C\epsilon_{\rm sch}\upsilon_{q-1,j}$, and both older families sum as geometric growth-scale series to $C\epsilon_{\rm sch}\upsilon_{q-1,j}$. At $j<q$, $h_q(j)=s_j$ and $\sigma_i^2 s_j\le C(\sigma_i/\sigma_{j-1})^2U_j$. Summing first over $i<j$ and then over $j\le q-1$ gives $C\sum_{j<q}U_j/U_q\le C\epsilon_{\rm sch}$. The A_i families give at most $C\epsilon_{\rm sch}\sum_{i\le q-2}U_{i+1}/U_q(1+\sum_{j>i}\upsilon_{i,j})\le C\epsilon_{\rm sch}^2$, besides the preceding layer entry $C\epsilon_{\rm sch}$ at $j=q-1$. The base contribution below q is at most $CU_q^{-1}\sum_{2\le j<q}s_j\le C\epsilon_{\rm sch}^2$, and above q at most $C\epsilon_{\rm sch}\upsilon_{q-1,j}$. This proves all three claimed bounds and the mass CU_q.

Finally $\alpha=d_{0c}-\varphi_c$. For $j\le c$, $h_c(j)=s_j\le\varepsilon_j$. The only other case is $j=l=c+1$, where $h_c(l)=s_{l-1}\upsilon_{l-2,l}\le C\varepsilon_l$. The controlled-angle contribution is at most $C\varepsilon_l@X_l$; on the first steering it is $\sin s_*@X_1=\varepsilon_1@X_1/(c_{\rm p}\Lambda_1)$. This proves the rotation estimate (5.21).

Choose C_{n+1} as the maximum of C_n and the finitely many constants just obtained. Every such constant is made from finite multinomial and set-partition counts, fixed-domain derivative norms, and constants already known through order n, except for the explicitly earlier estimates at the new order in the displayed sequence. Layer sums enter through a common constant and convergent geometric amplitude sums. Taking nondecreasing majorants in the profile norm therefore gives one sequence depending on the order, profile norms and fixed data. The threshold λ_{lo} remains the same throughout this derivative-order induction.

At each junction between intervals the controls are already smooth. At growth-to-transition the profile is flat at 0; at steering entry the transition is flat at 1, Z_l matches $\zeta_{l,1}$ and Z_l' is flat. The interior-supported negative pulse vanishes on a neighborhood of the end of steering, where the selected solution agrees with its stationary amplitude pair. Activation cutoffs are flat and enter none of their own background coefficients. Thus all differentiated identities extend to the endpoints of their subintervals, and their one-sided estimates apply to the global derivatives. This proves the proposition on all of $\mathcal{I}_l$. $\square$

5.4. **Summation over future stages.** We now pass from estimates on a finite stage to bounds over the full lifetime of an older layer. In the range $1 \le n \le Q_q$ of Lemma 5.4, we first bound a finite head and then estimate the first term beyond that head separately. The repeated frequency growth controls the remaining tail. For a fixed derivative order outside the uniform range, the finite head may be longer. Its terms need only be finite, not small.

Using the rates and amplitude weights in (5.3)–(5.4), for $n \ge 0$ and $i \ge 0$ put

$$\mathfrak{L}_n = \sum_{l\ge 1} \varepsilon_l X_l^n, \qquad \mathfrak{S}_{i,n} = \sum_{l\ge i+1} \upsilon_{i,l} X_l^n.$$

Lemma 5.4 (Cross-stage sums)**.** *For a constant C_Q depending only on Q_* and the fixed design, growth-scale and geometric bounds,*

$$\mathfrak{L}_n \le C_Q^n \Pi_q^n, \qquad \mathfrak{S}_{i,n} \le (1 + C_Q^n)\Pi_q^n \quad (1 \le n \le Q_q,\ 0 \le i \le q).$$

At $n = 0$, $\mathfrak{S}_{i,0} \le C$. For $1 \le n \le Q_q$, each of the following sums is bounded by $C(1+C_Q^n)\Pi_q^n$, where C depends only on the fixed design and bounds used above:

$$\sum_{l\ge i} \min(1, 4\sigma_{i-1}/\sigma_{l-1}) X_l^n \quad (1 \le i \le q+1), \qquad \sigma_i \sum_{l\ge i+1} \varepsilon_l X_l^{n-1} \quad (0 \le i \le q).$$

All four families are finite at every fixed positive order, without changing the threshold.

Proof. For later layers the exact identity is

$$\varepsilon_l X_l^n = \left(\frac{\sin s_l}{s_l}\right)^n L_l^{2n+2} \frac{\sigma_{l-2}^{n+1}}{\sigma_{l-1}} \le L_l^{2n+2} \frac{\sigma_{l-2}^{n+1}}{\sigma_{l-1}} \qquad (l \ge 2). \tag{5.25}$$

The first term is separately $\varepsilon_1(\Lambda_1\Gamma_1)^n$; the second term in (5.25) uses $\sigma_0 = 1$.

For $1 \le n \le Q_q$, split the sum at $q+1$ and $q+2$. The head $l \le q+1$ is at most Π_q^n by (5.12) and $\sum \varepsilon_l \le 1$. At the first term beyond this head,

$$\frac{\sigma_q^{n+1}}{\sigma_{q+1}} \le \frac{(\sqrt{C_\sigma})^{n+1}}{\sqrt{c_\sigma}} \lambda_q^{(n+1-Q_{q+1})/16} \le C^n.$$

This is exactly where $n \le Q_q$ is used. Also $L_{q+2} \le Q_{q+2}^3 \log \lambda_q$, $Q_{q+2} \le (Q_*+3)q$ and $q^2 \le \log \lambda_q$, so

$$L_{q+2}^{2n+2} \le Q_{q+2}^{12n} (\log \lambda_q)^{4n} \le (Q_*+3)^{12n} (\log \lambda_q)^{10n} \le (Q_*+3)^{12n} \Pi_q^n.$$

This includes $q = 1$ with the actual $\Pi_1 = (\log \lambda_1)^{10}$, independently of a zeroth frequency law.

For the remaining tail put $x = \log \lambda_{l-2}$ and $Q = Q_{l-1}$. The threshold gives $x \ge \epsilon_{\mathrm{sch}}^{-1}$, $Q \le x$, $L_l \le x^4$ and $x/\log x \ge 160Q$. The last comparison propagates because

$$\frac{Qx}{\log(Qx)} \ge \frac{Q}{2}\frac{x}{\log x} \ge 160(Q+1).$$

The condition $y \geq \max(1, C_\sigma^4)$ in (5.5), with $x \geq y$, gives $\log \max(1, \sqrt{C_\sigma}) \leq \frac{1}{8} \log x$. Also $c_\sigma \geq 1$ (the latter follows from $G_- > 0$). If $n \leq Q-3$, the logarithm of the upper bound in (5.25), multiplied by $\sigma_{l-2}^{1/4}$, is at most

$$(8Q-16)\log x + \tfrac{1}{8} Q \log x - \frac{7x}{64} \leq \left(8 + \frac{1}{8} - \frac{35}{2}\right) Q \log x < 0.$$

If $n \leq Q-2$, multiplication instead by $\sigma_{l-2}^{1/8}$ gives

$$(8Q-8)\log x + \tfrac{1}{8} Q \log x - \frac{7x}{128} \leq \left(8 + \frac{1}{8} - \frac{35}{4}\right) Q \log x < 0.$$

Thus $\varepsilon_l X_l^n \leq \sigma_{l-2}^{-1/4}$ in the first range and at most $\sigma_{l-2}^{-1/8}$ in the second. For $l \geq q+3$ we have $Q_{l-1} \geq Q_q + 2 \geq n+2$, so the second bound applies even at the first tail index. Its sum is at most a constant, since $\log \lambda_{l-2} \geq 201^{l-3} y$. This proves the bound for $\mathfrak{L}_n$.

For $\mathfrak{S}_{i,n}$ the head uses the geometric sum $\sum_{l \geq i+1} \upsilon_{i,l} \leq C$. For $l \geq q+2$, $i \leq q \leq l-2$, so $\upsilon_{i,l} \leq \epsilon_{\mathrm{sch}} \varepsilon_l$. The preceding future-stage argument applies. For the first auxiliary sum separate $l = i$ and compare the remaining terms with $4\mathfrak{S}_{i-1,n}$. For the second, $\sigma_i \varepsilon_l / X_l \leq C \upsilon_{i,l}$ for $l \geq 2$. The possible $l = 1, i = 0$ term uses $\varepsilon_1 / X_1 = c_{\mathrm{p}}$ exactly. Thus a fixed factor depending on c_{p} accounts for this possible base term.

Finally, for a fixed arbitrary n, eventually $Q_{l-1} \geq n+3$. The quarter-power estimate is then a summable tail. All preceding terms are finite in number; they need not be small. The comparisons just used prove finiteness of the other families. The same already chosen frequencies work at every fixed order. □

5.5. **Transfer to the coefficient fields.** There are two conclusions to retain. The coefficient estimates are uniform over layers only in the finite derivative range specified below. For each fixed layer, derivatives of every order remain bounded on its entire lifetime, with constants allowed to depend on that layer. The first conclusion controls the late force increments; the second handles the finitely many early increments at any requested order. Neither conclusion requires a new frequency choice.

The coefficients to be estimated are the material metric and transported vectors, followed by the two normalized amplitude couplings:

$$\begin{gathered}
\mathsf{C}_q = M_q^{-1} M_q^{-T}, \quad \widetilde{G}_q = M_q^T G_{<q}, \quad \widetilde{e}_{1,q} = M_q^{-1} e_1, \quad \kappa_q = |\zeta_q|^2, \\
g_{\perp,q} = J\zeta_q \cdot G_{<q}, \qquad g_{\parallel,q} = \zeta_{q,1}, \\
\widehat{a}_q = -\frac{g_{\perp,q}}{\sigma_{q-1}\kappa_q}, \qquad \widehat{b}_q = \sigma_{q-1} g_{\parallel,q}, \qquad \mathcal{B}_q = \begin{pmatrix} 0 & \widehat{a}_q \\ \widehat{b}_q & 0 \end{pmatrix}.
\end{gathered} \tag{5.26}$$

Thus $g_{\parallel,q}$ is the fixed laboratory component $\zeta_{q,1}$. The pair and its propagator are those of (3.17) and (3.21). Recall that $I_q = [t_q^{\mathrm{in}}, T_*)$ is the full lifetime and K_q is the supremum of the deformation and inverse-deformation norms there. The order Q_q and lifetime scale Π_q are fixed in (3.7). The older fields $D_{<q}, G_{<q}$ and transported phase $\zeta_q = M_q^{-T} p_q$ are defined in (3.3); $p_q = \mathsf{e}(s_q)$ is the unit activation direction, and w_q is the cutoff (3.11).

Theorem 5.5 (Lifetime derivative bounds)**.** *Assume λ_1 satisfies the enlarged threshold of Lemma 5.1. For the controlled evolution of Theorem 3.2, all time derivatives of α are bounded on $[0, T_*)$. For each fixed q, all time derivatives of*

$$D_{<q}, G_{<q}, M_q^{\pm 1}, \zeta_q, \Theta_q, \Omega_q, w_q$$

and the coefficients of (5.26) are bounded on I_q. These all-order bounds may depend on q and the order.

For $0 \le b \le Q_q - 1$, there are constants C_b independent of q, λ_1 and the admissible choices of k_j, depending only on b, fixed data and the C^{b+2} norms of the profiles, such that the following estimates hold. The first line controls deformation and strain; the next line controls the transported material coefficients; the last line controls the phase geometry, the amplitude matrix and a pointwise relative derivative bound for y_q^0:

$$\begin{gathered} \|\partial_t^b M_q^{\pm 1}\| \le C_b K_q \Pi_q^b, \qquad \|\partial_t^b D_{<q}\| \le C_b \Pi_q^{b+1}, \\ |\partial_t^b \widetilde{G}_q| \le C_b \Pi_q^{b+2}, \qquad |\partial_t^b (\mathsf{C}_q, \mathsf{C}_q p_q, \widetilde{e}_{1,q})| \le C_b \Pi_q^{b+1}, \\ |\partial_t^b (\zeta_q, \kappa_q, \kappa_q^{-1})| \le C_b \Pi_q^b, \qquad \|\partial_t^b \mathcal{B}_q\| \le C_b \Pi_q^{b+1}, \qquad |(y_q^0)^{(b)}(t)| \le C_b \Pi_q^b |y_q^0(t)|. \end{gathered} \tag{5.27}$$

In the same order range,

$$\begin{gathered} |\partial_t^b G_{<q}| + |\partial_t^b g_{\perp,q}| \le C_b \sigma_{q-1}^2 \Pi_q^b, \qquad |\partial_t^b g_{\parallel,q}| \le C_b \Pi_q^b, \\ |\partial_t^b w_q| \le \|\eta\|_{C^b} \Pi_q^b. \end{gathered} \tag{5.28}$$

The order-zero coefficient constant C_0 is a fixed generic constant, distinct from the multiplier C_P in the definition of S_q preceding (5.16). The size comparisons (5.13), the estimate $Y_q = \sup_{I_q} |y_q^0| \le C\lambda_q^{-7/8}$ and the actual weighted propagator bound (3.21), with C_R from (3.72), also hold.

Proof. Fix $t \in I_q \cap \mathcal{I}_l$, where $l \ge q$. *Angles and scalar geometry.* We first transfer (5.18)–(5.21) for $1 \le b \le Q_q$, using Lemma 5.4. The rotation estimate (5.21) sums directly to $C_b \mathfrak{L}_b$. For each $i \le q$, write φ_i as $\varphi_c - d_{ic}$ or, in its own growth or transition, as $\varphi_{q-1} + \vartheta_q$. The exceptional first growth and transition have constant angles. The controlled-angle list has amplitude at most $C\varepsilon_l$ at X_l. The difference-angle list has $h_c(j) \le C\epsilon_{\mathrm{sch}}$ for $j \le q+1$ and $h_c(j) \le C\varepsilon_j$ for $j \ge q+2$: either $j \le c$, or $j = l = c+1$ and (5.8) applies. The adjacent-angle list sums to $s_q \mathfrak{S}_{q-1,b}$. Thus

$$|\alpha^{(b)}| + |\varphi_i^{(b)}| \le C_b \Pi_q^b.$$

Composition gives the same unit-size bound for all $\mathsf{e}_i, J\mathsf{e}_i$ with $i \le q$.

For $i < q$, the vorticity and gradient-amplitude estimates in (5.18) and (5.20) give

$$|\Omega_i^{(b)}| \le C_b \sigma_i \mathfrak{S}_{i,b} \le C_b \sigma_i \Pi_q^b,$$

$$|A_i^{(b)}| \le C_b \sigma_i^2 \left[\Gamma_i \mathfrak{S}_{i-1,b-1} \mathbf{1}_{i \ge 2} + \frac{\sigma_{i-1}}{\sigma_i} (X_i^b + \mathfrak{S}_{i,b}) \right] \le C_b \sigma_i^2 \Pi_q^b.$$

When $b = 1$ and $i \ge 2$, use the bound $\mathfrak{S}_{i-1,0} \le C$ from Lemma 5.4. For the same reason the estimates for $e^{\pm \rho_q}$ give

$$|\partial_t^b e^{\pm \rho_q}| \le C_b \Gamma_q \mathfrak{S}_{q-1,b-1} \le C_b \Pi_q^b \quad (q \ge 2);$$

at $q = 1$ one has $e^{\pm \rho_1} = 1$. Products now bound $\zeta_q, J\zeta_q, \kappa_q, \kappa_q^{-1}$ at size C with the same Π_q^b rate.

Absolute amplitude bounds. On its own stage, y_q^0 satisfies the relative estimate of Proposition 5.3, with $X_q \le \Pi_q$ and $|y_q^0| \le C\lambda_q^{-7/8}$ from Theorem 3.2. On later stage intervals, the P_q estimate in (5.18) gives

$$|P_q^{(b)}| \le C_b S_q \frac{\sigma_{q-1}}{\sigma_q} (X_q^b + \mathfrak{S}_{q,b}) \le C_b \lambda_q^{-7/8} \Pi_q^b.$$

The normalized vorticity amplitude is zero on the holding interval. Afterwards, the estimates for Ω_q on later stages give its size and derivative bounds after multiplication by σ_{q-1}/λ_q; use $\sigma_{q-1}\sigma_q/\lambda_q \le C\lambda_q^{-7/8}$. This supplies the absolute amplitude estimates, but does not yet replace the relative estimate in (5.27).

Strain and material coefficients. All older activation cutoffs equal one on I_q. Hence

$$\partial_t^b D_{<q} = \alpha^{(b+1)} J + \sum_{i<q} \partial_t^b (\Omega_i J \mathsf{e}_i \otimes \mathsf{e}_i), \qquad G_{<q} = -\mathsf{e}_0 - \sum_{i<q} A_i \mathsf{e}_i.$$

The common constant in Leibniz's formula is independent of i. The growth-scale sums therefore give

$$\|\partial_t^b D_{<q}\| \le C_b \Pi_q^{b+1}, \qquad |\partial_t^b G_{<q}| \le C_b \sigma_{q-1}^2 \Pi_q^b.$$

In the first estimate $b+1 \le Q_q$ is required for the rotation term. This gives the uniform range $b \le Q_q - 1$; fixed-layer boundedness at higher orders is proved separately below.

Using the lifetime bound K_q of Theorem 3.2, differentiating $M_q' = D_{<q} M_q$ yields

$$M_q^{(b)} = \sum_{j<b} \binom{b-1}{j} D_{<q}^{(j)} M_q^{(b-1-j)}.$$

Induction on b gives $C_b K_q \Pi_q^b$. The inverse equation $(M_q^{-1})' = -M_q^{-1} D_{<q}$ gives the same bound. The product rule then gives

$$|\widetilde{G}_q^{(b)}| \le C_b K_q \sigma_{q-1}^2 \Pi_q^b \le C_b \Pi_q^{b+2}, \qquad \|\mathsf{C}_q^{(b)}\| \le C_b K_q^2 \Pi_q^b \le C_b \Pi_q^{b+1}.$$

For $\mathsf{C}_q p_q = M_q^{-1}\zeta_q$ and $\widetilde{e}_{1,q} = M_q^{-1} e_1$, use the same product bound. The denominator κ_q has the positive lower bound from Theorem 3.2. Consequently the two scalar coupling coefficients have their different natural sizes:

$$|g_{\perp,q}^{(b)}| = |(J\zeta_q \cdot G_{<q})^{(b)}| \le C_b \sigma_{q-1}^2 \Pi_q^b, \qquad |g_{\|,q}^{(b)}| = |\partial_t^b \zeta_{q,1}| \le C_b \Pi_q^b.$$

The lower coupling coefficient is a laboratory component, so the normalized matrix entries are exactly $\widehat{a}_q = -g_{\perp,q}/(\sigma_{q-1}\kappa_q)$ and $\widehat{b}_q = \sigma_{q-1} g_{\|,q}$. It follows that

$$|\widehat{a}_q^{(b)}| + |\widehat{b}_q^{(b)}| \le C_b \sigma_{q-1} \Pi_q^b \le C_b \Pi_q^{b+1}.$$

These are all normalized coefficient estimates. For the explicit activation cutoff, every integer $b \ge 0$ satisfies

$$\partial_t^b w_q(t) = \Gamma_q^b \eta^{(b)}\big(\Gamma_q(t - t_q^{\mathrm{in}})\big). \tag{5.29}$$

The same formula holds at the flat endpoints. Since $\Gamma_q \le \Pi_q$, it proves the activation cutoff estimate in (5.28); for fixed q it bounds all orders on the whole lifetime, without a restriction by Q_q.

Relative amplitude derivatives and fixed-layer bounds. Apply Lemma 5.2(v) globally to $(y_q^0)' = \mathcal{B}_q y_q^0$, with $R = \Pi_q$ and the just proved coefficient constants. It yields $(y_q^0)^{(b)} = \mathcal{P}_{q,b} y_q^0$ and $\|\mathcal{P}_{q,b}\| \le C_b \Pi_q^b$. The estimate remains relative to the pointwise $|y_q^0(t)|$ on the entire lifetime, including the initial activation interval. The streamfunction amplitude $B_q = \widehat{\Omega}_q/(\lambda_q \sigma_{q-1} \kappa_q)$ inherits size $C\lambda_q^{-7/8}/(\lambda_q \sigma_{q-1})$ and the same derivative rate.

For arbitrary fixed orders replace the uniform sums by their finite values in Lemma 5.4. One convenient finite bound is

$$2X_{q+1}^n + \mathfrak{L}_n + \max_{i \le q}\big(\mathfrak{S}_{i,n} + \Pi_q \mathfrak{S}_{i,n-1}\big).$$

The same argument, and its next order for α', gives all fixed-layer derivative bounds. Products, inverses with a fixed positive denominator and the deformation matrix equations preserve boundedness. The stage intervals tile $[1, T_*)$, while $\alpha = 0$ before activation, so $\sup_{[0,T_*)} |\alpha^{(n)}| \le C_n \mathfrak{L}_n < \infty$ for every $n \ge 1$. Smooth matching at the interval endpoints was checked in Proposition 5.3. These bounds concern each layer on its lifetime and the time-only control on $[0, T_*)$.

Theorem 3.2 supplies the full weighted propagator, including the first holding-interval bound $H_1 \le C_{\mathrm{hold}}$ rather than $H_1 = O(\epsilon_{\mathrm{sch}})$. We reuse its C_R from (3.72) unchanged. The present proof of the relative amplitude estimate is separate from that propagator argument.

Profile dependence. The coefficient estimate for $D^{(b)}$ uses Proposition 5.3 at order $b+1$, whose constants use profile derivatives through $(b+1)+1 = b+2$. Every subsequent operation is a fixed finite product or composition, so the same profile dependence holds for all estimates. The bounds hold uniformly over the entire admissible range of logarithmic gains before the common correction majorant of Theorem 7.3 and the final common frequency threshold are chosen. ☐

Thus (5.27) and (5.28) give the correction coefficients at their displayed natural sizes, uniformly through $Q_q - 1$ and at every order for each fixed layer. Together with the seed and propagator bounds of Theorem 3.2, these are the inputs for the local corrections. The uniform range covers the required derivative orders in (7.15), with two additional profile derivatives. The correction estimates use the actual constants C_b of (5.27), including C_0. Theorems 7.3 and 8.2 supply the bounds for the finite corrections and a parameter choice satisfying their hypotheses.

6. The finite local correction equations

Localization introduces errors into both equations of the exact wave. We reduce their oscillatory parts to higher order by a finite sequence of coupled corrections. Each correction solves the same amplitude system with a source formed from the earlier corrections, pointwise in the material coordinate. The higher corrections vanish on the entire envelope plateau, for every phase, and hence preserve the affine state used by all later layers. Activation errors and nonoscillatory phase averages are retained in the force and estimated directly.

6.1. **Material coordinates and the two residual equations.** Fix one layer and omit its subscript q. For this local calculation, assume the complete older state is $\theta_{\mathrm{old}} = G \cdot x$, $u_{\mathrm{old}} = Dx$ throughout the new layer's support for the whole time interval of the calculation. For the actual layers, Theorem 8.2 establishes this relation on each full lifetime by combining the nesting of Lemma 4.3 with the vanishing of corrections on the plateau proved in Proposition 6.1 below. For a periodic profile $A(s, a, t)$ its evaluation in physical coordinates is

$$A^{\natural}(x,t) = A(\lambda\zeta \cdot x, M^{-1}x, t). \tag{6.1}$$

The material space is $\mathbb{R}^2$; only the phase variable is periodic. The material coordinate records compact support transported by the affine flow. Evaluation in physical variables ties these two independent coordinates together. Averaging in phase before this evaluation is not integration over physical space; its result can still vary with the material coordinate.

The normalized phase average is

$$\langle A\rangle(a,t) = \frac{1}{2\pi}\int_0^{2\pi} A(s,a,t)\,ds.$$

The metric and transported vectors are those in (5.26); here write $C = \mathsf{C}_q$. For the local differential operators we use the following notation

$$C = M^{-1}M^{-T}, \quad \kappa = |\zeta|^2 = p \cdot Cp, \quad \widetilde{G} = M^T G, \quad \widetilde{e}_1 = M^{-1}e_1, \quad \partial_\perp = Jp \cdot \nabla_a.$$

The operator $\mathcal{L}_\lambda$ is the pulled-back Laplacian; $\mathcal{J}_s$ and $\{\cdot,\cdot\}_a$ are the fast and slow parts of the advective bracket, with $\partial_\perp$ transverse to the material phase:

$$\begin{aligned} &\mathcal{L}_\lambda = \lambda^2\kappa\partial_s^2 + \lambda\mathcal{D}_1 + \Delta_M, \qquad \mathcal{D}_1 = 2(Cp)\cdot\nabla_a\partial_s, \quad \Delta_M = \operatorname{div}_a(C\nabla_a),\\ &\mathcal{J}_s(A,B) = A_s\partial_\perp B - B_s\partial_\perp A, \qquad \{A,B\}_a = J\nabla_a A\cdot\nabla_a B. \end{aligned} \tag{6.2}$$

The chain rule and $\det M = 1$ give

$$\begin{aligned} \nabla_x A^{\natural} &= (\lambda\zeta A_s + M^{-T}\nabla_a A)^{\natural},\\ (\partial_t + Dx\cdot\nabla_x)A^{\natural} &= (\partial_t A)^{\natural}, \qquad \Delta_x A^{\natural} = (\mathcal{L}_\lambda A)^{\natural},\\ \nabla_x^{\perp}\Psi^{\natural}\cdot\nabla_x A^{\natural} &= (\lambda\mathcal{J}_s(\Psi,A) + \{\Psi,A\}_a)^{\natural}. \end{aligned} \tag{6.3}$$

For instance, the last identity follows by expanding the two gradients; $\det(M^{-T}v, M^{-T}w) = \det(v,w)$ converts both mixed terms into the stated $\partial_\perp$ bracket.

Adding $\theta = V^\natural$, $\psi = \Psi^\natural$ to the earlier state gives the exact force increments

$$\begin{aligned} E_\theta(V,\Psi) &= V_t + \lambda(J\zeta\cdot G)\Psi_s + J\nabla_a\Psi\cdot\widetilde{G} + \lambda\mathcal{J}_s(\Psi,V) + \{\Psi,V\}_a, \\ E_\omega(V,\Psi) &= \partial_t(\mathcal{L}_\lambda\Psi) + \lambda\mathcal{J}_s(\Psi,\mathcal{L}_\lambda\Psi) + \{\Psi,\mathcal{L}_\lambda\Psi\}_a - \lambda\zeta_1 V_s - \widetilde{e}_1\cdot\nabla_a V. \end{aligned} \tag{6.4}$$

Here $f_{\theta,q} = E_\theta^\natural$ and $E_q = E_\omega^\natural$ is the curl of the vector force, not that vector force itself. The time derivative differentiates both C and κ. The term $u_q\cdot\nabla\omega_{\text{old}}$ vanishes because the older vorticity is constant on the whole support.

The leading residual already explains why corrections and phase means are needed. In the next two displays only, T and Q denote the activated temperature and streamfunction amplitudes. Set $T = w\Theta$, $Q = w\Omega/(\lambda^2\kappa)$, $g = f(a/\ell)$, $h = (Cp)\cdot\nabla_a g$, and $d = \Delta_M g$. Direct expansion gives

$$\begin{aligned} E_\theta(V_0,\Psi_0) &= \dot{w}\Theta F g + QP(J\nabla_a g\cdot\widetilde{G}) + \lambda QT(F^2 - PF')g\partial_\perp g, \\ E_\omega(V_0,\Psi_0) &= \dot{w}\Omega F' g + \partial_t(2\lambda QFh + QPd) - TF(\widetilde{e}_1\cdot\nabla_a g) \\ &\quad + \lambda^3\kappa Q^2(FF' - PF'')g\partial_\perp g \\ &\quad + 2\lambda^2 Q^2\big[F^2 g\partial_\perp h - PF'h\partial_\perp g\big] + 2\lambda Q^2 PF\{g,h\}_a \\ &\quad + \lambda Q^2 PF\big[g\partial_\perp d - d\partial_\perp g\big] + Q^2P^2\{g,d\}_a. \end{aligned} \tag{6.5}$$

Every term except the activation terms contains a material-coordinate derivative of g and vanishes on the plateau for every s. If $\mu_2 = \langle F^2\rangle$, phase integration by parts gives

$$\langle E_\theta\rangle = 2\mu_2\lambda QTg\partial_\perp g, \qquad \langle E_\omega\rangle = 2\mu_2\lambda^2Q^2\partial_\perp(gh) + Q^2\langle P^2\rangle\{g,d\}_a. \tag{6.6}$$

In particular, a spatially localized exact wave is not an exact unforced solution. Its nonoscillatory terms persist and must be estimated.

6.2. **Phase inversion and the recursion.** We construct the corrections level by level using only a phase inverse. It is pointwise in the slow coordinate and therefore preserves material support. For a mean-zero periodic function $h(s,a,t)$, $\partial_s^{-1}h$ denotes its unique mean-zero periodic antiderivative. It has the formula

$$(\partial_s^{-1}h)(s,a,t) = \frac{1}{2\pi}\int_0^{2\pi}(\pi - v)h(s-v,a,t)\,dv, \qquad \left\|\partial_s^{-1}h\right\|_{C^r} \le \frac{\pi}{2}\left\|h\right\|_{C^r}.$$

This operator acts only in s: it does not spread material support. Write $\mathsf{P}_{\neq 0}h = h - \langle h\rangle$ and $\sigma = \sigma_{q-1}$. The coefficients $\widehat{a},\widehat{b},\mathcal{B}$ are those in (5.26), with layer indices omitted. For levels $n\ge 0$, introduce $\widetilde{W}_n$, $W_n = (\lambda/\sigma)\widetilde{W}_n$, and

$$\Psi_n = (\lambda^2\kappa)^{-1}\partial_s^{-1}W_n. \tag{6.7}$$

At level zero take $\widetilde{W}_0 = w(\sigma\Omega/\lambda)Fg$ and $V_0 = w\Theta F g$. Use zero for negative-index levels. The graded vorticity profile Z_j and its localization part $Z_j^\flat$ are defined by

$$Z_j = \partial_s W_j + \lambda\mathcal{D}_1\Psi_{j-1} + \Delta_M\Psi_{j-2}, \qquad Z_j^\flat = Z_j - \partial_s W_j.$$

At each correction level we collect the terms not canceled at earlier levels and remove their nonzero phase modes. The grade is the index assigned to a term by this finite expansion; it is distinct from the layer index, the derivative order, and the Fourier harmonic. The recursion below arranges that the source at a new level uses only earlier levels. Its phase average is estimated, not inverted.

The correction unknowns are the temperature profile V_n and the normalized streamfunction phase derivative $\widetilde{W}_n$. They depend on phase and material position, unlike the amplitude coefficients, which depend only on time. After undoing the displayed normalization, phase integration recovers the streamfunction. The sources determined by earlier levels are

$$\begin{aligned} S_n^\theta &= J\nabla_a \Psi_{n-1}\cdot \widetilde{G} + \lambda \sum_{i+j=n-1} \mathcal{J}_s(\Psi_i, V_j) + \sum_{i+j=n-2} \{\Psi_i, V_j\}_a, \\ S_n^\omega &= \partial_t(\lambda \mathcal{D}_1 \Psi_{n-1}) + \partial_t(\Delta_M \Psi_{n-2}) - \widetilde{e}_1 \cdot \nabla_a V_{n-1} \\ &\quad + \lambda \sum_{i+j=n-1} \mathcal{J}_s(\Psi_i, Z_j) + \sum_{i+j=n-2} \{\Psi_i, Z_j\}_a. \end{aligned} \tag{6.8}$$

For $1 \le n \le J_q$, solve, pointwise in (s,a),

$$\partial_t \begin{pmatrix} V_n \\ \widetilde{W}_n \end{pmatrix} = \mathcal{B}(t) \begin{pmatrix} V_n \\ \widetilde{W}_n \end{pmatrix} + \begin{pmatrix} -\mathsf{P}_{\neq 0} S_n^\theta \\ -\partial_s^{-1} \mathsf{P}_{\neq 0}((\sigma/\lambda) S_n^\omega) \end{pmatrix}, \qquad (V_n, \widetilde{W}_n)|_{t=t_q^{\mathrm{in}}} = 0. \tag{6.9}$$

Thus each step is an inhomogeneous two-dimensional linear ODE, with the same propagator as the principal amplitude pair. The inverse acts only on the one-dimensional periodic phase, so it does not spread material support. The finite cancellation statement below identifies exactly the activation terms, phase averages and final remainders that are left in the force.

Proposition 6.1 (Exact finite cancellation)**.** *Retain the fixed odd phase profile F in (4.1), its even primitive P, and the radial envelope f used in the leading fields (4.2), together with the activation cutoff (3.11). Use the homogeneous amplitude pair solving (3.2) for the leading profiles, and start the zero-data recursion (6.8)–(6.9) at t_q^{in}. Suppose its time coefficients are smooth on $[t_q^{\mathrm{in}}, \tau]$, $\tau < T_*$, and $\kappa > 0$ there. The recursion is defined to every finite depth. Its higher levels vanish for $|a| < \ell/2$ and for $|a| > \ell$, for every s. They are flat at activation and have zero phase means. Under simultaneous reflection $(s,a) \mapsto (-s,-a)$, V_n is odd and Ψ_n is even.*

For $V = \sum_{n=0}^{J_q} V_n$, $\Psi = \sum_{n=0}^{J_q} \Psi_n$, the exact residual is

$$E_\theta = \dot{w}\Theta F g + \sum_{n=1}^{J_q} \langle S_n^\theta \rangle + R_{>J_q}^\theta, \qquad E_\omega = \dot{w}\Omega F' g + \sum_{n=1}^{J_q} \langle S_n^\omega \rangle + R_{>J_q}^\omega. \tag{6.10}$$

The remainders consist of the sources (6.8) of grades $J_q + 1$ through $2J_q + 4$. In these sources the primary profiles V_j, W_j, Ψ_j are zero for $j > J_q$, while Z_j retains its defining shifted terms:

$$Z_{J_q+1} = \lambda \mathcal{D}_1 \Psi_{J_q} + \Delta_M \Psi_{J_q - 1}, \qquad Z_{J_q+2} = \Delta_M \Psi_{J_q}, \qquad Z_j = 0 \quad (j > J_q + 2).$$

Proof. For a fixed material point the solution is the Duhamel integral of the displayed source against the propagator of $\mathcal{B}(t)$. Every positive-grade source contains a material-coordinate derivative of a level, or of its localization term $Z^\flat$. At level zero all fields are independent of a on the plateau. Inductively every higher-level source is zero there for all phases; phase averaging, phase inversion, and the pointwise Duhamel integral preserve this property. The same argument outside the support preserves compactness. This is why the induction uses the full plateau, not merely the smaller linear phase cell.

Each inhomogeneous term has zero phase mean, so the mean of the solution satisfies a homogeneous ODE with zero data. Reflection in (s,a) sends the two source components to their negatives and preserves their equations. In (6.7) phase inversion converts the oddness of W_n under simultaneous reflection to evenness of Ψ_n. Differentiating (6.9) at activation proves flatness by induction, using the flat activation profile.

For the cancellation, the two principal expressions at higher level n are

$$\partial_t V_n - \widehat{a}\widetilde{W}_n, \qquad \partial_t(\lambda^2 \kappa \partial_s^2 \Psi_n) - \lambda \zeta_1 \partial_s V_n = \frac{\lambda}{\sigma} \partial_s(\partial_t \widetilde{W}_n - \widehat{b} V_n).$$

By (6.9) these are exactly $-\mathsf{P}_{\neq 0} S_n^\theta$ and $-\mathsf{P}_{\neq 0} S_n^\omega$. Notice that this calculation differentiates κ correctly, because $\lambda^2 \kappa \partial_s^2 \Psi_n = \partial_s W_n$ is an identity in time. The remaining linear terms shift the level by one or two as displayed in (6.8); the two brackets have grades $i+j+1$ and $i+j+2$. Expanding (6.4) therefore gives (6.10) term by term, including the stated maximal grade. At level zero (3.2) leaves just the two activation terms. $\square$

The forces during activation need not vanish on the core: $\dot{w}\Theta F$ and $\dot{w}\Omega F'$ are present there. What vanishes identically is the correction to the state. Nor will the compact inverse curl used below preserve a central region where the force vanishes. Neither fact changes the prescribed affine velocity and temperature.

6.3. **The nonoscillatory phase averages.** The phase average is a nonoscillatory profile, not a spatial constant. Its material derivatives remain. Because it is independent of phase, differentiation in physical coordinates produces no factor of the phase frequency. The symmetric cancellation below is a separate fact: it removes the leading phase-derivative part of the vorticity interaction from the mean. The remaining means are retained and estimated.

For periodic A, B, integration by parts gives $\langle \mathcal{J}_s(A,B)\rangle = \partial_\perp \langle A_s B\rangle$ and $\{A,B\}_a = \mathrm{div}_a(BJ\nabla_a A)$. The linear terms of (6.8) have zero phase average. Indeed V_i and Ψ_i have zero phase mean, including at grade zero, and the time-only coefficients commute with phase averaging:

$$\langle J\nabla_a \Psi_{n-1}\cdot \widetilde{G}\rangle = 0, \quad \langle \partial_t(\lambda \mathcal{D}_1\Psi_{n-1})\rangle = 0, \quad \langle \partial_t(\Delta_M \Psi_{n-2})\rangle = 0, \quad \langle \widetilde{e}_1\cdot \nabla_a V_{n-1}\rangle = 0.$$

Thus the surviving nonlinear averages are divergences:

$$\begin{aligned} \langle S_n^\theta\rangle &= \lambda\partial_\perp \sum_{i+j=n-1}\langle \Psi_{i,s}V_j\rangle + \mathrm{div}_a \sum_{i+j=n-2}\langle V_j J\nabla_a\Psi_i\rangle, \\ \langle S_n^\omega\rangle &= \lambda\partial_\perp \sum_{i+j=n-1}\langle \Psi_{i,s}Z_j^\flat\rangle + \mathrm{div}_a \sum_{i+j=n-2}\langle Z_j^\flat J\nabla_a\Psi_i\rangle. \end{aligned} \tag{6.11}$$

To check the second formula, the contribution of the leading phase-derivative part of the vorticity profile to the first sum is a symmetric sum of $\langle W_i\partial_s W_j\rangle$, hence a phase derivative with zero mean. The divergence of its contribution to the second vector sum is proportional to $\sum\langle J\nabla_a\Psi_i\cdot\nabla_a\partial_s^2\Psi_j\rangle$. Phase integration by parts makes it antisymmetric in i,j, so that it also vanishes. The vector primitive itself need not vanish; this cancellation concerns its divergence and uses the complete symmetric sum.

The scalar mean residuals are supported in the material annulus, but their displayed divergence primitives can be nonzero on the inner plateau $|a|<\ell/2$: the first one is proportional to $Jp\,g^2$. We only require compact support of the primitive, or directly the zero integral of its divergence. The means in (6.10) are retained as forces. They have no fast phase and thus produce no factor λ when differentiated in physical coordinates.

6.4. **Compact recovery without derivative loss.** The streamfunction has already fixed the velocity increment. We now recover an external velocity-force increment from the vorticity residual; this does not redefine the velocity. Compact recovery requires the spatial integral of the complete residual to vanish. Vanishing of a phase average alone would not suffice. The recovered force may be nonzero in the central affine region, even though the higher state corrections vanish there.

We use a rotated form of Bogovskiĭ's divergence construction [2]; see also [9] for general background. The support, parity and mixed derivative estimates used in this paper are proved directly in Lemma 6.2. Fix a radial $\phi_0\in C_c^\infty(B_{1/2})$ with $\int\phi_0 = 1$. The same profile will be used for the base and every layer. Define

$$\mathscr{K}(x,z) = z\int_1^\infty \phi_0(x+(s-1)z)s\,ds, \qquad \mathscr{B}E(x) = \int_{\mathbb{R}^2} E(x-z)\mathscr{K}(x,z)\,dz, \qquad \mathcal{R}_c E = J\mathscr{B}E. \tag{6.12}$$

Lemma 6.2 (A compact inverse curl)**.** *For smooth E supported in $\overline{B}_R$, put $R_* = \max\{R, 1/2\}$. Then*

$$\operatorname{div} \mathscr{B}E = E - \left(\int E\right)\phi_0, \qquad \operatorname{supp} \mathscr{B}E \subset \operatorname{conv}\big(\operatorname{supp} E \cup \overline{B}_{1/2}\big) \subset \overline{B}_{R_*}, \tag{6.13}$$
$$\|\mathscr{B}E\|_{C^k} \le C_{k,R}\|E\|_{C^k}.$$

The finite constant $C_{k,R}$ depends only on k, R and $\|\phi_0\|_{C^k}$, and may be chosen nondecreasing in k. If E is even and has zero integral, $\mathcal{R}_c E$ is odd and $\operatorname{curl} \mathcal{R}_c E = E$. The operator commutes with time differentiation, with the same estimates for mixed derivatives.

Proof. We first check the integrable singularity and the divergence identity. With $\widehat{z} = z/|z|$ and $r = (s-1)|z|$,

$$\partial_x^\gamma \mathscr{K}(x, z) = \widehat{z} \int_0^\infty (\partial^\gamma \phi_0)(x + r\widehat{z})(1 + r/|z|)\, dr.$$

The nonzero integration interval has length at most one, and there $r \le |x| + 1/2$. Hence the absolute value is at most $\|\partial^\gamma \phi_0\|_\infty [1 + (|x| + 1/2)/|z|]$. The singularity is locally integrable in two dimensions. Differentiation of $\mathscr{B}E$ differentiates $E(x - z)$ or the first argument of $\mathscr{K}$, and is justified by this bound.

Writing $y = x-z$, the kernel is $G_y(w) = w \int_1^\infty \phi_0(y+sw)s\, ds$, $w = x-y$. For $w \ne 0$, differentiation and integration by parts in s give $\operatorname{div}_w G_y = -\phi_0(y + w)$. Its outward flux through $|w| = \epsilon$ is $\int_{|v|\ge\epsilon} \phi_0(y+v)\, dv$, which tends to one. Gauss–Green on a punctured ball gives $\operatorname{div} G_y = \delta_0 - \phi_0(y+\cdot)$ in distributions. Integrating against $E(y)$ proves the divergence formula.

A nonzero integrand has $y + s(x - y) \in \operatorname{supp} \phi_0$ for some $y \in \operatorname{supp} E$ and $s \ge 1$. Thus x belongs to the stated convex hull. On $|x| \le R_*$, the kernel bound and Leibniz's formula give (6.13), using $\int_{|z-x|\le R} 1\, dz = \pi R^2$ and $\int_{|z-x|\le R} |z|^{-1}\, dz \le 2\pi(R + R_*)$. The finite binomial sum depends only on k; we use the maximum derivative convention for the C^k norm. These bounds prove the stated dependence of $C_{k,R}$ without a derivative loss. Outside the indicated support all derivatives vanish.

Radiality gives $\mathscr{K}(-x, -z) = -\mathscr{K}(x, z)$, so even E gives odd $\mathscr{B}E$. Finally $\operatorname{curl}(J\mathscr{B}E) = \operatorname{div} \mathscr{B}E$. Mean zero removes the correction term. Time differentiation acts only on E, proving the last assertion. □

For a layer supported in $B_{1/4}$ the recovery constant $C_{k,1/4}$ depends only on the order and the fixed profile, and the recovered force is supported in $\overline{B}_{1/2}$. This recovery may fill a central hole. It changes neither the prescribed streamfunction nor its velocity. The zero-integral condition is applied to the complete vorticity residual, not separately to its activation term.

The exact residual formula (6.10) now separates activation terms, divergence-form phase means and the grades above J_q. Lemma 6.2 recovers the compact vector force with the same mixed derivative order. Section 7 estimates these complete scalar and vector force increments.

7. Mixed derivative estimates for the forces

We estimate the complete pair $(f_{\theta,q}, \mathcal{R}_c E_q)$ in mixed space-time norms. The short activation interval retains the small seed-amplitude bound. Over the full lifetime, the relative propagator (3.21) controls corrections against the evolving homogeneous pair, avoiding the factor $e^{C\Pi_q|I_q|}$ from an unweighted coefficient estimate. An induction over finitely many correction levels, followed by evaluation in physical coordinates and recovery of the compactly supported force, gives the force bounds under the explicit hypotheses below; Section 8 realizes all these hypotheses simultaneously.

For a compactly supported field h and an interval I define

$$\|h\|_{\mathcal{X}^k(I)} = \max_{a+r\le k} \sup_{t\in I} \left(\|\partial_t^a h(t)\|_{C^r} + \|\partial_t^a h(t)\|_{W^{r,1}} + \sum_{|\gamma|\le r} \|x^\gamma \partial_t^a h(t)\|_{L^1} \right). \tag{7.1}$$

On a fixed support ball the last two terms are bounded by the first, with a constant depending on the ball and on k. The norm $\|h\|_{C^k_{x,t}(I)}$ is the maximum of $\|\partial_t^a\partial_x^\gamma h\|_{L^\infty(\mathbb{R}^2\times I)}$ over $a+|\gamma|\le k$; we omit I when it is understood.

Theorem 5.5 supplies the lifetime coefficient bounds. For the nonzero reference pair of Theorem 3.2, set $\mathfrak{w}_q(t)=|y_q^0(t)|/Y_q$. We use the full-lifetime estimate (3.21) in the inhomogeneous correction equations. Lemma 3.14 and its verification in Subsection 3.7 yield it from (3.71), including the separate first holding interval, without a lifetime derivative assumption. Division by $|y_q^0|$ cancels the frequency-dependent growth factor retained by an unweighted Grönwall estimate.

The layer-uniform bounds of Theorem 5.5 are used only through their stated order. Its separate all-order bounds for each fixed layer handle the finitely many old layers at any requested order. Lemma 5.4 proves the weighted future-stage summation that makes both statements possible.

For the local correction, let $\delta_q=\Pi_q^5\lambda_q^{-7/8}$. A useful normalized derivative norm is

$$[A]_{r,q}=\max_{b+i+|\gamma|\le r}\sup_{\substack{0\le s\le 2\pi\\ a\in\mathbb{R}^2,\ t\in I_q}}\frac{|\partial_t^b\partial_s^i\partial_a^\gamma A|}{\mathfrak{w}_q(t)\Pi_q^b\ell_q^{-|\gamma|}}. \tag{7.2}$$

The positivity of $|y_q^0|$ on its lifetime follows from Theorem 3.2. The phase inverse is bounded in this norm. Fix an order r for which the coefficient bounds through order r are available. For a two-component source whose component norms at order r are at most B, the zero-data solution of $\dot z=\mathcal{B}_q z+h$ satisfies

$$[z_1]_{r,q},[z_2]_{r,q}\le D_r\Pi_q B\le D_r\Pi_q^2 B. \tag{7.3}$$

Here $D_r\ge 1$ is a finite nondecreasing majorant depending only on r,C_R and the coefficient constants C_j through order r. The finite operations below define D_r for every fixed r; the estimate applies when those coefficient and source bounds are supplied. At order zero, (3.21) cancels the source's relative weight inside the Duhamel integral, leaving $C_R|I_q|B$ up to the fixed two-component norm factor. Use $|I_q|\le\Pi_q$. For higher time derivatives, differentiate the equation:

$$\partial_t^{b+1}z=\sum_{j=0}^{b}\binom{b}{j}\mathcal{B}_q^{(j)}\partial_t^{b-j}z+\partial_t^b h.$$

Leibniz's rule and $\Pi_q\ge 1$ give the bound by induction on b; spatial and phase derivatives commute with $\mathcal{B}_q$. Enlarge each D_r once to include the fixed costs of $\mathsf{P}_{\neq 0}$ and $\partial_s^{-1}\mathsf{P}_{\neq 0}$ in the two source components. This is a finite operation at each order. All choices are nondecreasing in their inputs, and C_R from (3.72) is retained in these constants and all later constants before any frequency parameter is chosen. In (7.3), D_r depends only on r,C_R and $C_0,\dots,C_r$; the lifetime length is accounted for by Π_q and the relative growth by $\mathfrak{w}_q$.

7.1. **Quantitative estimates for one layer.** The local estimate is conditional on explicit coefficient bounds. The constants for products, phase inversion, localization and the correction equations are defined at all orders; their uniform estimates use only the supplied derivative ranges. Theorem 8.2 verifies these hypotheses simultaneously for all layers.

In this subsection write $d=k_q$, $J_q=2d+8$, $r_*=9d+41$ and $I=I_q$. The required derivative order r_* is justified in (7.15). We omit q on $\lambda,\Pi,\ell,\Gamma,M,\zeta,p,Q,w,y^0,Y$ and on $\widetilde G,\widetilde e_1,\kappa,\mathcal{B},\mathcal{U}$. Precisely, $\sigma=\sigma_{q-1}$, $D=D_{<q}$ and $C=\mathsf{C}_q$ from (5.26); $\mathcal{U}$ is the full propagator in (3.21). We retain the index on the lifetime deformation bound K_q, distinct from the Biot–Savart operator K. Set $y^0=(\Theta,\widehat\Omega)$, $Y=\sup_I|y^0|$ and $\mathfrak{w}=|y^0|/Y$.

Definition 7.1 (Hypotheses for the one-layer estimate)**.** The layer satisfies the following conditions.

(i) *Scales and deformation.*

$$\lambda \geq 4, \quad Q \geq 201, \quad 120d \leq Q, \quad 1 \leq \ell^{-1} \leq \Pi \leq \lambda^{5/Q}, \\ 1 \leq \sigma \leq \Pi, \quad \Pi^{-1} \leq \Gamma \leq \Pi, \quad |I| \leq \Pi. \tag{7.4}$$

The unit vector p is constant, $M' = DM$, $\det M = 1$ and $\zeta = M^{-T}p$. For a fixed $K_q \geq 1$, $\sup_I \max(\|M\|, \|M^{-1}\|) \leq K_q$, $K_q^2 \leq \Pi$ and $K_q \ell \leq 1/4$.

(ii) *Coefficient derivatives.* For $0 \leq b \leq r_*$ the coefficient bounds are

$$\|(M^{\pm 1})^{(b)}\| \leq C_b K_q \Pi^b, \qquad \|D^{(b)}\| \leq C_b \Pi^{b+1}, \qquad |\zeta^{(b)}| \leq 2C_b \Pi^b, \\ |(\widetilde{G}_j)^{(b)}| \leq C_b \Pi^{b+2}, \qquad |C_{ij}^{(b)}| + |(Cp)_i^{(b)}| + |\widetilde{e}_{1,i}^{(b)}| \leq C_b \Pi^{b+1}, \\ |\kappa^{(b)}| \leq 4C_b \Pi^b, \qquad \kappa \geq \kappa_* > 0, \qquad \|\mathcal{B}^{(b)}\| \leq C_b \Pi^{b+1}. \tag{7.5}$$

Here κ_* is fixed independently of the layer, and $\mathcal{B}$ has exactly the entries in (6.9); in particular its lower entry is $\sigma\zeta_1$, with the fixed laboratory component ζ_1.

(iii) *Reference pair and activation.* The nonvanishing solution $(y^0)' = \mathcal{B}y^0$ satisfies

$$Y \leq C_Y \lambda^{-7/8}, \quad |(y^0)^{(b)}(t)| \leq C_b' \Pi^b |y^0(t)| \quad (0 \leq b \leq r_*), \\ |y^0(t)| \leq C_{\mathrm{seed}} \lambda^{-d-6} \quad \text{during activation}, \qquad \|\mathcal{U}(t,s)\| \leq C_R \frac{\mathfrak{w}(t)}{\mathfrak{w}(s)} \quad (s \leq t,\ s,t \in I). \tag{7.6}$$

The constants $C_Y, C_{\mathrm{seed}} \geq 1$ are fixed. The seed bound is for the nonzero homogeneous pair on the full closed activation interval in (3.20), including both endpoints. The activation is $w = \eta(\Gamma(t - t^{\mathrm{in}}))$, so $|w^{(b)}| \leq \|\eta\|_{C^b} \Pi^b$.

(iv) *Profiles, affine older fields and fixed-layer derivatives.* The profiles are the fixed profiles of the construction. Every older field is exactly affine on the full lifetime support of this layer, so (6.4) are its exact residual differences. For this fixed layer all derivatives of the coefficient functions are bounded on I, without an order restriction.

The fixed sequences (C_b) and (C_b') are specified in advance for all orders, but bound this layer only in the stated ranges. Their generic C_0 and the fixed reciprocal bound κ_*^{-1} are retained in every operation constant. In particular (7.3) applies for $0 \leq r \leq r_*$. Beyond these ranges the last hypothesis gives finiteness for each fixed layer, not the same uniform constants.

For a profile A, let $[A]_r = [A]_{r,q}$ be the normalized phase/material norm (7.2), and let $[A]_r^\sharp$ omit its factor $\mathfrak{w}$. Profiles in these norms are periodic in s, supported in $|a| \leq \ell$, and flat at t^{in}. The coefficient norm is $[c]_r^{\mathrm{time}} = \max_{b \leq r} \Pi^{-b} \|c^{(b)}\|_\infty$. There are constants $c_r^{\mathrm{alg}} \geq 1$, depending only on r, such that

$$[AB]_r \leq c_r^{\mathrm{alg}} [A]_r [B]_r, \qquad [cA]_r \leq c_r^{\mathrm{alg}} [c]_r^{\mathrm{time}} [A]_r, \\ [\partial_s A]_{r-1} \leq [A]_r, \quad [\partial_{a_j} A]_{r-1} \leq \ell^{-1} [A]_r, \quad [\partial_\perp A]_{r-1} \leq c_r^{\mathrm{alg}} \ell^{-1} [A]_r, \quad [\partial_t A]_{r-1} \leq \Pi [A]_r, \\ [a_j A]_r \leq c_r^{\mathrm{alg}} \ell [A]_r, \quad [\langle A \rangle]_r \leq [A]_r, \quad [\mathsf{P}_{\neq 0} A]_r + [\partial_s^{-1} \mathsf{P}_{\neq 0} A]_r \leq c_r^{\mathrm{alg}} [A]_r. \tag{7.7}$$

The derivative assertions are for $r \geq 1$. These rules also hold without the relative weight. Leibniz's rule proves the product and coefficient rules; the weighted product uses $\mathfrak{w}^2 \leq \mathfrak{w}$. Differentiating $a_j A$ proves its factor ℓ, while the displayed kernel of the phase inverse proves its boundedness. No operation here inverts a slow derivative.

At order zero, $|\kappa^{-1}| \leq \kappa_*^{-1}$. For $b \geq 1$, differentiating $\kappa\kappa^{-1} = 1$ gives

$$(\kappa^{-1})^{(b)} = -\kappa^{-1} \sum_{j=1}^{b} \binom{b}{j} \kappa^{(j)} (\kappa^{-1})^{(b-j)}.$$

Starting with that base bound defines a finite nondecreasing majorant at every order specified in advance, depending only on r, κ_*^{-1} and $(C_b)_{b\le r}$. It bounds $[\kappa^{-1}]_r^{\mathrm{time}}$ on this layer for $0 \le r \le r_*$. The generic coefficient constant C_0 is retained in all products.

7.1.1. *Frequency-independent level constants.* The constants are defined before the frequency λ is chosen, so that the common majorant can later satisfy $\mathcal{K}(d) \le \log \lambda$. The recurrence is finite at every prescribed order and level; its uniform estimates use the supplied derivative ranges. Increasing a correction level raises the remaining oscillatory error grade.

Put $\delta = \Pi^5 \lambda^{-7/8}$. Fix positive nondecreasing elementary majorants with the following roles and dependencies:

(7.8)

constant	operation bounded	orders of fixed inputs
E_r	level-zero products	$C'_b\ (b \le r),\ \eta, F, f$ through r
P_r	κ^{-1} and phase inversion	$C_b\ (b \le r),\ \kappa_*^{-1}$
B_r	two localization terms	$C_b\ (b \le r+2),\ \kappa_*^{-1}$
H_r	all source operations	$C_b\ (b \le r+3),\ \kappa_*^{-1}, C_Y$

Here P_r includes the algebra cost of multiplication by κ^{-1}. The constants B_r, H_r include the relevant P's and the finite product and component sums, but no level constants. They exist by (7.7) and the reciprocal induction. The two localization terms bounded by B_r are displayed in (7.13).

In particular the localization constant uses derivatives only through $r+2$, not through the source order $r+3$. The level induction verifies that all coefficient and relative-amplitude orders needed in an actual estimate lie within the supplied ranges.

The family $\mathcal{C}(j,r)$ bounds V_j and $\widetilde{W}_j$ at order r, $\mathcal{C}'(j,r)$ bounds the normalized localization part $\mathcal{Z}_j = (\sigma/\lambda) Z_j^\flat$, and $\mathfrak{s}(n,r)$ bounds the scalar and normalized vorticity sources. For all nonnegative orders and levels, define

$$
\begin{aligned}
\mathcal{C}(j,r) = \mathcal{C}'(j,r) = 0\ (j<0), \qquad \mathcal{C}(0,r) &= E_r, \qquad \mathfrak{s}(0,r) = 0,\\
\mathcal{C}'(j,r) &= B_r\left[\mathcal{C}(j-1,r+1) + \mathcal{C}(j-2,r+2)\right],\\
\mathfrak{s}(n,r) &= H_r\Big[\mathcal{C}(n-1,r+2) + \mathcal{C}(n-2,r+3)\\
&\quad + \sum_{i+j=n-1} \mathcal{C}(i,r+2)(\mathcal{C}(j,r+2) + \mathcal{C}'(j,r+2))\\
&\quad + \sum_{i+j=n-2} \mathcal{C}(i,r+3)(\mathcal{C}(j,r+3) + \mathcal{C}'(j,r+3))\Big],\\
\mathcal{C}(n,r) &= D_r \mathfrak{s}(n,r) \qquad (n \ge 1).
\end{aligned}
\tag{7.9}
$$

The sums run over nonnegative indices. Each recursive step uses a smaller level after $\mathcal{C}'$ is expanded. Hence these formulas define finite nonnegative constants at every prescribed order and level, nondecreasing in the fixed inputs and independent of the selected layer or admissible derivative order.

Proposition 7.2 (Quantitative level induction)**.** *Under Definition 7.1, the recursion (6.8)–(6.9) satisfies the following bounds in the weighted profile norm $[\cdot]_r = [\cdot]_{r,q}$ of (7.2). Here $d = k_q$, $Y = \sup_{I_q} |y_q^0|$, $\delta = \Pi^5\lambda^{-7/8}$, $J_q = 2d+8$ and $r_* = 9d+41$, as fixed at the start of this subsection; the level constants are those of (7.9):*

$$
[V_j]_r, [\widetilde{W}_j]_r \le \mathcal{C}(j,r) Y \delta^j, \qquad [\Psi_j]_r \le P_r \mathcal{C}(j,r) \frac{Y}{\lambda\sigma} \delta^j
\tag{7.10}
$$

for $0 \le j \le J_q$ and $r + 2j + 1 \le r_*$. A convenient combined bound for the pair and reconstructed streamfunction profile is

$$[V_j]_r + [\widetilde{W}_j]_r \le (2 + P_r)\mathcal{C}(j, r)Y\delta^j, \qquad [\Psi_j]_r \le \frac{(2 + P_r)\mathcal{C}(j, r)Y}{\lambda\sigma}\delta^j. \tag{7.11}$$

With $\mathcal{Z}_j = (\sigma/\lambda)Z^\flat_j$, one has

$$[\mathcal{Z}_j]_r \le \mathcal{C}'(j, r)Y\delta^j \quad (j \le J_q + 2,\ r + 2j \le r_*),$$

and

$$[S^\theta_n]_r, [(\sigma/\lambda)S^\omega_n]_r \le \mathfrak{s}(n, r)Y\delta^n\Pi^{-2} \quad (1 \le n \le J_q,\ r + 2n + 1 \le r_*). \tag{7.12}$$

Each remainder grade $J_q < g \le 2J_q + 4$ obeys the same estimate with $n = g$ and the primary profiles above J_q set to zero, with Z_j and $\mathcal{Z}_j$ retained through $J_q + 2$, whenever $r \le 5d + 18$.

Proof. We estimate the level pair, its localization terms, and then the sources and the corresponding Duhamel solutions. The final derivative count verifies all stated ranges, including the distinct truncated remainder range. The induction is on the level, with the estimates at each lower level available simultaneously at every order in its stated range. The level-zero formula and the relative amplitude estimate give $[V_0]_r, [\widetilde{W}_0]_r \le E_rY$ by Leibniz's rule. At every level, zero phase mean and (6.7) give

$$\Psi_j = (\lambda\sigma)^{-1}\kappa^{-1}\partial_s^{-1}\widetilde{W}_j, \qquad \lambda\Psi_{j,s} = (\sigma\kappa)^{-1}\widetilde{W}_j.$$

The first identity has cost $P_r/(\lambda\sigma)$ and the second has cost at most P_r/σ. These prove the bound for the streamfunction profile whenever the level-pair bound has been established.

The two localization pieces are

$$\mathcal{Z}_j = \frac{2}{\lambda\kappa}(Cp) \cdot \nabla_a\widetilde{W}_{j-1} + \frac{\sigma}{\lambda}\Delta_M\Psi_{j-2}. \tag{7.13}$$

The first piece uses level $j - 1$ at order $r + 1$ and the second uses level $j - 2$ at order $r + 2$. Their finite coefficient costs are included in B_r. The remaining scale factors, divided by $Y\delta^j$, are z_1, z_2 in the table below. For $j = 1$ the second piece is absent; in particular the allowed range $r + 2j \le r_*$ supplies every coefficient order used by B_r. For $j = 0$ both pieces vanish. This proves the bound for $\mathcal{Z}_j$.

For the sources the target is $T_n = Y\delta^n\Pi^{-2}$. The table records every scale quotient, separately from the Leibniz and coefficient constants already in H_r. The symbols z_1, z_2 refer to the two pieces of (7.13). The symbols b_1, b_2, b_3 refer, in order, to the older-gradient term and the fast and slow scalar brackets; b_4, b_5 refer to the two normalized time-derivative terms, and b_6 to the normalized slow buoyancy term.

(7.14)

scale quotient	upper bound
$z_1 = \Pi\ell^{-1}\lambda^{-1}\delta^{-1}$	$\Pi^{-3}\lambda^{-1/8}$
$z_2 = \Pi\ell^{-2}\lambda^{-2}\delta^{-2}$	$\Pi^{-7}\lambda^{-1/4}$
$b_1 = \Pi^4\ell^{-1}(\lambda\sigma)^{-1}\delta^{-1}$	$\lambda^{-1/8}$
$b_2 = Y\ell^{-1}\sigma^{-1}\delta^{-1}\Pi^2$	$C_Y\Pi^{-2}$
$b_3 = Y\ell^{-2}(\lambda\sigma)^{-1}\delta^{-2}\Pi^2$	$C_Y\Pi^{-6}\lambda^{-1/8}$
$b_4 = \Pi^4\ell^{-1}\lambda^{-1}\delta^{-1}$	$\lambda^{-1/8}$
$b_5 = \Pi^4\ell^{-2}\lambda^{-2}\delta^{-2}$	$\Pi^{-4}\lambda^{-1/4}$
$b_6 = \sigma\Pi^3\ell^{-1}\lambda^{-1}\delta^{-1}$	$\sigma\Pi^{-1}\lambda^{-1/8}$

Each last-column entry is at most 1 or C_Y. For example $b_1 \le \Pi^5\lambda^{-1}\Pi^{-5}\lambda^{7/8} = \lambda^{-1/8}$ and $b_3 \le C_Y\lambda^{-7/8}\Pi^2\lambda^{-1}\Pi^{-10}\lambda^{7/4}\Pi^2$.

The linear rows b_1, b_4, b_6 use level $n-1$ at order $r+2$; row b_5 uses level $n-2$ at order $r+3$. For b_4 and b_5, differentiate the two localization terms in (7.13) once in time, so the coefficient orders are included in H_r. For the fast bracket use the exact identity

$$\lambda \mathcal{J}_s(\Psi_i, V_j) = \frac{\widetilde{W}_i}{\sigma\kappa}\partial_\perp V_j - \lambda \Psi_{i,\perp} V_{j,s}.$$

It uses both levels at order $r+2$. The slow bracket uses both at order $r+3$. In the normalized vorticity brackets,

$$(\sigma/\lambda) Z_j = \partial_s \widetilde{W}_j + \mathcal{Z}_j,$$

so the second level constant becomes $\mathcal{C}(j, r+2) + \mathcal{C}'(j, r+2)$ in the fast row and its $r+3$ version in the slow row. All finite coefficient and Leibniz costs, including C_Y from b_2, b_3, are included in H_r. This accounts for all three scalar and five normalized vorticity source terms, proving (7.12) with the four groups in (7.9).

Applying (7.3) to the projected source gives the following bound for each component of the zero-data solution $(V_n, \widetilde{W}_n)$: $D_r \Pi^2 \mathfrak{s}(n,r) Y \delta^n \Pi^{-2} = \mathcal{C}(n,r) Y \delta^n$, since the projection and phase-inverse costs were included in D_r. This closes the level induction.

The support and flatness properties remain those of Proposition 6.1.

Required derivative orders. We distinguish the derivative orders used by the finite residual, whose primary profiles stop at level J_q, from those used to define its majorant $\mathfrak{s}(g,r)$ through the full recurrence (7.9). A source uses levels of index at most $n-1$ at order $r+2$, and levels at most $n-2$ at order $r+3$. Expanding the nested $\mathcal{C}'$ also uses level $n-3$ at order at most $r+4$ and level $n-4$ at order at most $r+5$. In each recursive step, decreasing the level index by j increases the required derivative order by at most $j+1 \le 2j$. Every recursive chain therefore reaches level zero at order at most $r+2n$, and defining the elementary coefficient constants requires derivatives only through order $r+2n+1$. This proves the stated level and source ranges and also bounds the inputs of $\mathcal{C}'(n,r)$.

The largest remainder grade is $2J_q+4$. Thus all formal constants used in the common majorant (7.21) of Theorem 7.3 below, with $n \le 2J_q+4$ and $r \le d$, are covered by the derivative order

$$r_* = d + 2(2J_q+4) + 1 = 9d + 41 \le \frac{3}{40} Q + 41 \le Q - 1. \tag{7.15}$$

The last inequalities use $120d \le Q$ and $Q \ge 201$. Theorem 5.5 supplies coefficient order b from profile norms through $b+2$, so the required order of profile derivatives is $9d+43$.

For an actual remainder grade g, no level beyond J_q is present and $\mathcal{Z}_j = 0$ for $j > J_q + 2$. These quantities are used only through order $r+3$, so it suffices that $r + 3 + 2(J_q+2) \le r_*$, equivalently $r \le 5d+18$. This smaller derivative count for the truncated expression does not replace the order required by the full recurrence: $\mathfrak{s}(g,r)$ remains defined by that recurrence even when $g > J_q$. $\square$

7.1.2. *Means and physical derivatives.* For the force estimates we need the first possible mean grades and the smaller physical differentiation cost of a phase-independent profile. Scalar means can survive at odd grades, and vorticity means at even grades; each derivative of such a mean costs Π^2 instead of $\lambda\Pi^2$. We prove both facts here. Separate reflections in phase and material coordinates give the following parities. Here parity (ς, ς') means $A(-s,a,t) = \varsigma A(s,a,t)$ and $A(s,-a,t) = \varsigma' A(s,a,t)$. Set $\varsigma_n = (-1)^n$. The profiles $V_n, \widetilde{W}_n, S_n^\theta$ have parity $(-\varsigma_n, \varsigma_n)$ in (s,a), while $\Psi_n, Z_n, \mathcal{Z}_n, S_n^\omega$ have parity $(\varsigma_n, \varsigma_n)$.

Indeed the assertion holds at level zero; ∂_s and ∂_s^{-1} reverse the first parity, ∇_a reverses the second, the fast bracket reverses both, and the slow bracket preserves both products. Inserting these rules into (6.8) proves the source parities, after which the two components of (6.9) have the same parity. The pointwise ODE preserves that parity. Consequently

$$\langle S_n^\theta \rangle = 0 \ (n \text{ even}), \qquad \langle S_n^\omega \rangle = 0 \ (n \text{ odd}). \tag{7.16}$$

In particular the higher scalar means start at grade 3, and the higher vorticity means at grade 4.

Let $b_\Omega = \Omega/(\lambda^2\kappa) = \widehat{\Omega}/(\lambda\sigma\kappa)$ and $h = (Cp)\cdot\nabla_a g$. The first nonzero grade means are

$$\langle S_1^\theta\rangle = 2\mu_2\lambda w^2\Theta b_\Omega g\partial_\perp g, \qquad \langle S_2^\omega\rangle = 2\mu_2\lambda^2 w^2 b_\Omega^2\partial_\perp(gh). \tag{7.17}$$

For the first identity use $\langle F^2 - PF'\rangle = 2\mu_2$. For the second, (6.11) has only the term with $(i,j) = (0,1)$, since $Z_0^\flat = 0$; then $Z_1^\flat = 2\lambda(Cp)\cdot\nabla_a\Psi_{0,s}$ gives the formula. These are individual grade means, not the full means in (6.6): its additional $(wb_\Omega)^2\langle P^2\rangle\{g,\Delta_M g\}_a$ belongs to grade 4. All surviving means are divergences in the material coordinate of compactly supported fields, as in (6.11). Their evaluations in physical coordinates have zero integral because $x = Ma$ has Jacobian one. They stay in the force, without a slow-coordinate inverse.

For physical derivatives put $D^\flat = M^{-1}DM$. The exact differential operators representing differentiation in physical coordinates are

$$\begin{aligned}\mathcal{X}_i &= \lambda\zeta_i\partial_s + \sum_j (M^{-1})_{ji}\partial_{a_j}, \quad i = 1,2,\\ \mathcal{X}_0 &= \partial_t - \sum_{j,l} D^\flat_{jl}a_l(\lambda p_j\partial_s + \partial_{a_j}).\end{aligned} \tag{7.18}$$

Indeed $a_t = -D^\flat a$ and $s = \lambda p\cdot a$, while $\partial_{x_i}a_j = (M^{-1})_{ji}$. Thus $\partial_{x_i}A^\natural = (\mathcal{X}_iA)^\natural$ and $\partial_t A^\natural = (\mathcal{X}_0A)^\natural$. Fix $0 \le r \le r_*$. The three-factor Leibniz rule gives

$$\|(D^\flat)^{(b)}\| \le c_r\Pi^{b+2} \quad (0 \le b \le r \le r_*),$$

where c_r depends only on $(C_b)_{b\le r}$; here $K_q^2 \le \Pi$. At remaining profile order $1 \le m \le r$, applying either spatial differentiation operator $\mathcal{X}_i$, $i = 1,2$, contributes a factor at most $c_r(\lambda + \Pi\ell^{-1})$, and applying the time differentiation operator $\mathcal{X}_0$ contributes a factor at most $c_r[\Pi + \Pi^2\ell(\lambda + \ell^{-1})]$. Each application lowers the available profile derivative order by one. Multiplication by a_l contributes the factor ℓ. Since $\ell^{-1} \le \Pi$, $\ell \le 1/4$ and $\lambda \ge 4$, both factors are at most a fixed order-dependent constant times $\lambda\Pi^2$. The finite iteration defines a nondecreasing constant for evaluation in physical coordinates, $C_r^{\mathrm{comp}} \ge 1$ for every fixed r, depending only on r and $(C_b)_{b\le r}$. On this layer it proves, for $0 \le r \le r_*$,

$$\|A^\natural\|_{C^r_{x,t}} \le C_r^{\mathrm{comp}}(\lambda\Pi^2)^r[A]^\sharp_r, \qquad \|A^\natural\|_{C^r_{x,t}} \le C_r^{\mathrm{comp}}\Pi^{2r}[A]^\sharp_r \quad (A_s = 0). \tag{7.19}$$

Indeed if $A_s = 0$, every intermediate profile is phase-independent; the $\lambda\partial_s$ terms vanish and $\ell\ell^{-1} = 1$ in the operator $\mathcal{X}_0$. Thus each derivative then costs only Π^2. This distinction is essential for the means.

7.1.3. *A common majorant for the force increments.* The level majorant collects the finitely many correction bounds needed at a given derivative order. The common majorant additionally includes the products of these constants with the constants for evaluation in physical coordinates and for the inverse curl. It is these products, not separate bounds for each factor, that will be compared with the logarithm of the frequency. The force consists of activation, nonoscillatory and remainder terms in each equation. There are eight contributions because each nonoscillatory term is split into its leading and higher grades.

Let $C_k^R = C_{k,1/4}$ be the $C^k \to C^k$ bound for the fixed inverse curl of Lemma 6.2. For each fixed k, choose a finite constant $c_k^{\mathrm{row}} \ge 1$ for both the activation and leading-mean estimates, obtained from the Leibniz expansions of the activation formulas and the two identities (7.17). It depends only on $C_{\mathrm{seed}}, C_Y, \kappa_*^{-1}, \mu_2, (C_b, C_b')_{b\le k}$ and profile norms through $k+2$. It includes the squared amplitude constant in each leading-mean estimate. These activation and leading-mean estimates apply when the stated bounds are available; their use below is restricted to $0 \le k \le d$. Define

$$C_k^{\mathrm{out}} = C_k^{\mathrm{comp}}(1 + c_k^{\mathrm{row}}). \tag{7.20}$$

These constants control products of the constants for these estimates with the constants for evaluation in physical coordinates, not just their individual sizes. For every nonnegative integer $d \geq 0$, set

$$
\begin{aligned}
\mathcal{M}(d) &= 1 + \max_{\substack{0\le n\le 4d+20\\ 0\le r\le d}} [\mathcal{C}(n,r) + \mathcal{C}'(n,r) + \mathfrak{s}(n,r) + \mathcal{C}(n,0)],\\
\mathcal{K}(d) &= \max_{0\le e\le d} C_*(1+\mathcal{M}(e)) \max_{0\le k\le e} (1+C_k^R) C_k^{\mathrm{out}}.
\end{aligned}
\tag{7.21}
$$

This is the function $\mathcal{K}$ used to choose the derivative orders in (3.8). Set $c_Y = \max\{1, 2C_Y\}$ and fix $C_* \geq 8c_Y$ once. The factor eight counts the raw scalar and vorticity contributions; c_Y covers their fixed geometric-series and amplitude allowance. Thus $\mathcal{K}(d)$ bounds the full product $8c_Y(1+C_k^R)C_k^{\mathrm{out}}(1+\mathcal{M}(d))$ for $k \leq d$, not merely its individual factors.

The function $\mathcal{K}$ is finite, nondecreasing and at least two, defined for every nonnegative integer d before any admissible derivative order is selected. By (7.15), its coefficient inputs at d run through $9d+41$ and its profile inputs through $9d+43$. Its other dependencies are the fixed C_R from (3.72), C_0, κ_*^{-1}, C_Y, C_{seed} and recovery constants through order d. No frequency, growth scale, realized gain, lifetime, stage index or selected derivative order enters this definition.

We now pass from correction profiles to physical increments. Restore the layer index and use evaluation in physical coordinates, (6.1), to define

$$
\begin{aligned}
\theta_q &= \left(\sum_{n=0}^{J_q} V_{q,n}\right)^{\natural}, \qquad \psi_q = \left(\sum_{n=0}^{J_q} \Psi_{q,n}\right)^{\natural},\\
u_q &= \nabla^{\perp}\psi_q, \qquad \omega_q = \Delta\psi_q.
\end{aligned}
\tag{7.22}
$$

Evaluate both full residuals (6.4) on these finite profile sums, including their activation terms, phase means and remainders. Denote their evaluations in physical coordinates by $f_{\theta,q} = E^{\natural}_{\theta,q}$ and $E_q = E^{\natural}_{\omega,q}$, and set $f_{u,q} = \mathcal{R}_c E_q$, using the fixed compact inverse curl of Lemma 6.2. The state and force increments are extended by zero for $t < t_q^{\mathrm{in}}$; flatness at activation makes these extensions smooth. In particular, E_q denotes the complete curl-form force, not only the uncancelled oscillatory remainder.

Theorem 7.3 (Estimates for the force increments)**.** *Assume Definition 7.1 and*

$$
\mathcal{K}(d) \leq \log \lambda. \tag{7.23}
$$

Then, with $\lambda = \lambda_q$ and $d = k_q$, the corrected physical fields (7.22) and their complete force increments satisfy the following conclusions.

Quantitative estimates. *Their complete force increments obey*

$$
\|f_{\theta,q}\|_{C^k_{x,t}} + \|\mathcal{R}_c E_q\|_{C^k_{x,t}} \leq \lambda_q^{-1/2} \quad (0 \leq k \leq k_q), \qquad \|\theta_q\|_\infty \leq C_\theta \lambda_q^{-7/8}, \tag{7.24}
$$

on $\mathbb{R}^2 \times [0, T_*)$, where C_θ is fixed before the frequencies.

Regularity and structure. *Each fixed layer and its complete force increments $f_{\theta,q}, \mathcal{R}_c E_q$ have bounded mixed derivatives of every order on that interval and are flat at activation. Temperature and streamfunction are supported in $\overline{B}_{1/4}$, and the force pair is supported in $\overline{B}_{1/2}$. The fields $\theta_q, u_q, f_{\theta,q}, f_{u,q}$ are odd, while ψ_q, ω_q, E_q are even. The vorticity and E_q have zero spatial integrals. On the affine region $\mathcal{A}_q$ defined in (4.3), the fields retain the exact identities (4.4). These structural conclusions use the full standing Definition 7.1; the additional condition (7.23) is used for the quantitative estimates.*

Proof. We first estimate temperature, then the eight force contributions, and finally prove fixed-layer regularity and structure.

Temperature and the finite sum. The scale bounds imply

$$\delta \le \lambda^{-3/4} \le \tfrac{1}{2}, \qquad \Pi^{2d+3} \le \lambda^{1/4}, \tag{7.25}$$

because $25/Q \le 25/201 < 1/8$ and $5(2d+3)/Q \le 1/12+15/201 < 1/4$. After bounding the coefficient constants uniformly, we estimate the finite sums of powers of δ below using $\sum_{n\ge n_0} \delta^n \le 2\delta^{n_0}$.

For temperature, Proposition 7.2 and the common majorant control every higher level:

$$\sum_{n=1}^{J_q} \|V_n\|_\infty \le 2Y\mathcal{M}(d)\delta \le 2Y(\log\lambda)\lambda^{-3/4} \le 2Y.$$

The last step uses $\log\lambda \le \sqrt{\lambda}$ for $\lambda \ge 1$. Adding the level-zero bound gives the assertion with the fixed $C_\theta = C_Y(\|F\|_\infty\|f\|_\infty + 2)$.

For the force estimates fix $0 \le k \le d$. The coefficient and profile bounds needed for the activation, leading-mean and physical-coordinate evaluation estimates at order k are therefore within the supplied ranges. By (6.10), the two exact residuals are

$$f_{\theta,q} = (\dot w\Theta F g + \mathfrak{m}_\theta + R^\theta_{>J_q})^\natural, \quad E_q = (\dot w\Omega F' g + \mathfrak{m}_\omega + R^\omega_{>J_q})^\natural,$$

where $\mathfrak{m}_\theta$ sums odd-grade scalar means and $\mathfrak{m}_\omega$ sums even-grade vorticity means. These are six complete pieces before inverse-curl recovery. The remainder consists of the higher-grade sources and may itself contain phase averages. We distinguish the leading part of each explicit mean sum from its higher grades.

Activation and the leading means. On the support of $\dot w$, the relative amplitude derivatives have the seed bound in (7.6). Hence

$$[\dot w\Theta F g]^\sharp_k \le c_k^{\mathrm{row}}\Pi\lambda^{-d-6}, \qquad [\dot w\Omega F' g]^\sharp_k \le c_k^{\mathrm{row}}\Pi\lambda^{-d-5}.$$

The second row uses $\Omega = (\lambda/\sigma)\widehat{\Omega}$, $\sigma \ge 1$ and the extra phase derivative of F. To estimate the leading means, write their exact formulas as

$$\langle S_1^\theta\rangle = \frac{2\mu_2}{\sigma}(w\Theta)(w\widehat{\Omega})\kappa^{-1} g\partial_\perp g,$$

$$\langle S_2^\omega\rangle = \frac{2\mu_2}{\sigma^2}(w\widehat{\Omega})^2\kappa^{-2}\sum_j (Cp)_j\partial_\perp(g\partial_j g).$$

Their amplitudes are quadratic in Y. The first has one slow derivative, and the second two slow derivatives and one coefficient of size Π. Thus Leibniz's rule gives

$$[\langle S_1^\theta\rangle]^\sharp_k \le c_k^{\mathrm{row}}\Pi\lambda^{-7/4}, \qquad [\langle S_2^\omega\rangle]^\sharp_k \le c_k^{\mathrm{row}}\Pi^3\lambda^{-7/4}.$$

Their evaluation in physical coordinates uses $C_k^{\mathrm{comp}}\Pi^{2k}$, with no λ^k.

Higher means and remainders. Every actual source constant at order $k \le d$ is bounded by $\mathcal{M}(d)$. The parity identities (7.16) and the source estimate of Proposition 7.2 give

$$\sum_{n=3}^{J_q} \|(\langle S_n^\theta\rangle)^\natural\|_{C^k} \le 2C_k^{\mathrm{comp}}\mathcal{M}(d)Y\Pi^{2k-2}\delta^3,$$

$$\sum_{n=4}^{J_q} \|(\langle S_n^\omega\rangle)^\natural\|_{C^k} \le 2C_k^{\mathrm{comp}}\mathcal{M}(d)\lambda Y\Pi^{2k-2}\delta^4.$$

The extra λ is the normalization factor $\lambda/\sigma \le \lambda$, not a cost of evaluation in physical coordinates. For remainders this evaluation does cost λ^k, but the first remaining grade is $J_q + 1 = 2d+9$:

$$\|(R^\omega_{>J_q})^\natural\|_{C^k} \le 2C_k^{\mathrm{comp}}\mathcal{M}(d)(\lambda\Pi^2)^k\lambda Y\delta^{2d+9}.$$

Here the favorable Π^{-2} was discarded; the scalar estimate omits the factor λ.

The joint force bound. Each of the eight contributions is bounded by its stated constant times a scale at most $\lambda^{-3/2}$. For activation and leading means, (7.19) bounds the respective scale factors by $\Pi^{2d+3}\lambda^{-5}$ and $\Pi^{2d+3}\lambda^{-7/4}$, so (7.25) applies. For the higher means and remainders, factor out C_Y using $Y \le C_Y\lambda^{-7/8}$. Their scales are all bounded by

$$\Pi^{2d+3}\lambda^{1/8}\delta^3 \le \Pi^{2d+3}\lambda^{-7/4} \le \lambda^{-3/2}.$$

Indeed $\Pi^{2k-2} \le \Pi^{2d+3}$ even at $k = 0$, and $\lambda^k\delta^{2d} \le 1$ since $k \le d$; the remaining powers δ^4 and δ^9 are at most δ^3. Thus each norm is at most $c_Y C_k^{\mathrm{out}}(1 + \mathcal{M}(d))\lambda^{-3/2}$, with exactly the geometric-series and amplitude allowance already included in c_Y. The common majorant includes the sum of these eight contributions and the $C^k \to C^k$ inverse-curl cost, giving

$$\begin{aligned} \|f_{\theta,q}\|_{C^k} + \|\mathcal{R}_c E_q\|_{C^k} &\le 8c_Y(1 + C_k^R)C_k^{\mathrm{out}}(1 + \mathcal{M}(d))\lambda^{-3/2} \\ &\le \mathcal{K}(d)\lambda^{-3/2} \le (\log\lambda)\lambda^{-3/2} \le \lambda^{-1/2}. \end{aligned} \tag{7.26}$$

The product of constants has already been included in the common majorant. Recovery is applied to the complete spatially mean-zero vorticity residual E_q. Its activation term is not asserted separately mean-zero, nor is the recovered force annular.

Fixed-layer regularity. To bound all derivatives of a fixed layer, use induction on the finite level and then on the time derivative. Every source is a finite differential polynomial in earlier levels and bounded coefficient derivatives. At level zero the homogeneous amplitude equation and the fixed-layer coefficient bounds give all amplitude derivatives, and the exact activation cutoff formula supplies every activation derivative. The unweighted bound $\|\mathcal{U}(t,s)\| \le e^{C_0\Pi|I|} \le e^{C_0\Pi^2}$ is finite for this fixed layer, and bounds all material-coordinate and phase derivatives of its Duhamel integral. The identity

$$\partial_t^{b+1}y_n = \sum_{j=0}^{b}\binom{b}{j}\mathcal{B}^{(j)}\partial_t^{b-j}y_n + \partial_t^b f_n$$

then bounds every time derivative. The phase inverse and the differentiation operators (7.18) transfer these bounds to all mixed physical derivatives; the fixed inverse curl preserves their order.

Structure and zero spatial integral. Flatness follows from the flat activation cutoff and the same recurrence at activation. Compact support, the exact affine core, parity and mean zero follow from Proposition 6.1, (6.11), and the compact-streamfunction identity. More explicitly the complete physical vorticity residual is

$$E_q = \partial_t\Delta\psi_q + (Dx + \nabla^\perp\psi_q)\cdot\nabla\Delta\psi_q - \partial_1\theta_q.$$

Every term integrates to zero: ψ_q and θ_q are compact, $\operatorname{tr} D = 0$, and $\nabla^\perp\psi_q$ is divergence-free. This proves the required spatial mean condition for the complete E_q directly and justifies compact inverse-curl recovery. On the plateau intersected with the linear phase cell the unchanged level-zero fields are exactly

$$\theta_q = w_q\Theta_q\lambda_q\zeta_q\cdot x, \qquad u_q = w_q\Omega_q(\mathsf{e}_q\cdot x)J\mathsf{e}_q, \qquad \omega_q = w_q\Omega_q.$$

The vector-force support is $\overline{B}_{1/2}$ by Lemma 6.2; the force need not vanish on the plateau. □

Theorem 7.3 bounds the force increments under the explicit hypotheses for one layer. The following lemma on derivative orders and Section 8 verify all its hypotheses with one frequency choice, using the generic constants already supplied.

Lemma 7.4 (Derivative orders chosen before the controlled evolution)**.** *For any finite nondecreasing function $\mathcal{K}$ independent of the frequencies, use $Q_q = Q_* + q$ and the increasing frequencies of (3.7), with $\log\lambda_1 \ge \mathcal{K}(0)$. Then (3.8) defines nondecreasing integers $k_q \to \infty$ without using any realized stage time or growth scale.*

Proof. The admissible sets are nonempty and increase, because Q_q and $\log \lambda_q$ increase. Given any fixed integer k, eventually $Q_q \geq 120k$ and $\log \lambda_q \geq \mathcal{K}(k)$. Then $k_q \geq k$. The growth scales are used only afterward in the successive steering definitions. □

8. One common choice of parameters

We now make the preceding estimates hold for one sequence of layers. The order of choice is essential: first fix the geometric bounds with their strict inequalities and the single low-order error tolerance, then all coefficient and correction constants, and finally the initial frequency. That frequency determines the later frequencies, derivative orders and seeds before the controlled evolution is constructed. We will verify every local hypothesis on the resulting full lifetimes. Increasing the requested derivative order will not change this construction.

Fix the normalized cutoff and the other profiles, Q_*, s_0, C_ℓ and the design (3.9), then the fixed low-order bounds, K_{sch} and the tolerances $\epsilon, \epsilon_{\mathrm{sch}}$ in (3.16). The low-order proof fixes $C_K, C_\alpha, C_\Pi, C_\Theta$ and C_R. Next fix the sequence of coefficient constants in Theorem 5.5, with a finite enlargement for each row of (7.5); fix also the constants C_b' in the relative amplitude-derivative estimate (7.6), $\kappa_* = z_*^2$, the proved seed-amplitude constant C_{seed}, and a fixed $C_Y \geq C_\Theta$ in the normalization $Y_q \leq C_Y \lambda_q^{-7/8}$. Retain the actual generic order-zero constant C_0. These constants are supplied by estimates proved before the frequencies are chosen and uniform over all admissible derivative orders, not by suprema over an unconstructed trajectory. The reciprocal recurrence, pointwise Duhamel recurrence, phase inverse and compact-curl constants then determine the level recurrences, $\mathcal{M}(d)$, and the single function $\mathcal{K}$ of (7.21). In particular K_{sch} depends on none of these later constants.

8.1. A finite logarithmic threshold.

The preceding estimates require only finitely many low-order frequency conditions: separated rates and persistent gain, the first-layer exceptions, and nesting over full lifetimes. We show that one finite logarithmic threshold satisfies all of them, along with the condition on the admissible derivative orders.

Put

$$\Gamma_1 = \sin s_*, \quad B = \sqrt{C_\sigma}, \quad C_\zeta = \zeta_+, \quad a_L = \frac{Q_1}{120} + \frac{41}{8}.$$

Here B, C_ζ are the parameters of Lemma 4.3, and the fixed coefficient a_L gives $L_1 \leq a_L y$ by (3.8). These quantities and $\ell_1 = s_0/C_\ell$ are fixed data. Let y_{free} be the maximum of the logarithms of sufficient initial-frequency thresholds in Theorem 3.2, Lemma 3.13 for this ϵ_{sch}, and Lemma 5.1. These logarithmic thresholds are independent of $\mathcal{K}$ and of the chosen derivative orders. They already include the finite scalar comparisons proved in their schedule arguments, including the exceptional second stage and the first term beyond the finite head in the future-stage sum.

For $y = \log \lambda_1$, the schedule, derivative-order and deformation bounds require

$$\begin{gathered} y \geq \max\{y_{\mathrm{free}}, (Q_* + 3)^3, \mathcal{K}(0), \epsilon_{\mathrm{sch}}^{-1}, C_K/5\}, \\ \frac{y}{\log y} \geq 160 Q_2. \end{gathered} \tag{8.1}$$

The separation of the first growth scales needed for the geometric-sum bounds requires

$$c_\sigma e^{y/8} \geq 16/\epsilon_{\mathrm{sch}}^2. \tag{8.2}$$

The first-layer coefficient scale $\Pi_1 = y^{10}$ must dominate

$$y^{10} \geq \max\{C_\Pi, C_\ell/s_0, (\sin s_*)^{-1}, 2C_\alpha\}. \tag{8.3}$$

The first lifetime bound $|I_1| \leq \Pi_1$ and the bound $X_2 \leq \epsilon_{\mathrm{sch}} \Pi_1$ for the second-stage rate follow from

$$y^9 \geq \frac{a_L + 2 + \Lambda_1^{-1}}{\sin s_*} + C_T, \qquad y^8 \geq \epsilon_{\mathrm{sch}}^{-1} Q_2^4. \tag{8.4}$$

The exceptional first-layer inequality $\Pi_1 \le \lambda_1^{5/Q_1}$ is

$$\frac{5y}{Q_1} \ge 10 \log y. \tag{8.5}$$

Finally, the sufficient inequalities for spatial and linear-phase nesting in Lemma 4.3 require

$$\begin{gathered} Be^{-47y/16} \le \ell_1/4, \qquad B^2 e^{-y} \le 1/4, \\ C_\zeta B e^{-31y/16} \le s_0/2. \end{gathered} \tag{8.6}$$

Together with the schedule comparisons, the inequality $y/\log y \ge 160Q_2$ in (8.1) gives the later-layer bound $\Pi_q \le \lambda_q^{5/Q_q}$; (8.5) supplies its first-layer case. All coefficients have already been fixed. The finitely many scalar bounds from the control and derivative schedule proofs are already included in y_{free}, as are $X_2/X_1 \ge \epsilon_{\text{sch}}^{-1}$ and $\log \max(1, \sqrt{C_\sigma}) \le (\log y)/8$. The first-design ratio $\Gamma_1/X_1 = \Lambda_1^{-1}$ remains a fixed design contribution. The threshold y_{free} is fixed independently of the correction constants, before $\mathcal{K}$ is defined; the only added common-majorant condition is $y \ge \mathcal{K}(0)$. This threshold is used for all derivative orders.

Lemma 8.1 (A common threshold)**.** *There is a finite y_* such that every $y \ge y_*$ satisfies (8.1)–(8.6). For the associated frequencies in (3.7), writing $x_q = \log \lambda_q$, one has*

$$x_q \ge 201^{q-1} y, \qquad q^2 \le x_q, \qquad Q_q \le x_{q-1} \ (q \ge 2), \qquad \Pi_q \le \lambda_q^{5/Q_q} \ (q \ge 1). \tag{8.7}$$

Moreover $x_{q-1}/\log x_{q-1} \ge 160 Q_q$ for $q \ge 2$.

Proof. The schedule thresholds are finite by their normalized ratio bounds and (3.51). In the displayed list, positive powers and exponentials tend to infinity, while the three nesting expressions tend to zero. The functions $y/\log y$ and $5y/Q_1 - 10\log y$ are increasing once $y > 3$ and $y > 2Q_1$, respectively; both ranges follow from the cubic lower bound in (8.1). Taking the maximum of these finitely many initial thresholds gives y_*. The only condition involving the common majorant is $y \ge \mathcal{K}(0)$; no threshold is chosen by derivative order.

The recursion $x_q = Q_q x_{q-1}$ gives $x_q \ge 201^{q-1} y$. Since $y \ge 1$ and $201 q^2 \ge (q+1)^2$, it also gives $q^2 \le x_q$. The deterministic comparisons in the proof of Lemma 3.13 give $Q_q \le x_{q-1}$ and $\Pi_q \le \lambda_q^{5/Q_q}$ for $q \ge 2$. At $q = 1$ the latter is exactly (8.5); the frequency recursion is only used for $q \ge 2$.

Finally, for $Q \le x$ and $x/\log x \ge 160Q$,

$$\frac{Qx}{\log(Qx)} \ge \frac{Q}{2}\frac{x}{\log x} \ge 80Q^2 \ge 160(Q+1).$$

The last inequality holds for $Q \ge 201$. Starting from (8.1), this proves $x_{q-1}/\log x_{q-1} \ge 160 Q_q$ for every $q \ge 2$. Every decreasing generic schedule term is evaluated at $x \ge y$; its exceptional initial terms were included in y_{free}. Thus y_* depends only on the fixed data and $\mathcal{K}(0)$. ☐

8.2. **Realization on the full lifetimes.** The threshold just constructed is now applied once. The proof first constructs the controlled trajectories and their coefficient bounds, then checks support nesting and the local correction hypotheses. This order avoids defining any frequency or derivative order from a selected steering time.

Theorem 8.2 (One choice of parameters for all layers)**.** *For the fixed data and profiles above, let $\lambda_* = e^{y_*}$, where y_* is supplied by Lemma 8.1. For every $\lambda_1 \ge \lambda_*$ the schedule (3.7)–(3.8) defines the frequencies, derivative orders and seeds before the controlled evolution is run. The activated affine-wave evolution of Theorem 3.2, with the stage protocol of Subsection 3.3, exists on $[0, T_*)$. Its compact layers are obtained from the finite corrections of Proposition 6.1 by (7.22). They satisfy Definition 7.1 and (7.23) on every full lifetime $I_q = [t_q^{\text{in}}, T_*)$. Consequently all conclusions of Theorem 7.3, in particular both estimates in (7.24), hold for those actual layers. The derivative* *orders are nondecreasing and tend to infinity. The same choice of frequencies works for every derivative order.*

Proof. The proof verifies first the scales, then lifetime nesting, and finally the coefficient and residual hypotheses. Since $\log \lambda_1 \geq \mathcal{K}(0)$, Lemma 7.4 defines k_q before the controlled evolution is constructed and proves $k_q \uparrow \infty$. They obey $120k_q \leq Q_q$, so the theorems independent of the correction constants apply to this sequence. Theorem 3.2 constructs all stages, growth scales, growth threshold crossings and least admissible pulse amplitudes, and Theorem 5.5 supplies the full-lifetime estimates.

Scales: Definition 7.1(i). For the first layer, (8.2) and the strict lower growth-scale bound give $\sigma_1 \geq 4\epsilon_{\mathrm{sch}}^{-1}$ and $\sigma_1^{-1} \leq \epsilon_{\mathrm{sch}}/4$. The schedule threshold included in y_{free} gives $1 \leq \epsilon_{\mathrm{sch}}\lambda_1^{1/8}$. The first gain satisfies $L_1 \leq a_L y$. Thus (8.4) yields

$$|I_1| \leq \frac{L_1 + 2 + \Lambda_1^{-1}}{\Gamma_1} + C_T \leq y\left(\frac{a_L + 2 + \Lambda_1^{-1}}{\Gamma_1} + C_T\right) \leq \Pi_1.$$

For the second-stage rate, we separately check $X_2 \leq \epsilon_{\mathrm{sch}}\Pi_1$: $X_2 = L_2\Gamma_2 \leq L_2^2 \leq Q_2^4 y^2 \leq \epsilon_{\mathrm{sch}}\Pi_1$; here $L_2 \leq Q_2 \log \lambda_2 = Q_2^2 y$. All remaining first-layer sizes, including $C_\Pi, \Gamma_1^{\pm 1}, \ell_1^{-1}$ and $2C_\alpha$, are bounded by Π_1 through (8.3); $\Gamma_1 < 1 < C_\Pi$ makes the omitted unit and positive-rate bounds automatic.

For $q \geq 2$, Lemmas 3.13 and 5.1 give

$$C_\Pi \sigma_{q-1}^2 \leq \Pi_q, \qquad 1 \leq \sigma_{q-1} \leq \Pi_q, \qquad \Gamma_q^{\pm 1} \leq \Pi_q, \qquad \ell_q^{-1} = \lambda_{q-1}^3 \leq \Pi_q.$$

Also $|I_q| \leq \Pi_q$ by (5.13). Also $\ell_1 < 1/4$ and $\ell_q = \lambda_{q-1}^{-3} < 1$ for $q \geq 2$, so $1 \leq \ell_q^{-1}$ for every layer. Together with (8.7), these estimates give all numerical inequalities in (7.4).

The deformation-matrix bound is $K_1 = 1$ and $K_q \leq \sigma_{q-1} \leq B\lambda_{q-1}^{1/16}$ for $q \geq 2$, by Theorem 3.2; thus $K_q^2 \leq \Pi_q$. Here K_q bounds $\sup_{I_q} \max(\|M_q\|, \|M_q^{-1}\|)$. With $B = \sqrt{C_\sigma}$ and $C_\zeta = \zeta_+$, (8.6) is exactly the sufficient condition (4.7) of Lemma 4.3. That lemma gives (4.8), the common support radius strictly below $1/4$, and shows that the base is affine on every support. It places each later support in every older material plateau and linear phase cell over its entire lifetime, not merely at activation.

Coefficients: Definition 7.1(ii). The remaining structural identities in item (i) follow from the material flow. The activation vector p_q is a fixed unit vector; $M_q' = D_{<q}M_q$ and $\zeta_q = M_q^{-T}p_q$. Since $\operatorname{tr} D_{<q} = 0$ and $M_q(t_q^{\mathrm{in}}) = \mathrm{Id}$ by (3.3), one has $\det M_q = 1$. The bound $\zeta_{q,2} \geq z_*$ gives $\kappa_q \geq \kappa_*$, uniformly over the lifetime. The coefficient range required by the correction recursion is available by (7.15) with $d = k_q$ and $Q = Q_q$. Its two additional profile derivatives are already included in $\mathcal{K}$. Thus (5.27) and (5.28) give every row of (7.5). In particular the lower off-diagonal entry remains $\sigma_{q-1}\zeta_{q,1}$, with the laboratory component. *Reference pair: Definition 7.1(iii).* The same order range supplies the relative estimate $|(y_q^0)^{(b)}(t)| \leq C_b'\Pi_q^b|y_q^0(t)|$. It is relative to the pointwise pair throughout activation and the later lifetime. The nonzero reference solution, both normalized amplitude components during activation, $Y_q \leq C_Y\lambda_q^{-7/8}$ and the weighted propagator come directly from (3.20)–(3.21). Their proof uses the separate fixed first holding-interval bound $H_1 \leq C_{\mathrm{hold}}$. The activation has the prescribed form and is flat at activation. The arbitrary-order coefficient bounds are the first assertion of Theorem 5.5, with no new frequency threshold. The constants used throughout are exactly those fixed before (7.21), including their actual generic order-zero values.

It remains to verify the common-majorant condition and construct the physical layers. The function $\mathcal{K}$ is fixed before k_q is selected, and (3.8) gives directly

$$\mathcal{K}(k_q) \leq x_q = \log \lambda_q.$$

This is precisely (7.23). By (7.26), it bounds together the constants for all correction grades, evaluation in physical coordinates and inverse-curl recovery at every order $k \leq k_q$. No additional absorption of constants is required. Also $\lambda_q \geq 4$ and

$$\delta_q = \Pi_q^5 \lambda_q^{-7/8} \leq \lambda_q^{-7/8+25/Q_q} \leq \lambda_q^{-3/4} \leq \frac{1}{2}.$$

The temperature proof uses this geometric ratio and $\mathcal{K}(k_q) \leq \log \lambda_q$, without an additional normalization condition on the initial frequency.

Affine older fields: Definition 7.1(iv). With the whole controlled evolution now fixed, construct the layers in increasing q by the finite local recursion. The base interpolation has finished at $t_1^{\mathrm{in}} = 1$ and is affine on the first support. Inductively, every previously constructed correction vanishes on its entire material plateau for all phase values, by Proposition 6.1. The lifetime inclusions above therefore make the actual older fields exactly $G_{<q} \cdot x$ and $D_{<q}x$ on the complete support of layer q. The older vorticity is constant there. Consequently (6.4) are its exact residual differences, including the laboratory term $-\partial_1 \theta_q$; the interaction with the gradient of the older vorticity vanishes identically. All premises of Theorem 7.3 are now available. It produces the stated mixed derivative bounds for the forces, all-order bounds for each fixed layer, flat activation, parity, complete spatial mean cancellation, and the original compact recovery for each layer. Recovery is applied to the complete spatially mean-zero E_q, not inferred from zero phase mean of an oscillatory profile. The geometric bounds put temperature and streamfunction in $\overline{B}_{1/4}$ and their force pair in $\overline{B}_{1/2}$ throughout their lifetimes. This completes the layer construction. Section 9 assembles the finite preterminal state sums and extends the force series smoothly to the terminal time. ☐

Every choice has thus been made in the order: profiles and data, fixed design and low-order bounds, the single low-order error tolerance and schedule factor, uniform coefficient and recovery constants, positive correction constants and $\mathcal{K}$, the finite threshold, the frequencies, derivative orders and seeds, the controlled evolution, and finally the compact layers. In particular a stage endpoint or a selected pulse amplitude is never used to define a frequency parameter or its admissible derivative order.

9. Construction of the solution and smoothness of the forces

Two different convergence arguments are needed. Before the blowup time only finitely many layers are active, so the equations follow from exact identities between smooth finite sums. At the blowup time the force is an infinite series. Its extension requires uniform convergence of every mixed derivative and, separately, the estimate at the next order to obtain terminal limits. No terminal extension of the state or pressure is part of this argument.

Fix once and for all the parameters and actual layers of Theorem 8.2. Their activation times increase to $T_* < \infty$, and their full-lifetime supports lie inside every older affine region. No further enlargement of a frequency parameter is made in this section.

9.1. The base and pressure recovery.

The base supplies a smooth contribution to the force series. We first record its all-order bounds and exact origin values, then recover pressure from the complete preterminal residual.

Lemma 9.1 (All-order bounds for the base). *Write $\mathsf{e}_0(t) = R_{\alpha(t)} e_2$, and let $\mathcal{R}_c$ be the fixed compact inverse curl of Lemma 6.2. The base fields in (4.9) have bounded mixed derivatives of every order on $\mathbb{R}^2 \times [0, T_*)$. The temperature is odd, the streamfunction is even, and their supports lie in $\overline{B}_{R_0}$ and $\overline{B}_{R_H}$, respectively. Their exact residuals are (4.10); E^b is even with zero integral, and $f_u^b := \mathcal{R}_c E^b$ is odd and supported in $\overline{B}_{R_H}$. Both base forces have bounded mixed derivatives of all orders, and*

$$\nabla \theta^b(0,t) = -\mathsf{e}_0(t), \qquad Du^b(0,t) = \dot{\alpha}(t) J, \qquad \omega^b(0,t) = 2\dot{\alpha}(t). \tag{9.1}$$

In addition,

$$\|\theta^b\|_\infty \leq C_b^\theta := \|\chi_0\|_\infty (2\|F\|_\infty + 1).$$

Proof. Theorem 5.5 bounds every derivative of α on $[0, T_*)$, while $\alpha = 0$ on $[0, 1]$. Each derivative of $R_\alpha e_2$ is a finite polynomial in derivatives of α with bounded trigonometric coefficients. Differentiating (4.9) therefore gives bounded products of those coefficients, fixed profile derivatives, and powers of x restricted to a fixed support. This proves the mixed bounds. On $[0, 1]$ the base is exactly $\eta(t)\theta^{\mathrm{ref}} + (1 - \eta(t))\theta_{\mathrm{in}}$ and $\psi^b = 0$. Every time derivative of positive order of η and $1 - \eta$ vanishes at 0 and 1. The interpolation satisfies $\theta^b(\cdot, 0) = \theta_{\mathrm{in}}$, and its pure spatial derivatives at 0 are those of the prescribed initial temperature. At 1 the value and all spatial derivatives agree with θ^0, and every time derivative of positive order of α vanishes. Thus all mixed derivatives agree at the join. Oddness of F and of sine and radiality of the cutoffs give the stated parities and support. The displayed supremum bound follows from $0 \le \eta \le 1$.

For $t \ge 1$, $u^b = \dot\alpha J x$ on the support of θ^0. Since $(R_\alpha e_2)' = \dot\alpha J R_\alpha e_2$, $J^T = -J$ and $Jx \cdot \nabla\chi_0 = 0$, differentiation gives $(\partial_t + u^b \cdot \nabla)\theta^0 = 0$. This proves the first residual formula in (4.10). The second follows from $J\nabla H \cdot \nabla\Delta H = 0$, again by radiality. Its integral is zero because both ΔH and $\partial_1\theta^b$ are derivatives of compactly supported functions. Lemma 6.2, with $R = R_H > 1/2$, gives the support, parity and $C^k \to C^k$ mixed bounds for $\mathcal{R}_c E^b$. These bounds need not be small. Finally $H = |x|^2/2$ near zero and $F'(0) = 1$ give the values at the origin after the interpolation. During the interpolation both sine and F have derivative one at zero, so the convex combination has the same gradient $-e_2$. This proves (9.1) for every preterminal time. □

Lemma 9.2 (Substitution and pressure)**.** *Let θ be odd and ψ even, both smooth and supported in one fixed spatial ball on a preterminal slab. Let $\mathcal{R}_c$ be the fixed compact inverse curl of Lemma 6.2. Set $u = \nabla^\perp\psi$, $\omega = \Delta\psi$ and*

$$f_\theta = \partial_t\theta + u \cdot \nabla\theta, \qquad E = \partial_t\omega + u \cdot \nabla\omega - \partial_1\theta, \qquad f_u = \mathcal{R}_c E.$$

Then E is even with zero spatial integral, f_θ, f_u are odd, and $u = K\omega$. For $Z = \partial_t u + u \cdot \nabla u - \theta e_2 - f_u$, the function

$$p(x,t) = -\int_0^1 Z(rx,t) \cdot x \, dr \tag{9.2}$$

is smooth and even, satisfies $p(0,t) = 0$, and makes (1.1) hold classically.

Proof. Lemma 4.1 gives $\operatorname{div} u = 0$, $u = K\omega$ and $\int \omega = 0$. Integrating the transport divergence and the derivative $\partial_1\theta$ proves $\int E = 0$. The parities follow by differentiation and products, and Lemma 6.2 gives $\operatorname{curl} f_u = E$. For divergence-free planar u, direct component differentiation gives $\operatorname{curl}(u \cdot \nabla u) = u \cdot \nabla\omega$. Consequently $\operatorname{curl} Z = 0$. In components, $\partial_i Z_j = \partial_j Z_i$, whence

$$\partial_i \int_0^1 Z(rx,t) \cdot x \, dr = \int_0^1 \big[r x_j \partial_j Z_i(rx,t) + Z_i(rx,t) \big] \, dr = Z_i(x,t).$$

Thus $\nabla p = -Z$, with the required sign in the momentum equation. Oddness of Z gives evenness of p, and its value at zero is zero by definition. The smooth potential is defined only on preterminal slabs. Its gradient is compactly supported; the pressure itself is not required to be compact or in L^2. □

9.2. **Exact residual differences and finite sums.** Defining increments as differences of the full residuals includes all cross interactions automatically. The local calculation applies because the entire older state is affine on the new support. We then recover the velocity-force increment from the complete vorticity residual, whose spatial integral vanishes, and telescope the finite sums.

For an earlier state $(\theta^{\mathrm{old}}, u^{\mathrm{old}}, \omega^{\mathrm{old}})$ and increment (ϑ, v, ϖ), direct subtraction of the two equations gives

$$\begin{aligned} \Delta f_\theta &= \partial_t\vartheta + u^{\mathrm{old}} \cdot \nabla\vartheta + v \cdot \nabla\theta^{\mathrm{old}} + v \cdot \nabla\vartheta, \\ \Delta E &= \partial_t\varpi + u^{\mathrm{old}} \cdot \nabla\varpi + v \cdot \nabla\omega^{\mathrm{old}} + v \cdot \nabla\varpi - \partial_1\vartheta. \end{aligned} \tag{9.3}$$

On the new support Theorem 8.2 gives $\theta^{\mathrm{old}} = G_{<q} \cdot x$, $u^{\mathrm{old}} = D_{<q}x$ and spatially constant ω^{old}. Hence $v \cdot \nabla\omega^{\mathrm{old}} = 0$ there, and these are precisely (6.4). Outside that support every increment and its derivatives vanish. Thus the identities hold globally, including both inter-layer scalar interactions and the fixed laboratory buoyancy derivative.

Using the corrected physical layers defined in (7.22), put

$$\theta = \theta^b + \sum_{q\geq 1}\theta_q, \qquad \psi = \psi^b + \sum_{q\geq 1}\psi_q, \qquad u = \nabla^\perp\psi, \quad \omega = \Delta\psi. \tag{9.4}$$

The state and force increments here are the same zero-extended increments defined before Theorem 7.3. Set

$$\begin{gathered} f_\theta = f_\theta^b + \sum_{q\geq 1} f_{\theta,q}, \qquad E = E^b + \sum_{q\geq 1} E_q, \\ f_u = \mathcal{R}_c E^b + \sum_{q\geq 1}\mathcal{R}_c E_q. \end{gathered} \tag{9.5}$$

Each complete E_q is spatially mean-zero and even, by Theorem 7.3. The same is true of E^b. Thus the original fixed operator $\mathcal{R}_c$ applies to each summand. This uses spatial mean cancellation of the complete residual, not zero phase mean of an oscillatory profile.

For a fixed $\tau < T_*$ there is q_τ with $t_q^{\mathrm{in}} > \tau$ for $q > q_\tau$. Every sum just displayed is finite on $\mathbb{R}^2 \times [0,\tau]$. Telescoping (9.3) through q_τ therefore identifies f_θ and E with the exact scalar and vorticity residuals of (9.4). Linearity gives $f_u = \mathcal{R}_c E$ on this slab. Every state derivative and every nonlinear product here is a finite smooth expression.

9.3. **Extension of the forces to the blowup time.** Fix a mixed derivative order. All sufficiently late layers satisfy the summable uniform estimate at that order, while the finitely many remaining layers have bounded derivatives of every order. The same split at the next derivative order makes the partial sums uniformly Lipschitz in time, giving their terminal limits. The following lemma records this step and the smooth extension of the force alone.

Lemma 9.3 (Smooth extension of a summable series)**.** *Let $0 < T_* < \infty$. Suppose smooth functions h_q on $\mathbb{R}^2 \times [0, T_*)$ have one fixed compact spatial support, and each fixed h_q has bounded mixed derivatives of all orders. Suppose $k_q \to \infty$ and nonnegative summable numbers ε_q satisfy $\|h_q\|_{C^k_{x,t}} \leq C_k\varepsilon_q$ whenever $k \leq k_q$. Then $\sum_q h_q$ is smooth on the closed slab with the same spatial support. It has a smooth extension to a neighborhood of the closed slab without enlarging that support.*

Proof. For each k, the set $\{q : k_q < k\}$ is finite. Its sum has bounded derivatives of order k, while the remaining derivative series converges uniformly by the summable bound. Doing this for every k identifies the sum and all its derivatives on the half-open slab. To pass to the terminal time at order k, enlarge the finite head to $\{q : k_q < k + 1\}$. The same argument at order $k+1$ gives a finite global bound for the time derivative of every mixed derivative of order at most k of the sum. Thus, uniformly in x, each such derivative is Cauchy as $t \uparrow T_*$. Extend it by that limit. The spatial and temporal fundamental theorems of calculus pass to these uniform limits and identify them with all the derivatives of the extended sum. This proves closed-slab smoothness and preserves the fixed support. Reflection parities are preserved whenever present.

For an extension beyond T_*, write h for the extended sum and let $a_j(x) = \partial_t^j h(x, T_*)$. These are smooth functions with the same compact support. Choose a smooth cutoff $\chi = 1$ near zero and, for $s \geq 0$, use

$$\sum_{j\geq 0}\frac{a_j(x)s^j}{j!}\chi(s/\delta_j).$$

Take $\delta_j \downarrow 0$ so that for $j \geq 1$ the jth term has norm at most 2^{-j} in every mixed order at most $j-1$. This is possible: a time derivative of order $b \leq j-1$ leaves a factor δ_j^{j-b}, and at each j only finitely many spatial derivatives are requested. For each fixed derivative order the tail converges uniformly. Since $\chi = 1$ near zero, each summand has the same derivatives at $s = 0$ as its polynomial factor; the jth time derivative of the sum there is therefore a_j. Joining at $s = 0$ gives the desired extension. The same construction at the initial boundary completes an extension to a neighborhood of the closed slab, always with the original spatial support. $\square$

Proposition 9.4 (Construction from the compact layers)**.** *For the parameters and layers of Theorem 8.2, the state (9.4) and forces (9.5) have the following properties.*

(a) They satisfy (1.1) classically on every $\mathbb{R}^2 \times [0, \tau]$, $\tau < T_*$, with the unit data (1.8), $\operatorname{div} u = 0$, $u = K\omega$ and $\omega = \operatorname{curl} u$.
(b) Both forces are smooth on $\mathbb{R}^2 \times [0, T_*]$, including all mixed derivatives.
(c) The forces are odd and the state has (1.9). One fixed compact ball contains the supports of θ, u, ω at preterminal times and both forces through T_*.
(d) The pressure (9.2) is smooth on every preterminal slab with $p(0, t) = 0$; the pair (θ, u) belongs to the finite-energy, locally Lipschitz class X_τ for every $\tau < T_*$.

Proof. Theorem 3.2 gives $0 < T_* < \infty$, and Theorem 8.2 supplies the nesting over full lifetimes used above. On each closed preterminal slab the sums are finite smooth sums, flat at every activation join. The telescoping argument and Lemma 9.2 prove both classical equations, the vorticity equation with laboratory ∂_1, the pressure gauge and Biot–Savart compatibility. These statements extend to $t = 0$ by the smoothness of all the finite expressions.

It remains to check that the forces retain their regularity at the accumulation time. Write $\mathbf{F}_q = (f_{\theta,q}, \mathcal{R}_c E_q)$, $\mathbf{F}^b = (f_\theta^b, \mathcal{R}_c E^b)$ and use the sum norm for these pairs. For every fixed k, the actual estimate (7.24) gives

$$\|\mathbf{F}^b + \sum_q \mathbf{F}_q\|_{C^k_{x,t}} \leq \|\mathbf{F}^b\|_{C^k_{x,t}} + \sum_{q: k_q < k} \|\mathbf{F}_q\|_{C^k_{x,t}} + \sum_{q \geq 1} \lambda_q^{-1/2} < \infty. \tag{9.6}$$

The layer head is finite by Lemma 7.4; its all-order mixed bounds come from the fixed-layer conclusion of Theorem 7.3. The base pair $\mathbf{F}^b$ is an additional finite-head summand, with all-order bounds from Lemma 9.1. Thus the full finite head consists of the base pair together with these finitely many layer pairs. To see summability directly, $\lambda_q \geq \lambda_1^{2^{q-1}}$ and $2^{q-1} \geq q$ imply $\lambda_q^{-1/2} \leq \lambda_1^{-q/2}$, a geometric majorant. Apply (9.6) also at $k+1$ and then Lemma 9.3 to the base pair, the finite layer head and the summable tail. This proves simultaneous uniform terminal limits for every mixed derivative and hence smoothness of both forces on the closed slab. Only force series and their derivatives have been evaluated at T_*.

The curl residual. Its extension also records the extra derivative needed when recovering E from f_u. Since $E_q = \operatorname{curl} \mathcal{R}_c E_q$,

$$\|E_q\|_{C^k_{x,t}} \leq 2\|\mathcal{R}_c E_q\|_{C^{k+1}_{x,t}} \leq 2\lambda_q^{-1/2} \quad \text{if } k+1 \leq k_q.$$

Thus its finite head at order k is $\{q : k_q < k+1\}$, not $\{q : k_q < k\}$; using $k+2$ for the terminal limit gives the same conclusion. Alternatively the extended residual is $\operatorname{curl} f_u$. The identity and zero spatial integral pass to the terminal time by uniform convergence on the fixed support.

The supports in Theorem 7.3 are $\overline{B}_{1/4}$ for temperature, streamfunction and scalar/curl residuals, and $\overline{B}_{1/2}$ for recovered vector increments. Lemma 9.1 puts the base and its recovered force in $\overline{B}_{R_H}$. Derivatives do not enlarge support. Taking $R_F = R_H + 2$ therefore gives one fixed ball for all the stated fields and forces. On every preterminal slab, θ, u, f_θ, f_u are odd and ψ, ω, E are even. The uniform terminal limits preserve the supports and parities of f_θ, f_u, E. For $a + r \leq k$, a field in that fixed ball has

$$\|\partial_t^a h(t)\|_{W^{r,1}} + \sum_{|\gamma|\le r} \|x^\gamma \partial_t^a h(t)\|_1 \le C'_{k,R_F}\|h\|_{C^k_{x,t}}.$$

Here C'_{k,R_F} depends only on the fixed support radius and k. Consequently both force norms in (7.1) are finite at every order, including $t = T_*$.

All positive layers vanish on $[0,1]$, and $\alpha = 0$ there. The base interpolation therefore gives $\theta(\cdot,0) = \theta_{\rm in}$ with its prescribed sine and $u(\cdot,0) = 0$. Smoothness and fixed support imply the strong L^2 traces. At $t = 0$, $E = -\partial_1\theta_{\rm in}$, so $\operatorname{curl}(\theta_{\rm in}e_2 + f_u) = 0$, consistently with the initial momentum equation $\nabla p = \theta_{\rm in}e_2 + f_u$. Finally, on each fixed preterminal slab the state and all its derivatives are bounded on a fixed compact support. This gives local space-time Lipschitz continuity, finite energy uniformly in time and bounded spatial gradients, exactly the conditions of X_τ. No pressure integrability or terminal state is used. □

10. The two blowup conclusions and uniqueness

The origin identities now give the two different blowup conclusions. The cone condition prevents cancellation of temperature gradients at every sufficiently late time. Vorticity is evaluated only at the ends of the steering intervals, when the newest vorticity amplitude is zero and the older contribution is positive. The first argument gives a full limit; the second gives a limsup.

Recall from Subsection 3.3 that t_n^a is the growth threshold crossing and t_n^b the end of steering. The frequencies are (3.7), and σ_{n-1} is the growth scale of the completed preceding background. The fixed vorticity lower-bound constant c_W is specified in (3.14) and used in (3.19).

Proposition 10.1 (Temperature-gradient and vorticity blowup)**.** *The solution of Proposition 9.4 satisfies all three conclusions in* (1.5)*. In fact there is a fixed $c_* > 0$ such that for every $n \ge 1$,*

$$\|\nabla\theta(t)\|_\infty \ge c_*\lambda_n^{1/8} \quad (t_n^a \le t < T_*), \qquad \|\omega(t_n^b)\|_\infty \ge c_W\sigma_{n-1} \quad (n \ge 2).$$

Proof. The temperature bound in (7.24) and Lemma 9.1 give

$$\sup_{t<T_*}\|\theta(t)\|_\infty \le C_b^\theta + C_\theta\sum_{q\ge1}\lambda_q^{-7/8} < \infty.$$

The series has the geometric majorant used in the assembly proof. After the initial interpolation, the finitely many active affine regions have a common ball around zero at each preterminal time. Their corrections vanish there. The exact core fields and (9.1) give

$$\nabla\theta(0,t) = -\mathsf{e}_0(t) - \sum_j w_jA_j\mathsf{e}_j(t), \qquad \omega(0,t) = 2\dot\alpha(t) + \sum_j w_j\Omega_j(t). \tag{10.1}$$

Both sums are finite at each time. The first formula holds also during the initial interpolation because $\sin'(0) = F'(0) = 1$.

Let $c_{\rm cone} > 0$ be the uniform lower bound for the projection onto $\mathsf{e}_1(t)$ from Theorem 3.2, including the base direction. Since every $w_jA_j \ge 0$, for $t \ge t_n^a$ the origin formula yields

$$\|\nabla\theta(t)\|_\infty \ge -\mathsf{e}_1(t)\cdot\nabla\theta(0,t) \ge c_{\rm cone}w_nA_n(t) \ge c_{\rm cone}c_A\lambda_n^{1/8}.$$

Activation is complete at t_n^a, so $w_n = 1$ on this entire remaining interval. Set $c_* = c_{\rm cone}c_A > 0$, with the already fixed cone and persistent-gradient lower-bound constants. Given any $M > 0$, choose n so that $c_*\lambda_n^{1/8} > M$. The bound then holds for every $t \in [t_n^a, T_*)$, proving $\|\nabla\theta(t)\|_\infty \to \infty$ as $t \uparrow T_*$.

At $t_n^b = t_{n+1}^{\rm in}$, the physical layer $n+1$ vanishes at its activation time, and layers $j > n+1$ have not yet been activated. Thus (10.1) and the vorticity lower bound (3.19) give $\omega(0,t_n^b) \ge c_W\sigma_{n-1}$ for $n \ge 2$. The growth scales tend to infinity and $t_n^b \uparrow T_*$, which proves the vorticity limsup. This vorticity lower bound applies at the steering endpoints t_n^b; between them a growing layer can have oppositely signed vorticity, so the endpoint argument supplies the stated limsup conclusion. □

We now give the larger finite-energy class used in the uniqueness statement and its proof. For $0 < \tau < T_*$ let X_τ be the class of pairs (ϑ, v) locally Lipschitz on $\mathbb{R}^2 \times [0, \tau]$, with $\operatorname{div} v = 0$,

$$\vartheta, v \in L^\infty(0, \tau; L^2(\mathbb{R}^2)), \qquad \nabla\vartheta, \nabla v \in L^\infty(\mathbb{R}^2 \times (0, \tau)).$$

A competing solution in this regularity class must satisfy the scalar equation almost everywhere and the momentum equation in distributions for some distributional pressure, with the same forces. It must attain the prescribed initial values strongly in L^2, meaning

$$\lim_{t \downarrow 0} \Big(\|\vartheta(t) - \theta_{\mathrm{in}}\|_{L^2(\mathbb{R}^2)} + \|v(t) - u_{\mathrm{in}}\|_{L^2(\mathbb{R}^2)} \Big) = 0. \tag{10.2}$$

It remains to prove uniqueness in this class, without assuming spatial integrability of the competitor's pressure. We first derive bounded velocities and justify eliminating the pressure in an energy argument. Continuity in the energy space then permits spatial cutoffs and Grönwall's inequality for the difference of two solutions. All estimates are on a closed time interval strictly before blowup.

Proposition 10.2 (Preterminal uniqueness)**.** *For every $\tau < T_*$ the solution in Proposition 9.4 is unique among pairs $(\vartheta, v) \in X_\tau$ with the same forces and strong L^2 initial traces. Competitors need only be locally Lipschitz on $\mathbb{R}^2 \times [0, \tau]$, divergence-free, bounded in $L^\infty_t L^2_x$ with bounded spatial gradients, and satisfy the scalar equation almost everywhere and the momentum equation distributionally with a distributional pressure.*

Proof. Bounds at every time. Fix $\tau < T_*$, let (ϑ, v) be such a competitor and put $\rho = \vartheta - \theta$, $z = v - u$. Local continuity and Fatou's lemma transfer the essential uniform L^2 bounds to every time. Spatial mollification on interior compact subsets and continuity at the time endpoints transfer the spatial Lipschitz bounds to every time. Testing divergence against a compact spatial function and using time continuity gives $\operatorname{div} v(t) = \operatorname{div} z(t) = 0$ for every $t \in [0, \tau]$. For any such Lipschitz field ϕ, averaging over a unit ball gives

$$|\phi(x)| \le \pi^{-1/2} \|\phi\|_2 + \|\nabla\phi\|_\infty.$$

In particular v, ρ, z are bounded in space-time. These statements require no support assumption on the competitor.

Difference equations and curl-free orthogonality. Subtract the equations, writing $\mathfrak{p}$ for the difference of the distributional pressures:

$$\rho_t = -v \cdot \nabla\rho - z \cdot \nabla\theta, \qquad z_t + Y + \nabla\mathfrak{p} = 0, \qquad Y = v \cdot \nabla z + z \cdot \nabla u - \rho e_2. \tag{10.3}$$

The scalar right-hand side and Y belong to $L^\infty(0, \tau; L^2)$: for example, $\|v \cdot \nabla z\|_2 \le \|v\|_2 \|\nabla z\|_\infty$. Denote a uniform L^2 bound for Y by M_Y. We first remove the pressure by taking the curl, before taking the inner product used in the energy estimate. For every $0 \le s \le t \le \tau$, define the L^2 field

$$G_{s,t} = z(t) - z(s) + \int_s^t Y(r)\, dr. \tag{10.4}$$

Then $\operatorname{curl} G_{s,t} = 0$. Indeed, pair the distributional curl equation with a fixed compact smooth spatial test function. Its pairing with z is continuous up to both time endpoints, and its distributional time derivative is the integrable pairing with $-Y$. The scalar fundamental theorem of calculus therefore gives the identity for every s, t, including the endpoints. The integral in (10.4) is a Bochner L^2 integral; measurability follows from the measurable fields and separability of L^2.

A curl-free L^2 field is orthogonal to every divergence-free L^2 field. For almost every $\xi \neq 0$, their Fourier transforms are respectively parallel and perpendicular to ξ; the distributional curl and divergence identities give these relations as almost-everywhere identities, and Plancherel proves orthogonality. The single point $\xi = 0$ contributes no L^2 mass. *Strong time regularity.* Pairing $G_{s,t}$ with $z(t) - z(s)$ yields

$$\|z(t) - z(s)\|_2^2 = -\left\langle \int_s^t Y(r)\,dr, z(t) - z(s) \right\rangle, \qquad \|z(t) - z(s)\|_2 \le M_Y|t - s|.$$

Thus z is Lipschitz into L^2. For ρ, the pointwise fundamental theorem for its locally Lipschitz dependence on time at each spatial point and the scalar right-hand side in $L_t^\infty L_x^2$ give the same conclusion by Fubini and Minkowski's inequality.

Energy identities. Pairing instead with $z(t) + z(s)$ gives

$$\|z(t)\|_2^2 - \|z(s)\|_2^2 = -\left\langle \int_s^t Y(r)\,dr, z(t) + z(s) \right\rangle.$$

Sum this identity over a partition of $[s, t]$ of mesh Δt. Replacing each endpoint sum by $2z(r)$ produces an error at most $2M_Y^2 \Delta t(t - s)$, by the just proved L^2 Lipschitz bound. Letting $\Delta t \downarrow 0$ gives

$$\|z(t)\|_2^2 - \|z(s)\|_2^2 = -2 \int_s^t \langle Y(r), z(r)\rangle\,dr.$$

The scalar energy identity follows by applying the pointwise fundamental theorem to ρ^2 and then Fubini. Its integrands are in $L^1_{x,t}$, since $\rho, v, z \in L_t^\infty L_x^2$ and the spatial gradients are bounded. No pressure representation, pressure integrability or energy test against the pressure was used.

Transport cancellation and uniqueness. For $W = \rho$ or $W = z$, transport cancels. Let χ_R be a smooth cutoff equal to one on B_R, supported in B_{2R}, with $|\nabla\chi_R| \le C/R$. Testing $\operatorname{div} v = 0$ with a compact smooth approximation to $\chi_R|W|^2/2$ gives

$$\int \chi_R(v \cdot \nabla W) \cdot W = -\frac{1}{2} \int |W|^2 v \cdot \nabla\chi_R.$$

The right side is bounded by $CR^{-1}\|v\|_\infty \|W\|_2^2 \to 0$; the integrand on the left is integrable by the L^2 and gradient bounds. The approximation is justified on its compact support by local Lipschitz continuity. Thus the untruncated transport contribution vanishes. Put $g_1 = \|z\|_2^2$, $g_2 = \|\rho\|_2^2$, $L_u = \|\nabla u\|_\infty$ and $L_\theta = \|\nabla\theta\|_\infty$ on this slab. The two energy identities now imply almost everywhere

$$g_1' \le (2L_u + 1)g_1 + g_2, \qquad g_2' \le L_\theta(g_1 + g_2),$$

and therefore

$$\frac{d}{dt}\big(\|\rho\|_2^2 + \|z\|_2^2\big) \le (2L_u + 1 + L_\theta)\big(\|\rho\|_2^2 + \|z\|_2^2\big). \tag{10.5}$$

In (10.5), both squared norms are absolutely continuous. The coefficient is finite on the fixed preterminal slab, and the common strong initial traces give $g_1(0) + g_2(0) = 0$. Gronwall implies $\rho = z = 0$ in L^2 at every time. Their local continuity gives pointwise equality on the entire closed preterminal slab. □

Proof of Theorem 1.1. Restore the original A_0, λ_0, χ_0 of (1.4). Apply the normalized construction with $\widetilde{\chi}_0(X) = \chi_0(X/\lambda_0)$, choosing the parameters once by Theorem 8.2, and denote its fields, terminal time and support radius by tildes. Theorem 3.2 gives $0 < \widetilde{T}_* < \infty$. Proposition 9.4 proves both classical coefficient-one equations on every closed preterminal slab, the unit initial datum, Biot–Savart compatibility, parity, one fixed compact support ball and the pressure gauge. It also gives both forces smooth through $\widetilde{T}_*$ with every mixed derivative and every norm in (7.1) finite. The constructed pair belongs to each preterminal X_τ and is unique there by Proposition 10.2, with no extra condition on a competitor's pressure or spatial support. Proposition 10.1 gives bounded temperature, the full gradient blowup limit and the vorticity limsup. Now use (1.6) with $\mu = \lambda_0$, $\nu = \sqrt{A_0}$. Lemma 1.2 transfers all these conclusions to the original datum, with $T_* = \widetilde{T}_*/\sqrt{A_0}$ and

$R_F = \widetilde{R}_F/\lambda_0$. Its force-derivative formulas preserve the closed terminal slab, and its bijection of competitor classes proves uniqueness in X_T on each original preterminal slab.

On each such slab the constructed state and all its mixed derivatives are bounded on one fixed compact spatial support by Proposition 9.4. Hence $(\theta, u) \in \mathcal{Y}_T$. Conversely, a competing solution in $\mathcal{Y}_T$ with the same forces and prescribed initial values has bounded spatial gradients and uniform L^2 bounds, from compact scalar support and (1.3). Dominated convergence gives its strong L^2 initial attainment, so it belongs to X_T and the preceding uniqueness result applies. Once θ, u agree, the momentum equation makes the pressure difference spatially constant; $p(0, t) = 0$ makes that difference zero.

Finally, Lemma 9.3 extends both forces to a neighborhood of the closed time interval $[0, T_*]$ with the same spatial support and parity. Multiply these extensions by a smooth time cutoff supported in that neighborhood and equal to one on $[0, T_*]$, and take zero outside. The resulting forces belong to $C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$, are unchanged on $[0, T_*]$, and retain their parity and the same fixed spatial support. Only the forces are extended through T_*; the state, pressure and equations are used only before blowup. □

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