Quiz 9 Rubric

Problem

Let $f_n(x) := \frac{nx^4}{1+nx^4}$.

(a) Show that f_n converges pointwise on $[0, 1]$.

(b) Is it true that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx?$$

Rubric

- The argument for (a) can be very brief as long as it is not wrong.
  - If $x = 0$ is not considered, and there is no argument that we can ignore one point, comment but do not deduct.
  - If (a) is missing or very incorrect, deduct **1 to 2 points**.
- For full credit the student must choose a convergence theorem (likely one of: bounded convergence, monotone convergence, dominated convergence) and justify why it can be used.
  - If the student chooses bounded convergence theorem:
    * Deduct up to **2 points** for not obtaining a bound on $f_n(x)$ which is uniform in n and x.
    * Deduct **0.5 points** for not observing that $[0, 1]$ is bounded.
    * Deduct **0.5 points** for not observing that f_n and the pointwise limit are Riemann integrable.
  - If the student chooses monotone convergence theorem:
    * Deduct up to **2 points** for not showing that the sequence of functions is monotone.
    * Deduct **0.5 points** for not observing that f_n and the pointwise limit are Riemann integrable.
  - If the student chooses dominated convergence theorem:
    * Deduct up to **2 points** for not showing that the sequence is dominated by an integrable function.
    * Deduct **0.5 points** for not observing that f_n and the pointwise limit are Riemann integrable.
  - It is not true that $f_n \to f$ uniformly.

Solution 1

When $x > 0$, we have $nx^4 \to \infty$ as $n \to \infty$, so it is clear that $\frac{nx^4}{1+nx^4} \to 1$ for all $x > 0$. For $x = 0$ we have $f_n(0) = 0$ for all n. So f_n converges pointwise to $f(x) := \begin{cases} 0, & x = 0 \\ 1, & 0 < x \leq 1 \end{cases}$.

Now note that

$$|f_n(x)| \leq 1$$

for all n and $x \in [0, 1]$. Hence f_n is dominated by 1, which is integrable over $[0, 1]$. Moreover both f and f_n are Riemann integrable. Thus by dominated convergence the interchange of limit and integral is justified.

Solution 2

Instead of dominated convergence we can use the bounded convergence theorem, thanks to the fact that $[0, 1]$ is bounded..

Solution 3

We claim $\{f_n\}_n$ is a monotone sequence. Indeed, we claim it is increasing in n, in that

$$\frac{nx^4}{1+nx^4} \overset{?}{\leq} \frac{(n+1)x^4}{1+(n+1)x^4}.$$

Dividing by x^4 and cross-multiplying, we wish to show that

$$n + n^2x^4 + nx^4 \overset{?}{\leq} n + 1 + n^2x^4 + nx^4,$$

which simplifies to $0 \leq 1$, so the claim is proven. Now by the monotone convergence theorem the interchange is justified.

Solution 4

We have that

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{nx^4}{1+nx^4}\,dx = 1 - \int_0^1 \frac{1}{1+nx^4}\,dx.$$

It is clear that $\int_0^1 f(x)\,dx = 1$ where $f_n \to f$ so it suffices to show that $\int_0^1 \frac{1}{1+nx^4}\,dx \to 0$. Indeed,

$$\begin{aligned}
0 \leq \int_0^1 \frac{1}{1+nx^4}\,dx &= \int_0^{\frac{1}{n^{1/8}}} \dots\,dx + \int_{\frac{1}{n^{1/8}}}^1 \dots\,dx \\
&\leq \frac{1}{n^{1/8}} + \int_{\frac{1}{n^{1/8}}}^1 \frac{1}{1+n\left(\frac{1}{n^{1/8}}\right)^4}\,dx \\
&= \frac{1}{n^{1/8}} + \int_{\frac{1}{n^{1/8}}}^1 \frac{1}{1+\sqrt{n}}\,dx \\
&\leq \frac{1}{n^{1/8}} + \int_0^1 \frac{1}{\sqrt{n}}\,dx = \frac{1}{n^{1/8}} + \frac{1}{\sqrt{n}} \to 0.
\end{aligned}$$