NYU PDE Recitation

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Contents

Remarks

These notes are an abbreviated take on what was done in recitation.

Notational conventions:

- f_x is the partial derivative of f with respect to x. I prefer writing $\partial_x f$ or $\frac{\partial f}{\partial x}$, but we decided in recitation to use f_x throughout so we will generally be doing that.
- $u\,|_{\partial\Omega}$ is the restriction of a function u to the set $\partial\Omega$, which is the boundary of the domain $\partial\Omega$. Read this as "u evaluated on the boundary of Ω". This expression comes up when we need to assign boundary conditions to certain PDEs on domains.
- Similarly $u_{\,t=0}$ can be thought of as simply being the function $u(x, 0)$.

1 Change of Variables and Method of Characteristics

1.1 Chain Rule

We recall the chain rule,

$$\frac{d}{dt}f(x(t),y(t),z(t)) = f_x(x(t),y(t),z(t))x'(t)+f_y(x(t),y(t),z(t))y'(t)+f_z(x(t),y(t),z(t))z'(t)$$

for a differentiable function $f = f(x,y,z) : \mathbb{R}^3 \to \mathbb{R}$ and differentiable functions $x, y, z : \mathbb{R} \to \mathbb{R}$. But I don't like using the same letters for dependent variables and independent variables.

1.1.1 A Case Study

Define

$$f(x,y,z) := x + 2y + 3z.$$

We ask two questions:

1. What is $\frac{\partial f}{\partial x}(x,y,x)$?
2. What is $\frac{\partial}{\partial x}f(x,y,x)$?

For the first question, we know that $\frac{\partial f}{\partial x}(x,y,z) = 1$, and when plugging in (x,y,x) into (x,y,z), we get $\frac{\partial f}{\partial x}(x,y,x) = \boxed{1}$.

For the second question, we know that $f(x,y,x) = 4x + 2y$, and now $\frac{\partial}{\partial x}f(x,y,x) = \frac{\partial}{\partial x}(4x+2y) = \boxed{4}$.

The difference arises in the order in which we plug values in and differentiate, so it is important to be very vigilant. Instead I prefer to rewrite the two questions as follows:

1. What is $\frac{\partial f}{\partial x}(s,t,s)$?
2. What is $\frac{\partial}{\partial s}f(s,t,s)$?

The first question is taking the partial of f with respect to the input variable whose name is x, i.e. f_x. *Then* I plug in (s,t,s) into (x,y,z).

The second question is plugging in (s,t,s) into (x,y,z), and *then* I differentiate with respect to s (not x! there are no more x's after I plug stuff in).

Note that it makes no sense to write f_s because f does not "know" who s is. f only knows x, y, and z, which are the input names that I've implicitly defined when I wrote "$f(x, y, z) = x + 2y + 3z$".

One way to explicitly name the variables without writing down a formula for f is to say something like "$f = f(x, y, z)$", i.e. "f is a function of the input variables whose names are x, y, and z".

1.1.2 Quick Example

Let's say we have a function $f = f(x, y)$ of two input variables, defined as $f(x, y) = x+2y$. Let's say that a problem is asking you to find f_θ, where (r, θ) are polar coordinates. Morally speaking, f_θ makes no sense because f doesn't know who θ is. What this is really asking is, if we defined a new function

$$g(r, \theta) := f(r\cos\theta, r\sin\theta),$$

which is basically f but in new coordinates, then what is g_θ? By the chain rule,

$$g_\theta(r, \theta) = f_x(r\cos\theta, r\sin\theta) \cdot (-r\sin\theta) + f_y(r\cos\theta, r\sin\theta) \cdot (r\cos\theta).$$

Note that when taking partials of f, I have to use f_x or f_y, because writing f_θ technically makes no sense.

Anyways, the "answer" to what "f_θ" is in terms of f_x and f_y would be basically $f_x \cdot (-r\sin\theta) + f_y \cdot r\cos\theta$.

In general I vastly prefer using different letters for functions when I'm introducing new coordinates. I will be carrying this philosophy forward because I think this can really reduce confusion.

1.2 Rewriting a PDE in different coordinates

Example 1.1: We have a PDE in $u = u(x, t)$, given by

$$-u_{xx} + au_x + u_t = 0$$

for a constant $a \in \mathbb{R}$. Show that under a new coordinate system (y, s) given by $(x, t) = (y + as, s)$, this PDE transforms into a heat equation (which basically looks like $-u_{yy} + u_s = 0$).

So I'm trying to show that "$-u_{yy} + u_s = 0$", but this makes no sense to me because u doesn't know who y and s are. u only knows x and t. Me, the person using the u, is the one

introducing the y and s in the first place. To wit, what I'm really doing by introducing this new coordinate system is that I'm defining a new function $v = v(y, s)$, which, in terms of u, is given by

$$v(y, s) = u(y + as, s).$$

Then, I'm asking what PDE v solves. In particular, I want to show that $-v_{yy} + v_s$. (Now it makes sense to write v_y because v knows who y is, they're childhood friends!)

By the chain rule:

$$v_y(y, s) = \partial_y u(y + as, s) = u_x(y + as, s) \cdot 1 + u_t(y + as, s) \cdot 0 = u_x(y + as, s)$$

and similarly

$$v_{yy}(y, s) = u_{xx}(y + as, s).$$

(It's good to keep writing the inputs if you're confused, because just writing "$v_{yy} = u_{xx}$" looks kinda weird and misleading. But if you're tryna save time it can be ok.)

For the s partial,

$$v_s(y, s) = u_x(y + as) \cdot a + u_t(y + as, s) \cdot 1.$$

Therefore,

$$-v_{yy}(y, s) + v_s(y, s) = -u_{xx}(y + as, s) + au_x(y + as, s) + u_t(y + as, s) = 0.$$

Tada!

1.2.1 What if we don't know the answer ahead of time?

Example 1.2: We have a PDE in $u = u(x, t)$, given by

$$-u_{xx} + au_x + u_t = 0$$

for a constant $a \in \mathbb{R}$. Transform the PDE under the change of coordinates $(x, t) = (y + as, s)$.

This is the exact same problem, but oh no, they didn't tell me the answer, so it's a bit harder. Briefly speaking, what you'd do here instead is, instead of writing v in terms of u like $v(y, s) = u(y + as, s)$, we'd go the other way, writing u in terms of v, like

$$u(x, t) = v(x - at, t).$$

(Why is this true and how did I come up with it?) Now I can directly take partials of u and see what they come out to in terms of v.

$$-u_{xx}(x, t) = -v_{yy}(x - at, t)$$

$$au_x(x,t) = av_y(x-at,t)$$
$$u_t(x,t) = -av_y(x-at,t) + v_s(x-at,t)$$

Adding these equations up gets me what I got before.

Generally this approach requires no guesswork or hope, but it can be a bit messier (e.g. trying to invert polar coordinates is slightly painful).

1.3 Method of Characteristics

The whole theory of the *method of characteristics* is motivated by the *transport equation*:

$$\begin{cases} u_t + c \cdot u_x = 0, \\ u(x,0) = f(x) \end{cases}$$

The resolution of many a PDE is obtained by hopeful guessing. The story here is that for this PDE, we made the hopeful guess that there are certain lines/curves in the space-time continuum, called *characteristics*, over which u is constant. Then, to find the value of u at some (x,t), we *follow the characteristic* back to the "base line" back at $t = 0$, where we know what u is.

Step 1

Let's follow this story. We guess that, starting from a point on the "base line" $(x_0, 0)$, there is a curve $(X_{x_0}(s), T_{x_0}(s))$ over which u is constant, meaning as I go along the curve (i.e. vary s), the value of u does not change:

$$u(X_{x_0}(s), T_{x_0}(s)) \stackrel{?}{=} \text{Constant in } s.$$

For this hope to come true, the derivative in s has to be zero, which by the chain rule is:

$$u_x(X_{x_0}(s), T_{x_0}(s))X'_{x_0}(s) + u_t(X_{x_0}(s), T_{x_0}(s))T'_{x_0}(s) \stackrel{?}{=} 0.$$

Step 2

Can we choose a good X_{x_0} and T_{x_0} to make the above equality true? Well it almost looks like the original PDE, $cu_x + u_t = 0$, and it would match that PDE exactly if we knew that:

$$X'_{x_0}(s) = c$$

and

$$T'_{x_0}(s) = 1.$$

These are two ODE's. But to solve ODEs I need initial conditions. But remember, I am starting at $(x_0, 0)$, and we can say that the "start" is when $s = 0$. So the initial conditions are given by $(X_{x_0}(0), T_{x_0}(0)) = (x_0, 0)$. This gives us two ODEs:

$$\begin{cases} X'_{x_0}(s) = c \\ X_{x_0}(0) = x_0 \end{cases}$$

and:

$$\begin{cases} T'_{x_0}(s) = 1 \\ T_{x_0}(0) = 0 \end{cases}$$

Solving gives $X_{x_0}(s) = cs + x_0$ and $T_{x_0}(s) = s$.

This means that the characteristic $(X_{x_0}(s), T_{x_0}(s)) = (cs + x_0, s)$ charts out a *line* in space-time.

Step 3

Why did we just do that? Remember that u is constant along the characteristic, so

$$u(cs + x_0, s) = \text{Constant in } s.$$

Now we are ready to move x_0 around, meaning that we are now shooting many different characteristics. To make the above equation less misleading we now write it as

$$u(cs + x_0, s) = g(x_0)$$

for some function g of x_0. In English, this equation says, "The value of u over the characteristic that starts at x_0 depends *only* on x_0."

What is g? Well, now we *go back to the base line* by setting $s = 0$, giving us

$$u(x_0, 0) = g(x_0)$$

for all x_0. But wait, $u(x_0, 0)$ is just the initial data $f(x_0)$, so $f = g$. That makes a lot of sense! In English: "The value of u at a point is the value of f at the part of the base line that we traced back the characteristic line to."

So $u(cs + x_0, s) = f(x_0)$ for all x_0 and s. But we wanted $u(x, t)$. How can we write $f(x_0)$ in terms of (x, t), where $x = cs + x_0$ and $t = s$? Well, we can solve for x_0 in terms of x and t as $x_0 = x - ct$, giving us the final answer,

$$\boxed{u(x, t) = f(x - ct).}$$

So that's the story of how the method of characteristics began. In later problems it may be a bit more complicated and hopefully we will be doing more of that in the next recitation.

2 Method of Characteristics

2.1 The Recipe

Let's summarize the method! Let's assume that the PDE problem looks like

$$\begin{cases} A(x,y)\cdot u_x + B(x,y)\cdot u_y = 0 \\ u = f \text{ on } S \end{cases}.$$

Here, "$A(x,y)\cdot u_x + B(x,y)\cdot u_y = 0$" is the PDE part of the problem, for some expressions A and B. For example, in the transport equation, $A = 1$ and $B = c$ (if we were to think of x as time and y as space).

The "$u = f$ on S" is the *initial data* part of the problem, where S is some curve or line in the plane and f is a given function whose domain is S. For example, in the transport equation, S is one of the axes, i.e. the x axis $\{(x,0) : x \in \mathbb{R}\}$ or the y axis $\{(0,y) : y \in \mathbb{R}\}$, and the "initial data" then can be written more simply as something like "$u(x,0) = f(x)$". I like to think of S as the "base line", where all the characteristics begin. **For this recipe, let's assume for now that S is actually an axis, say, the x axis,** and I'll talk about *later* what changes we'd need to make if it wasn't that simple.

Some problems won't give you initial data and instead ask you to find a *general solution* to the PDE part. When that happens, we'll have to invent our own initial data and our own "base line" S. We'll do an example like that.

Instead of u_x and u_y you might see u_x and u_t, in which case you'll have to change some variable names in the recipe.

Step 1: Write down the "hope".

Our hope is that there is a curve or line $(X_{x_0}(t), Y_{x_0}(t))$ over which u is constant. The x_0 subscript indicates that this curve starts at $(x_0, 0)$ at $t = 0$. (Remember that we're assuming the "base line" is the x axis, for now.)

Step 2: Use the chain rule and compare with the original PDE to get a condition that can make our hope come true.

Our hope is true if

$$\frac{d}{dt}u(X_{x_0}(t), Y_{x_0}(t)) \stackrel{?}{=} 0.$$

By the chain rule, this is equivalent to

$$X'_{x_0}(t)u_x(X_{x_0}(t), Y_{x_0}(t)) + Y'_{x_0}(t)u_y((X_{x_0}(t), Y_{x_0}(t))) \stackrel{?}{=} 0.$$

Our original PDE said that

$$A(x,y)\cdot u_x(x,y)+B(x,y)\cdot u_y(x,y)=0.$$

So by comparing the above two equations, our hope will become true if $X'_{x_0}(t)=A(X_{x_0}(t),Y_{x_0}(t))$ and $Y'_{x_0}(t)=B(X_{x_0}(t),Y_{x_0}(t))$.

Step 3: Deduce a system of ODEs for the X and Y, and solve it.

Combining the fact that X_{x_0} and Y_{x_0} start at $(x_0,0)$ with what we got in the previous step, we get this system of ODEs:

$$\begin{cases} X_{x_0}(0)=x_0 \\ Y_{x_0}(0)=0 \\ X'_{x_0}(t)=A(X_{x_0}(t),Y_{x_0}(t)) \\ Y'_{x_0}(t)=B(X_{x_0}(t),Y_{x_0}(t)) \end{cases}$$

Often enough, if this method is supposed work, these ODEs will be simple. For example, in the transport equation, we end up with like $X'_{x_0}(t)=1$ and $Y'_{x_0}(t)=c$, which are super easy to solve.

Step 4: Reel points back to the "base line" to finish.

Now, to find $u(x,y)$, I have to figure out which $(x_0,0)$ I have to start at for the characteristic $(X_{x_0}(t),Y_{x_0}(t))$ to pass through (x,y). Since u is constant along this characteristic, this will let me write u in terms of the initial data f. Voila!

More formally, we have

$$u(X_{x_0}(t),Y_{x_0}(t))=u(X_{x_0}(0),Y_{x_0}(0))=u(x_0,0)=f(x_0),$$

and now to get this into $u(x,y)=f(\ldots)$ form, we set $x=X_{x_0}(t)$ and $y=Y_{x_0}(t)$ and solve for x_0.

Step 5 (Optional): Review what you've done and make sure it makes sense.

For example you can plug in your solution into the original PDE to make sure it works, or plot the characteristics to make sure they don't intersect weirdly or something. If something weird happens, that's a sign that you should check your work!

2.2 Examples

Example 2.1: Solve the PDE

$$\begin{cases} yu_x + u_y = 0, \\ u(x,0) = x^{9001}. \end{cases}$$

Solution. Follow the recipe!

Step 1: We hope that $u(X_{x_0}(t), Y_{x_0}(t))$ is constant as t varies, for a characteristic $(X_{x_0}(t), Y_{x_0}(t))$ starting at $(x_0, 0)$.

Step 2: Differentiating,

$$X'_{x_0}(t)u_x((X_{x_0}(t), Y_{x_0}(t))) + Y'_{x_0}u_y((X_{x_0}(t), Y_{x_0}(t))) = 0,$$

so looking back at $yu_x + u_y = 0$, we now want $X'_{x_0}(t) = Y_{x_0}(t)$ and $Y'_{x_0}(t) = 1$. *(It's not "$X'_{x_0}(t) = y$", because we replaced the x and y with X and Y.)*

Step 3: Since the ODE for X has Y in it, we'll solve the ODE for Y first, which is

$$\begin{cases} Y'_{x_0}(t) = 1 \\ Y_{x_0}(0) = 0 \end{cases}.$$

This solves as $Y_{x_0}(t) = t$. Now the ODE for X is

$$\begin{cases} X'_{x_0}(t) = t \\ X_{x_0}(0) = x_0 \end{cases}.$$

This solves as $X_{x_0}(t) = \frac{1}{2}t^2 + x_0$.

Step 4: So

$$u(\frac{1}{2}t^2 + x_0, t)$$

is constant for every x_0. Particularly it only depends on which characteristic we're on, i.e. $u(\frac{1}{2}t^2 + x_0, t) = u(x_0, 0) = x_0^{9001}$.

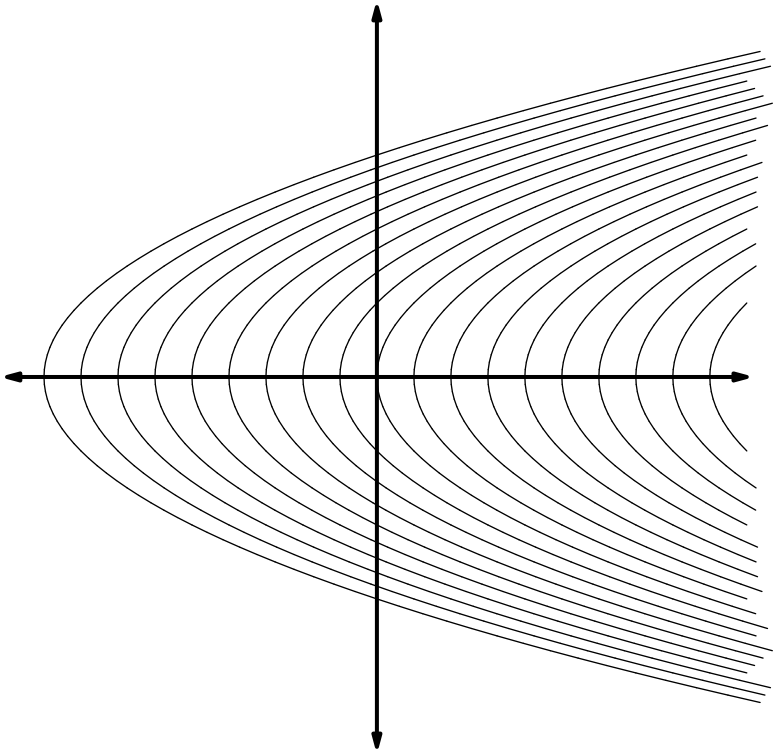
Now let's get this into $u(x, y)$ form, so we set

$$\begin{cases} x = \frac{1}{2}t^2 + x_0, \\ y = t \end{cases}$$

and solve for x_0 in terms of x and y.

Fortunately this isn't so bad — It's just $x_0 = x - \frac{1}{2}t^2 = x - \frac{1}{2}y^2$. Thus our solution is $\boxed{u(x,y) = \left(x - \frac{1}{2}y^2\right)^{9001}}$.

Step 5: Let's graph the characteristics because why not. Let's first take a simple one, like the one associated with $x_0 = 0$. This characteristic is parametrized as $(\frac{1}{2}t^2, t)$, which can be written as $x = \frac{1}{2}y^2$. This is a *parabola* that opens rightward. Other values of x_0 will also give parabolas that open up rightward, so we expect that the characteristics are "horizontally stacked parabolas that open up rightward". I'll let you check that this is true.



■

Remarks: What if the initial data was along the y-axis instead? If instead the initial data was $u(0,y) = y^{9001}$, then there are NO solutions, whereas if the initial data was $u(0,y) = y^{420}$, then there are actually infinitely many solutions. Using the discussion thus far, determine why this is the case!

2.3 Complications: When no initial data given

What if we were to try and solve the above PDE without the initial data, i.e. just solving

$$yu_x + u_y = 0$$

and finding the *general* solution to this PDE? In short, we will have to figure out our own "base line" to start the characteristics. (We can't just choose any line though! By the remark, taking the y-axis will end up being disastrous! So we need to figure out how to pick one that doesn't suck.)

To do this, we'll forgo trying to set the starting point for our characteristics and instead figure out what they look like generally. To wit, we make the hope that $u(X(t), Y(t))$ is constant (no subscript this time), and differentiate to find these ODEs:

$$\begin{cases} X'(t) = Y(t) \\ Y(t) = 1 \end{cases}$$

We no longer have initial conditions for these ODEs because we haven't assigned starting points for the characteristics. Anyways, we find that $Y(t) = t + c_2$ for a constant c_2 and $X(t) = \frac{1}{2}t^2 + c_2 t + c_1$ for a constant c_1.

Each choice of c_1 and c_2 will give a characteristic, but different choices won't necessarily give different characteristics. In fact we should expect that a single characteristic is given by many many possible (c_1, c_2) choices (why?). Anyways, how can we now pick a good "base line" to start characteristics from?

One way is to plot a bunch of characteristics to give us a very good idea. For example, you can write $t = Y(t) - c_2$ and plug it into the equation for $X(t)$ to get a relationship between $X(t)$ and $Y(t)$, which you can plot and/or analyze as-is (once you replace $X(t)$ with x and $Y(t)$ with y). Or you can just pull up Desmos. Either way, you'll end up getting the picture on the previous page. Now we want to pick a line that intersects every characteristic exactly once. A great choice for that is the x-axis! (We also see why the y-axis is a poor choice: It doesn't intersect all the characteristics, and it intersects some of them twice!)

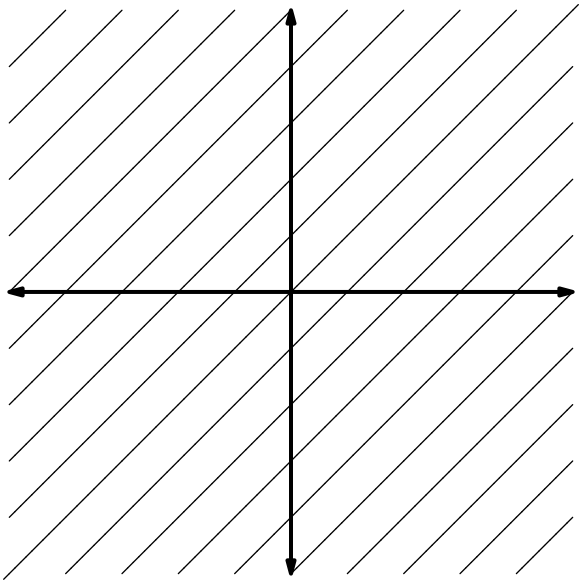
(If we do the $t = Y(t) - c_2$ substitution mentioned in the previous paragraph, we'd get $X(t) = \frac{1}{2}(Y(t) - c_2)^2 + c_2(Y(t) - c_2) + c_1 = \frac{1}{2}Y(t)^2 + (-c_2^2/2 + c_1)$, so the characteristics should look like $x = \frac{1}{2}y^2 + c$ for a constant c.)

If you can think of a more concrete way do let me know, though I suspect there might not be a very good one.

2.4 More Examples

Example 2.2: Solve $u_x + u_y = 1$.

Solution. If there are characteristics $(X(t), Y(t))$ then they'll have to satisfy $X'(t) = 1$ and $Y'(t) = 1$. This solves as $X(t) = t + c_1$ and $Y(t) = t + c_2$. What do the characteristics look like for varying choices of c_1 and c_2? With some thought, you'll find that they're just diagonal lines going northeast:



Time to choose a base line! What's a line or curve that intersects every characteristic exactly once? Hm, both the x-axis and y-axis work fine. Let's be quirky and choose the y-axis.

Now we can more or less run the recipe. We'll invent our own initial data: A function $f(y)$ that specifies what u is on the y-axis. Then the characteristic that starts at $(0, y_0)$ is given by $(t, t + y_0)$. So $u(t, t + y_0)$ is constant and is equal to $u(0, y_0)$ which we've defined to be $f(y_0)$.

So $u(t, t + y_0) = f(y_0)$ for all t and y_0. Setting $x = t$, $y = t + y_0$ and solving for y_0, we get $y_0 = y - x$. So our general solution is $\boxed{u(x, y) = f(y - x)}$ for any differentiable function f. ■

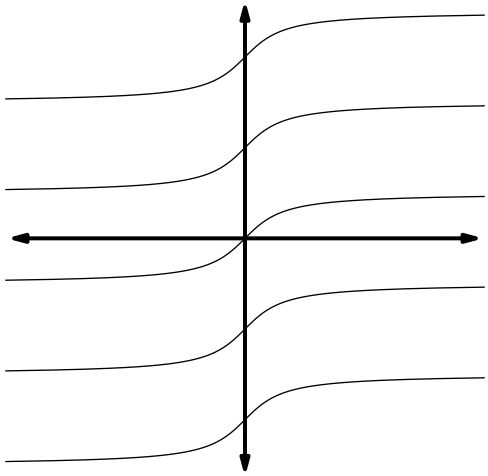
Exercises: Check that this indeed satisfies the PDE. Also, try picking a different base line, like the x-axis, or even the line $\{y + x = 0\}$.

Example 2.3: Find the general solution of the PDE

$$(1 + x^2)u_x + u_y = 0.$$

Solution. The characteristics $(X(t), Y(t))$ will satisfy $X'(t) = 1 + X(t)^2$ and $Y'(t) = 1$. So $Y(t) = t + c_2$ and $X(t) = \tan(t + c_1)$ for constants c_1 and c_2.

Hold on now — if we graph just *one* characteristic, we'll have multiple branches because tan is discontinuous...



What does this mean for our recipe?

Basically, each of these branches will be its own characteristic. This is because at a point of discontinuity in the characteristic $(\tan(t + c_1), t + c_2)$, the value of u is allowed to jump while still allowing the derivative in t to be 0. (As an analogy, $f'(x) = 0$ usually implies that f is constant, but if we poke a hole in the domain so that it's now $(-\infty, 0) \cup (0, \infty)$, then f can "jump" at 0 while still having derivative 0 wherever it's defined.)

In short, we see that a good "base line" is the y-axis and the characteristics are given by, for each choice of y_0,

$$(\tan t, t + y_0), \qquad t \in (-\pi/2, \pi/2).$$

(By restricting t to an interval we've selected one branch of tan. Note that we're not leaving out anyone by forgetting about the other branches, since we get those branches back anyway by taking different values of y_0.)

So, $u(\tan t, t + y_0) = f(y_0)$ for any f we want, and now writing this in $u(x, y)$ form we get $\boxed{u(x, y) = f(y - \tan^{-1} x)}$. ■

There is another possible complication that may occur but you won't need to deal with it in this week's homework. I hope you find these examples helpful! If you're getting stuck or something weird is happening, remember to do Step 5 of the recipe, and also do note that this method isn't necessarily that straightforward and you may have to think on your feet if a curveball gets thrown at ya. (There may or may not be at least one curveball on the homework...)3

3 More Characteristics and Energy

3.1 Characteristics and stuff

We did more stuff with Method of Characteristics. Here's one example we did (slightly modified).

Example 3.1: Find the general solution to $xu_x+yu_y = u$ over $(x, y) \in \mathbb{R}^2\backslash\{(0, 0)\}$.

Solution. The right side is no longer $= 0$, it's $= u$, so along the characteristics we no longer can really expect u to be constant. Instead it will change as a function that we'll call $f(s)$:

$$f(s) = u(X(s), Y(s))$$

Differentiating both sides,

$$f'(s) = u_x(X(s), Y(s))X'(s) + u_y(X(s), Y(s))Y'(s). \qquad (*)$$

We want to get this to match the original PDE, so we'll pick $X(s)$ and $Y(s)$ so that they satisfy the following:

$$\begin{cases} X'(s) = X(s) \\ Y'(s) = Y(s) \end{cases}$$

(If this happens, then $f'(s) = X(s)u_x + Y(s)u_y = u(X(s), Y(s)) = f(s)$, giving us an ODE for f! We'll come back to this.)

No initial conditions for these ODEs because the original problem didn't give us any. So, solving these we get $X(s) = c_1e^s$ and $Y(s) = c_2e^s$ for some constants c_1 and c_2.

Before we proceed let's figure out what this does for us. If we go back to $(*)$

What this means is that our characteristics are of the form (c_1e^s, c_2e^s), which, for each choice of c_1 and c_2, look like lines radiating from the origin (check this for yourself!).

Now to choose a good "base line" to set our own initial conditions: A good one to choose is the circle $x^2 + y^2 = 1$ since all the characteristic lines intersect it exactly once. Let's say that our new PDE with "initial conditions" is now,

$$\begin{cases} xu_x + yu_x = u \\ u = g \text{ on the circle } x^2 + y^2 = 1 \end{cases}$$

for a function g taking values on the circle $x^2 + y^2 = 1$.

Let's look at the characteristic line that goes through a starting point (x_0, y_0) on this circle. To find it, we pick the c_1 and c_2 so that $X(0) = x_0$ and $Y(0) = y_0$. This gives us

$$X(s) = x_0e^s$$

and

$$Y(s) = y_0 e^s.$$

So the characteristic line is $(x_0 e^s, y_0 e^s)$.

Now what? Well the whole point of finding a characteristic line is that u should be nice on it. Going back to equation $(*)$ with the $X(s)$ and $Y(s)$ we found, and remembering that $X' = X$ and $Y' = Y$, gives us:

$$\begin{aligned} f'(s) &= X'(s)u_x + Y'(s)u_y && \text{(From the chain rule we did before)} \\ &= X(s)u_x + Y(s)u_y \\ &\quad \text{(Because we chose } X \text{ and } Y \text{ so that } X' \text{ and } Y' \text{ match the original PDE)} \\ &= u(X(s), Y(s)) && \text{(From the original PDE)} \\ &= f(s) && \text{(By definition of } f(s)) \end{aligned}$$

So $f(s) = ce^s$ for a constant c. We can find this constant: Plugging in $s = 0$ gives $f(0) = c$, and we know that $f(0) = u(X(0), Y(0)) = g(x_0, y_0)$, which "known" because it's part of the "initial condition" that we invented. So therefore $f(s) = g(x_0, y_0)e^s$.

Now to wrap up,

$$u(x_0 e^s, y_0 e^s) = f(s) = g(x_0, y_0)e^s.$$

To simplify things let $t = e^s$ $(t > 0)$ to turn this into

$$u(x_0 t, y_0 t) = g(x_0, y_0)t.$$

Now we get this into $u(x, y)$ form. If $x = x_0 t$ and $y = y_0 t$, we want to solve for x_0, y_0, and t all in terms of x and y. But we should remember that $x_0^2 + y_0^2 = 1$.

All that's left is algebraic parlor tricks so it's ok if this seems like black magic. If we square each equation,

$$x^2 = x_0^2 t^2$$
$$y^2 = y_0^2 t^2$$

then add them up,

$$x^2 + y^2 = (x_0^2 + y_0^2)t^2$$

we can apply the $x_0^2 + y_0^2 = 1$ condition to get $x^2 + y^2 = t^2$, so $t = \sqrt{x^2 + y^2}$. Now $x = x_0 t = x_0\sqrt{x^2 + y^2}$ so that $x_0 = \frac{x}{\sqrt{x^2+y^2}}$, and similarly $y_0 = \frac{y}{\sqrt{x^2+y^2}}$. So our final solution is

$$\boxed{u(x, y) = g\left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right)\sqrt{x^2 + y^2}.}$$

(Do you see now why I decided to exclude $(0, 0)$ from the domain in the original problem?) ■

We did more examples but these notes are supposed to be abbreviated.

3.2 Energy

We did not have time for this but here's what I would have said.

3.2.1 What is energy, anyway?

It's just some integral quantity that behaves nicely.

3.2.2 Huh?

See it's just a lot better if you saw examples since in my view it's more of a methodology than a formally defined property.

3.2.3 But let's talk about integration by parts first.

You probably know that

$$\int_a^b u'v\,dx = uv\,|_{x=a}^b - \int_a^b uv'.$$

If for some reason u and/or v happen to be zero at the endpoints $x = a$ and $x = b$ (which is pretty common!), then this just becomes

$$\int_a^b u'v\,dx = -\int_a^b uv'\,dx.$$

I'll write this in two other different ways for fun.

$$\int_a^b u_x v\,dx = -\int_a^b uv_x\,dx$$

$$\int_a^b \partial_x uv\,dx = -\int_a^b u\partial_x v\,dx$$

Motto #1: Integrationg by parts lets you move derivatives from one term to the other!

Here's an important identity we're going to be using a lot, when in higher dimensions:

$$\int u\Delta v = -\int \nabla u \cdot \nabla v \text{ (If } u \text{ decays to 0)}$$

The reason why this is true is because in the ith dimension,

$$\int uv_{ii} = -\int u_i v_i,$$

and now if you add this up over all dimensions i, you get the important identity.

3.2.4 Energy for the Wave Equation

The wave equation is

$$u_{tt} - c^2 \Delta u = 0.$$

To get an "energy result", you follow these steps:

1. Multiply each side by something (usually u_t, sometimes u).
2. Integrate over space.
3. Use integration by parts and chain rule until every term looks like $\int |\text{something}|^2$, or maybe $\int \partial_t |\text{something}|^2$.

Ok, let's multiply by u_t.

$$u_t u_{tt} - c^2 u_t \Delta u = 0$$

Now integrate over space. Let's say space is $\mathbb{R}^2$. Doesn't really matter though.

$$\int_{\mathbb{R}^2} u_t u_{tt} - c^2 \int_{\mathbb{R}^2} u_t \Delta u = 0$$

Now we need each integrand to be nice.

- The $u_t u_{tt}$ can be written as $\frac{1}{2}\partial_t |u_t|^2$ by the chain rule, so that's fine. (This trick is VERY IMPORTANT.)
- $u_t \Delta u$ is not so nice. Let's use integration by parts, particularly that **important identity**, to move derivatives around:

  $$-c^2 \int_{\mathbb{R}^2} u_t \Delta u = c^2 \int_{\mathbb{R}^2} \nabla u_t \cdot \nabla u$$

  (If a minus sign goes away, that's a VERY GOOD SIGN!)

  The new integrand is $\nabla u_t \cdot \nabla u$. This still isn't that nice, but if we write it as $(\nabla u)_t \cdot (\nabla u)$, it now looks very similar to the other term we did, which was $(u_t)_t \cdot u_t$. So we can actually write this as

  $$(\nabla u)_t \cdot (\nabla u) = \frac{1}{2}\partial_t |\nabla u|^2.$$

  That looks good, yay!

So therefore,

$$\boxed{\int_{\mathbb{R}^2} \partial_t \frac{1}{2}|u_t|^2 + \frac{c^2}{2}\int_{\mathbb{R}^2} \partial_t |\nabla u|^2 = 0}.$$

This is our energy result, which is essentially telling us that if we call this expression

$$E(t) := \int_{\mathbb{R}^2} \frac{1}{2}|u_t|^2 + \frac{c^2}{2}\int_{\mathbb{R}^2} |\nabla u|^2$$

the "energy", then energy stays constant over time!

(Caveat: I technically made a mistake in the above derivation: You can only apply the important identity if u goes to 0 as you shoot off to infinity. The "energy" expression we get from making this generous assumption is very important nevertheless!)

3.2.5 Energy gives Uniqueness

Now for my next motto.

Motto #2: Energy results give uniqueness results

A *uniqueness result* basically means that if you know what u is like at the start of time $t = 0$, then you can determine what it does for all time. In other words, its behavior over time is unique.

More generally, uniqueness results take the following form: "If we knew that $u = v$ at some time and in some region, then $u = v$ at [some later time] at [some point]."

The recipe for proving a uniqueness result is as follows:

1. Get a result about the energy.
2. Using that result, demonstrate that if the energy within [some region] at [the earlier time] is zero, then the energy at [the later time] must also be zero.
3. Therefore, if a solution is zero in [some region] at [the earlier time], then it's still zero at [the later time].
4. Conclude that if $u - v = 0$ in [some region] at [the earlier time], then it's still still zero at [the later time].

Here's an example using the energy for the wave equation.

Example 3.2: Show that the solution to the PDE

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

is unique *(over all solutions that decay to zero at spatial infinity, but that's a technicality that's just there to make my point easier to make.)*

Solution. Let's remember the energy result we obtained.

$$\text{The energy } E(t) = \int_{\mathbb{R}^2} \frac{1}{2}|u_t|^2 + \int_{\mathbb{R}^2} \frac{c^2}{2}|\nabla u|^2 \text{ is constant in time.}$$

Step one done!

Step two: Let's prove that the solution to the "zero problem" is unique:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

The solution $w \equiv 0$ is definitely a solution, but I want to show that nothing else is possible. We can use the energy to prove this:

- At time $t = 0$, the energy of w, i.e. $\int_{\mathbb{R}^2} \frac{1}{2}|w_t|^2 + \int_{\mathbb{R}^2} \frac{c^2}{2}|\nabla w|^2$, is zero.
- But according to our energy result, the energy is constant.
- So the energy of w is always zero.
- The energy of w could only be zero if it's constant.
- So the initial conditions for w mean that w must be zero everywhere for all time!

Tada.

Last step(s): Let's suppose that u and v are two solutions to the PDE. Then they must have the same initial conditions, so $u = v$ and $u_t = v_t$ at time $t = 0$. That means that if we set $w = u - v$, then w satisfies the "zero PDE", meaning that w must always be zero. So $u - v = 0$ always. So $u = v$ everywhere, always! ■

3.2.6 Energy for the Beam Equation

Here's a very silly example just so you get more input on what energy looks like. The *beam equation* is given by

$$\begin{cases} u_{tt} + u_{xxxx} = 0 \\ u(x,0) = f(s) \\ u_t(x,0) = g(s) \\ u \text{ and } u_t \text{ are zero at the endpoints } x = 0, 1 \end{cases}$$

over the domain $x \in (0,1)$. (Yes, that's four derivatives.)

Example 3.3: Show that the solution to the beam equation (if there is one...) is unique.

Solution. Following the recipe, multiply by u_t to get

$$u_t u_{tt} + u_t u_{xxxx} = 0.$$

Integrate in space to get

$$\int_0^1 u_t u_{tt}\, dx + \int_0^1 u_t u_{xxxx}\, dx = 0.$$

Now we fix each term to make it square.

- $u_t u_{tt}$ is the time derivative of $\frac{1}{2}|u_t|^2$, so that's fine.
- $u_t u_{xxxx}$ is not so nice, but we can move some derivatives over using integration by parts *(note that the zero boundary conditions let us do this!)*, giving us

$$\begin{aligned} \int_0^1 u_t u_{xxxx}\, dx &= -\int_0^1 u_{xt} u_{xxx}\, dx && \text{(Integrated by parts)} \\ &= \int_0^1 u_{xxt} u_{xx}\, dx && \text{(Integrated by parts again!)} \end{aligned}$$

  Oh look now we can write this new integrand as

$$u_{xxt} u_{xx} = \partial_t |u_{xx}|^2.$$

So we have the energy result

$$\int_0^1 \partial_t |u_t|^2\, dx + \int_0^1 \partial_t |u_{xx}|^2\, dx = 0.$$

The energy result tells us that if we set

$$\boxed{E(t) := \int_0^1 |u_t|^2 \, dx + \int_0^1 |u_{xx}|^2 \, dx}$$

as the energy, then this is constant in time.

How does this give uniqueness? Well, let me repeat the argument but written with a slightly different rhythm:

- If u and v are two solutions, then they have the same initial conditions.
- So $u - v$ is also a solution, but with initial conditions being zero instead.
- So the energy of $u - v$ at time $t = 0$ is zero.
- By the energy result, we find that the energy of $u - v$ at all later times is zero.
- So $u - v$ has to be constant and thus zero at all times *(I skipped some reasoning here, convince yourself this is true).*
- So $u = v$ everywhere, always.

■

3.2.7 Energy for the Heat Equation

The heat equation is

$$u_t - c\Delta u = 0.$$

I'll skip the derivation for brevity, but essentially if you multiply each side by u and do some (comparatively easy) math, you deduce that

$$\boxed{\int \partial_t |u|^2 + c \int |\nabla u|^2 = 0}$$

where the integrals are over space. If we toss out the ∇u term, we get an *inequality*:

$$\int \partial_t |u|^2 \leq 0$$

This is an energy result: The "mass" part of the energy, $\int |u|^2$, can only decrease in time *(...provided that u decays to 0 at infinity / at the boundary of the domain over which we're integrating. small caveat so i don't get sued!)*

This gives a uniqueness result!

- If $u = v$ at the start of time, then $u - v$ has zero energy.
- The energy result says energy can only *decrease*.
- But energy can't go below zero.
- So since the energy of $u - v$ starts at zero, it must stay at zero!
- So you can deduce that $u = v$ everywhere, always.

3.2.8 Back to the Wave Equation: Causality

Now that we know a bit more about what energy looks like, and what it can do, let's try and prove a more "advanced" uniqueness property for the wave equation: the *causality principle*.

Well, actually you've already seen the proof in lecture, so to save myself some time and not upset my advisor I'll just include some notes on what exactly the proof is doing.

Chiefly, the proof is using an *energy result*, comparing the energies of $u(x, t)$ at two different time slices of a *cone*.

The "sketch" of the proof is actually not that complicated after all that energy discussion. Let's assume that space has two dimensions, so that spacetime has three dimensions (x, y, t).

- We start with an upright cone in spacetime with vertex (x_0, y_0, t_0).
- (What that means is that the bottom of the cone is the beginning of time, and the time-cross-section of the cone shrinks to a point as time moves up towards t_0.)
- I claim that the energy in this cone('s cross section) can only go down with time.
- Let's pick two slices of the cone: Call them the "earlier time" and the "later time". The "earlier time" is a bigger slice of the cone.
- Then the claim is that

$$\text{Energy(later time)} \leq \text{Energy(earlier time)}.$$

- *Proving that is the messy part! But this is our* ***energy result.***
- Now we want to prove that if $u = v$ in the cone at the earlier time, then $u = v$ in the cone at the later time.
- Well, the energy of $u - v$ in the cone is zero at the earlier time.

- By the energy result, the energy can only go down, so it's still zero at the later time. So $u - v = 0$ at the later time.
- Letting "earlier time" be time $t = 0$ and "later time" be all the later times up to t_0, we deduce that $u - v = 0$ within the entire spacetime cone!
- So $u(x_0, y_0, t_0) = v(x_0, y_0, t_0)$.

How do we prove that energy result? This is the messier part. The idea is to apply the divergence theorem (in spacetime!) to the part of the cone between the earlier time and the later time. But how?

Ok, let's remember that the energy is this guy:

$$\int \frac{1}{2}|u_t|^2 + \int \frac{c^2}{2}|\nabla u|^2$$

That means that we need to be integrating $\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2$, and it needs to over both the top part (the later time) and the bottom part (the earlier time).

Let's also remember that the divergence theorem says that

$$\int_{\text{inside}} \text{div}(F) = \int_{\text{surface}} F \cdot n,$$

where the surface consists of the top part, bottom part, and the curved part. Which vector field F should we choose so that the surface integral part will have us integrating $\frac{1}{2}|u_t|^2 + |\nabla u|^2$?

Well, note that on the top and bottom, the unit outward normal n is going to either point straight up or straight down, i.e. is either $(0, 0, 1)$ and $(0, 0, -1)$. So, a first attempt is to take the vector field

$$(0, 0, \frac{1}{2}|u_t|^2 + |\nabla u|^2)$$

so that the dot product with n spits out $\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2$.

Does this work? Well, not really, and that's because I really want the whole inside integral $\int_{\text{inside}} \text{div}(F)$ to be zero. That means that I really want the divergence of the vector field I choose to be zero, and $(0, 0, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2)$ doesn't look very divergence-free to me.

So what would we need to replace the first two components with to make the divergence equal to zero? Well, the divergence would take the time derivative of $\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2$ (remember we're doing calculus in spacetime and the third dimension is time), and this would give

$$u_t u_{tt} + c^2 \nabla u_t \cdot \nabla u$$

which is equal to

$$u_t u_{tt} + c^2 u_{tx} u_x + c^2 u_{ty} u_y.$$

Hold on, from the original wave equation I know that $u_{tt} = c^2 \Delta u = c^2 u_{xx} + c^2 u_{yy}$. So this is all equal to

$$c^2 u_t u_{xx} + c^2 u_t u_{yy} + c^2 u_{tx} u_x + c^2 u_{ty} u_y.$$

Whoa hold on! By some miracle this is all equal to

$$c^2 (u_t u_x)_x + c^2 (u_t u_y)_y$$

by the product rule! That sure looks like a divergence to me! This tells us that to make the divergence zero, we should slot in a $-c^2 u_t u_x$ into the x-component and a $-c^2 u_t u_y$ into the y-component. Finally our left side of the divergence theorem says this:

$$\int_{\text{inside}} \operatorname{div}(-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + |\nabla u|^2)$$

Of course, this is equal to zero because we forced the integrand to be completely zero. But the divergence theorem says this is equal to this:

$$= \int_{\text{surface}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n$$

So

$$0 = \int_{\text{top}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n + \int_{\text{bottom}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n$$

$$+ \int_{\text{curvy part}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n.$$

The top and bottom parts should spit out the energy we want, as we devised from the very beginning. Indeed, the normal on the top is $n = (0, 0, 1)$, giving us an integrand of $\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2$. And, the normal on the bottom is $n = (0, 0, -1)$, giving us an integrand of $-\left(\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2\right)$. So

$$\int_{\text{top}} \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2 + \int_{\text{curvy part}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n = \int_{\text{bottom}} \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2.$$

In other words,

$$\text{Energy at later time} + \int_{\text{curvy part}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n = \text{Energy at earlier time.}$$

This almost looks like our energy result! Now we just need to prove that the curvy part term is ≥ 0.

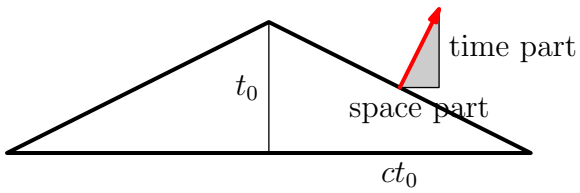
...

Ok screw it I've already spent two hours writing this so I may as well finish the proof.

I'm going to make this as simple as I possible can. So **shift the whole cone so that it's centered at** $(0,0)$. Now take a point (x,y,t) on the curvy part of the cone. What is the unit outward normal?

Let's not worry about finding the exact coordinates of the outward normal. In fact, I'm not even going to worry about making it have length 1, because if I'm proving that something is ≥ 0 then it doesn't matter if I multiply it by 3 or something. I just care about how the length of the "space" part of the normal compares with the length of the "time" part of the normal.

Draw this funny diagram.



This is a side-view of the cone. The red line is the outward normal. Now look at the gray triangle. By geometry, it's similar to a right t_0 by ct_0 triangle. So the space and time parts of the normal have to be in a 1 to c ratio.

Therefore, the normal at (x,y,t) looks like $(\hat{x},\hat{y},c)$ where $(\hat{x},\hat{y})$ is a vector with length 1. That's it. If you don't mind, I'm going to call this $\hat{r} := (\hat{x},\hat{y})$ and write the normal as $(\hat{r},c)$.

We're almost done. The integral we're wrestling is

$$\int_{\text{curvy part}} (-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n.$$

Actually I only care about the integrand,

$$(-c^2 u_t u_x, -c^2 u_t u_y, \frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2) \cdot n.$$

We agreed that the normal looks like $(\hat{x},\hat{y},c)$, so this is just

$$-c^2 u_t u_x \hat{x} - c^2 u_t u_y \hat{y} + \frac{c}{2}|u_t|^2 + \frac{c^3}{2}|\nabla u|^2.$$

Let me rewrite this as

$$-c^2 u_t \nabla u \cdot \hat{r} + \frac{c}{2}|u_t|^2 + \frac{c^3}{2}|\nabla u|^2.$$

Let me toss a common factor of c so that it's clearer what's going on. And also multiply by 2 for fun. Why not.

$$-2cu_t\nabla u \cdot \hat{r} + |u_t|^2 + c^2|\nabla u|^2.$$

Now, I'm going to marry c with ∇u. This is looking nicer and nicer!

$$|u_t|^2 - 2u_t(c\nabla u \cdot \hat{r}) + |c\nabla u|^2.$$

This almost looks like a perfect square trinomial! How do I make this actually factor correctly? My next trick is to marry the $\hat{r}$ with u_t to get

$$|u_t|^2 - 2(u_t\hat{r}) \cdot (c\nabla u) + |c\nabla u|^2.$$

And now for my final magic trick! Sneak in a $\hat{r} \cdot \hat{r}$ into the first term, because $|\hat{r}|^2 = 1$!!!

$$|u_t\hat{r}|^2 - 2(u_t\hat{r}) \cdot (c\nabla u) + |c\nabla u|^2.$$

At last, this factors as:

$$|u_t\hat{r} - c\nabla u|^2,$$

which is ≥ 0.

4 Heat and Wave

FYI I call $u_t - ku_{xx} = 0$ the **heat equation** rather than what the book calls it.

4.1 Solving the Heat Equation

In lecture we derived a formula for solving the heat equation

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

with initial condition ϕ. It turns out that the solution is quite "simple", at least when compared to the wave equation. It's given by the following formula:

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t)\phi(y)\,dy$$

where $s(x,t)$ is the *heat kernel* or *Green's function*,

$$s(x,t) := \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

(In other texts you might see this denoted as $K(x,t)$ or $G(x,t)$ or $\Phi(x,t)$.)

I said this was relatively simple, but it might look intimidating. To help with intuition, here are the main points I brought up in recitation.

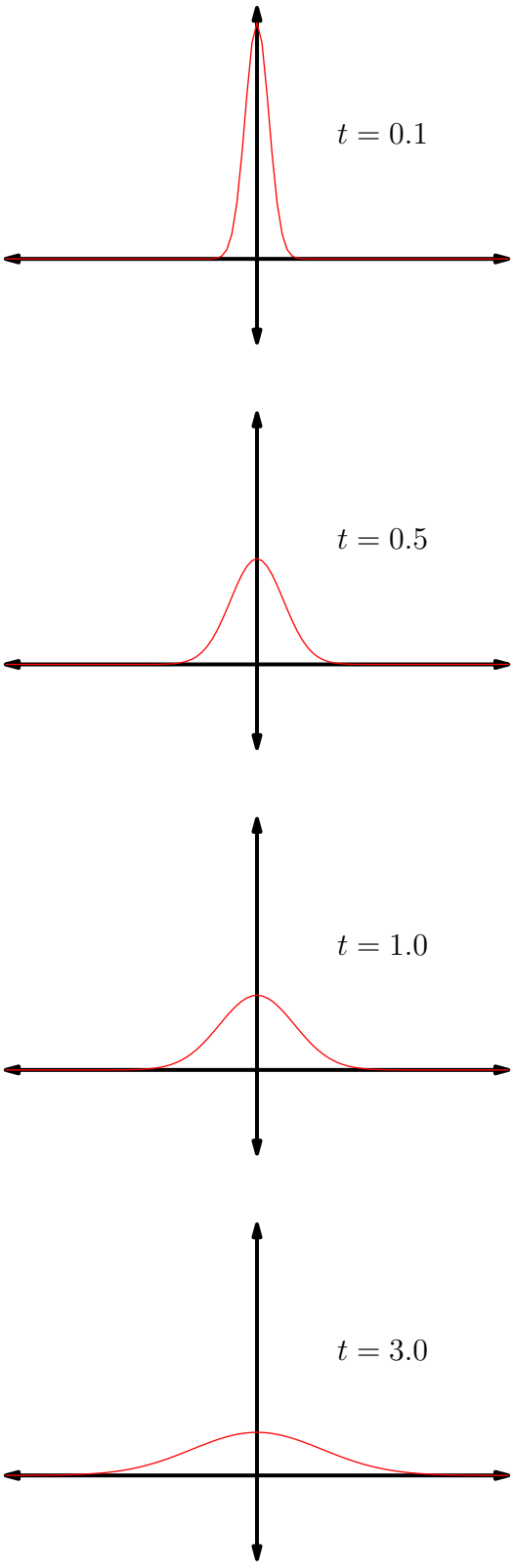
4.1.1 The Heat Kernel describes how heat flows

Let's take a closer look at this $s(x,t)$ function, the heat kernel. The point of the $\frac{1}{\sqrt{4\pi kt}}$ factor is to *normalize* the function to always have a total mass of 1. That is, we always want

$$\int_{-\infty}^{\infty} s(x,t)\,dx$$

to be true for all time. *(The way we actually find that factor to be $\frac{1}{\sqrt{4\pi kt}}$ is by doing quirky Gaussian integral stuffs.)*

Now, if you graph the heat kernel for certain values of t, you'll see that it basically looks like a lump of heat that's dissipating and spreading out!


$t = 0.1$
$t = 0.5$
$t = 1.0$
$t = 3.0$

Now you might be wondering, what happens at $t = 0$? Well, when $t = 0$ in the equation

$$s(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},$$

we kinda get nonsense. It's simply not defined! But judging from the graphs I drew, if we really wanted to make $s(x,0)$ make "sense", I think we can make a little guess...

Vague Concept: $s(x,0)$ is a "delta function", which has a "point mass" at $x = 0$ and is zero everywhere else.

(NOTE: Now that the cat's out of the bag I'm obligated to mention: (1) The delta function is NOT a function. (2) Nothing I say here is rigorous. (3) Actually understanding what a delta function actually is requires advanced real analysis that I will not be discussing whatsoever!)

Thus, in essence, we can think of the heat kernel as "solving" the following "PDE" with a rather weird "initial condition":

$$\begin{cases} s_t - k s_{xx} = 0 \\ \text{“}s(x,0) = \delta_{x=0}\text{”} \end{cases}$$

It is possible to make this discussion mathematically correct but I'm not going to do it. Here's the takeaway that I want you to get from this:

Key Point: The heat kernel $s(x,t)$ describes how a single "point mass of heat" will spread out over time.

$s(x,t)$ simply says that if you put a bunch of heat at a single point, then it will spread out smoothly as described by the pictures on the previous page. This is the picture I want you to have in mind.

4.1.2 Solution via Convolution

The solution of the heat equation with initial data ϕ is

$$u(x,t) = \int_{-\infty}^{\infty} s(x-y,t)\phi(y)\,dy.$$

What exactly is this? The intuition here is that we are using the heat kernel and "*applying it to every part of* ϕ", and this thus spreads out the initial "heat" that is described by ϕ.

This integral expression is known as a *convolution*, and is a way of essentially combining two functions in a nice way. Here, we are using it to mix the heat kernel with the initial condition ϕ, and the convolution happens to be the correct way to mix them together so that the "rules for spreading heat" as described by $s(x,t)$ are applied to ϕ.

(I would write more to explain the convolution more, but I told myself to keep these notes brief, so at the very least I hope you can kinda see the picture I'm trying to paint here even if the exact mechanisms and details are fuzzy. If you're lost, I encourage you to stare at the convolution and process what exactly it's doing...)

4.2 Practice with waves

We did two examples in recitation.

Example 4.1: Let $u(x,t)$ solve the 1D wave equation with initial conditions $\phi \equiv 0$ and

$$\psi(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$$

where $a > 0$ is a constant. Sketch a graph of $u(x,t)$ (horizontal axis x, vertical axis u) at the following fixed times: $t = \frac{a}{2c}$, $t = \frac{2a}{2c}$, and $t = \frac{3a}{2c}$.

Example 4.2: Where does a three-dimensional wave *have* to vanish if its initial data ϕ and ψ vanish outside a sphere?

I think the first exercise is great for understanding how to use the formulas we've derived in class to determine what solutions should look like.

The second exercise is great for your conceptual understanding of how waves propagate.

4.2.1 Odd versus Even dimensions

I won't include my solutions to these exercises in these notes (at least, for now; I'd be happy to include some upon popular demand). So I'll just briefly mention the interesting stuff that happens with waves in different dimensions.

In odd dimensions ≥ 3, the value $u(x_0,t_0)$ for u a solution to the wave equation depends *only* on the values of the *boundary* of the cone's base, i.e. only depends on the value of the initial data (ϕ and ψ) on $\partial B(x_0, ct_0)$. Roughly speaking, this means that sound travels in one direction and never echoes back.

In even dimensions, this is not true: $u(x_0,t_0)$ depends on the whole part of the cone's

base, i.e the initial data inside the ball $B(x_0, ct_0)$. So living in 2 dimensions kinda sucks — if you say anything, you'll hear yourself forever! *(But it will get quieter at an exponential rate.)*

That's why it's a miracle that our reality is in 3 dimensions — it's the smallest possible dimension for which talking to each other isn't terrible.

One dimension is a bit of an exception: If we look at the formula

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy,$$

we see that $u(x_0, t_0)$ depends on ϕ's value on the *boundary* of the interval $(x - ct, x + ct)$, but also ψ's value in the *interior* of this interval.

5 Some Practice with the Heat Equation

5.1 Fire Drill

At the beginning of class there was an announcement over the speakers that was like, we're gonna have a fire drill please leave when you hear alarm! And we all thought that the alarm would happen soon so we decided to leave. So we're standing outside for like 5 minutes and I decide to go back in and ask the security lady when the drill would end, and she's like, oh actually it didn't start yet it, it starts like half an hour into class. So I'm like, dang they did not make that clear at all, so now we all go back and talk about math anticipating the very loud fire alarm to happen and destroy our eardrums. But then like 40 minutes later the fire people come and they're like, yeah people are taking an exam so we won't sound the fire alarm, so instead of a drill we'll just tell you that if you ever hear an alarm you should walk out and turn left. Unless there's smoke there, then you should turn right. Then they turned left and left leftwards.

5.2 The Inhomogenous Problem

How do you solve

$$\begin{cases} u_t - k\Delta u = f \\ u\,|_{t=0} = g \end{cases} ?$$

Now there's an annoying f there.

Important Trick: To solve this, you can solve it for when $f = 0$, solve it for when $g = 0$, and then add up these two solutions.

This is because if $u^{(1)}$ solves the PDE

$$\begin{cases} u_t^{(1)} - k\Delta u^{(1)} = f \\ u^{(1)}\,|_{t=0} = 0 \end{cases},$$

and $u^{(2)}$ solves the PDE

$$\begin{cases} u_t^{(2)} - k\Delta u^{(2)} = 0 \\ u^{(2)}\,|_{t=0} = g \end{cases},$$

then it turns out that $u = u^{(1)} + u^{(2)}$ solves the PDE that has both the f and g? Why? Well, $u(x,0) = u^{(1)}(x,0) + u^{(2)}(x,0) = 0 + g = g$ and

$$u_t - k\Delta u = u_t^{(1)} - k\Delta u^{(1)} + u_t^{(2)} - k\Delta u^{(2)} = f + 0 = f.$$

This is a very important prank that is used for other PDEs! Keep it in mind.

Why is this trick important?

Key Point: This trick lets us assume that $g = 0$.

This is because if we solve the PDE for $g = 0$, then we can add the solution for when $f = 0$, which we *know how to solve!*

So how do we solve this PDE?

$$\begin{cases} u_t - k\Delta u = f \\ u\,|_{t=0} = 0 \end{cases} \quad ?$$

This requires **Duhamel's Principle**. We didn't have time to really explore this though. Instead...

5.3 What the f does f do?

Often you'll see the f in

$$u_t - k\Delta u = f$$

be referred to as a *forcing term*. Essentially this means that f is constantly adding heat. Why does it do that though?

It's important to remember where the heat equation comes from. Going back to just $u_t - k\Delta u = 0$ for now, let's integrate both sides over some domain Ω just for fun. This gives us

$$\partial_t \int_\Omega u(x,t)\,dx = k \int_\Omega \Delta u(x,t)\,dx.$$

The left side is the rate of change in the total amount of heat inside Ω. What about the right side? By the Divergence Theorem,

$$k \int_\Omega \Delta u(x,t)\,dx = k \int_\Omega \operatorname{div} \nabla u(x,t)\,dx = k \int_{\partial\Omega} \nabla u(x,t) \cdot \vec{n}\,dS$$

where $\vec{n}$ is the unit outward normal. This expression is basically how "steep" the temperature difference is between the inside and outside of Ω, along the boundary.

Thus, the heat equation just says (roughly) that if the inside is really cold (near the boundary) and the outside is really hot (near the boundary), then a lot of heat will come rushing in. The constant k controls how much faster it will come in.

Now let's add in the f term. This gives us an integral equation that looks like

$$\partial_t \int_\Omega u(x,t)\,dx = k\int_{\partial\Omega} \nabla u(x,t)\cdot \vec{n}\,dS + \int_\Omega f(x,t).$$

Aha! Now this equation says, in addition to how much heat is flowing across the boundary, a function f is adding in some heat to Ω at time t. Interesting!

Having this picture and concept in mind is useful for understanding Duhamel's formula. Perhaps we'll talk about that more next week.

5.4 Some Exercises

For the following examples that we talked about, the main idea is to do the following:

- Use your *intuition* about how heat works to try and guess the answer. (In my opinion, this is half the battle!)
- Try to prove it by using what we know about the heat equation.

Example 5.1 (Cold half-line vs. Warm half-line): Let u solve

$$\begin{cases} u_t - u_{xx} = 0 \\ u\,|_{t=0} = \begin{cases} 1, & x > 0 \\ 0, & x \le 0. \end{cases} \end{cases}.$$

What is the long-time behavior of $u(x,t)$? For example, can you tell me what $\lim_{t\to\infty} u(3,t)$ is?

Example 5.2 (Evolution of the maximum): Let's assume that the solution $u(x,t)$ to the problem

$$\begin{cases} u_t - u_{xx} = 0 \\ u\,|_{t=0} = \phi \end{cases}.$$

is smooth and has a maximum $\max_{x\in\mathbb{R}} u(x,t)$ for all times $t \ge 0$.
Is this maximum increasing with time, or decreasing with time? Or neither?

Example 5.3 (Lighting a never-ending fire inside $(0,1)$): Let u solve

$$\begin{cases} u_t - u_{xx} = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \quad . \\ u\,|_{t=0} = 0 \end{cases}$$

What is the long-time behavior of $u(x,t)$?

The last exercise is a bit harder than the other two, and the correct answer may go against your intuition! I didn't solve this last exercise in depth, but I did spoil the answer :) I think it's super interesting.

6 Duhamel and Midterm Review

6.1 Midterm Review

We talked about some of the problems.

6.2 Another Example of Method of Characteristics

I saw there was some confusion as to how to apply this methodology when the RHS is not zero (i.e. we're not necessarily constant along characteristics). So I wanted to include one more example for your reference.

Example 6.1: Please solve this PDE:

$$\begin{cases} u_t + xu_x + x = tu \\ u\,|_{t=0} = \sin x \end{cases}$$

Solution. To **find the characteristics, you just need the derivative terms**. That means $u_t + xu_x$. You'd find them the same way as in the PDE $u_t + xu_x = 0$.

Let's start our characteristic $(X(s), T(s))$ at $(x_0, 0)$. Then to match the derivative terms of our PDE, we have the ODEs

$$\begin{cases} X'(s) = X(s) \\ X(0) = x_0 \end{cases}$$

and

$$\begin{cases} T'(s) = 1 \\ T(0) = 0 \end{cases},$$

which solves as $X(s) = x_0 e^s$ and $T(s) = s$.

Now we **analyze behavior along the characteristic**. Meaning, we look at $u(x_0 e^s, s)$. The derivative of this in s is

$$\begin{aligned} \frac{d}{ds} u(x_0 e^s, s) &= u_t(x_0 e^s, s) + x_0 e^s u_x(x_0 e^s, s) \\ &= su(x_0 e^s, s) - x_0 e^s \end{aligned}$$

where we got the last equality from $u_t + xu_x = tu - x$, with $x_0 e^s$ in place of x and s in place of t.

Now, we **recognize this as an ODE**. If we define $g(s) := u(x_0 e^s, s)$ then g satisfies the ODE

$$g'(s) = sg(s) - x_0 e^s.$$

We now solve this ODE. We move $sg(s)$ to the other side and multiply by the integrating factor $e^{-\frac{s^2}{2}}$ to get

$$\frac{d}{ds}\left(g(s)e^{-\frac{s^2}{2}}\right) = -x_0 e^{s-\frac{s^2}{2}}.$$

Integrating each side from 0 to s gives

$$g(s)e^{-\frac{s^2}{2}} - g(0) = -x_0 \int_0^s e^{r-\frac{r^2}{2}}\, dr.$$

The RHS here is doomed and we will not simplify it. To finish solving for g it remains to figure out what $g(0)$ is. Looking back at how we defined g, we see that $g(0) = u(x_0, 0) = \sin x_0$. Therefore

$$g(s) = e^{\frac{s^2}{2}}\sin(x_0) - x_0 e^{\frac{s^2}{2}} \int_0^s e^{r-\frac{r^2}{2}}\, dr.$$

Finally we **use g to solve for u**. Since we solved for g, we look back at how we defined g. This gives

$$u(x_0 e^s, s) = g(s) = e^{\frac{s^2}{2}}\sin(x_0) - x_0 e^{\frac{s^2}{2}} \int_0^s e^{r-\frac{r^2}{2}}\, dr.$$

This holds for all x_0 and s. So now we replace $x_0 e^s$ with x and s with t. This means replacing x_0 with xe^{-t} and s with t. Thus

$$u(x,t) = e^{\frac{t^2}{2}}\sin(xe^{-t}) - (xe^{-t})e^{\frac{t^2}{2}} \int_0^t e^{r-\frac{r^2}{2}}\, dr.$$

■

...I will not be checking if this is actually right since it's a mess, I just want to showcase the method.

6.3 Duhamel's Formula

Alright so one problem on the midterm uses this so we should chat about this. From the previous section in this notes, we found that the $f(x,t)$ term in the PDE

$$\begin{cases} u_t - k\Delta u = f \\ u\,|_{t=0} = 0 \end{cases}$$

is a *forcing term*. Intuitively, this means that at every small bit of time, f adds a bit of heat. Specifically, we can imagine that at time s, f adds some heat to the real line according to the function $f(x,s)$.

How would this heat disperse? Naturally, it would be according to the solution to the standard heat equation if the initial data were $f(x,s)$, i.e. solving

$$\begin{cases} v_t - kv_{xx} = 0 \\ v\,|_{t=0}(x) = f(x,s) \end{cases}.$$

I will now introduce some silly notation that nobody uses. Let

$$\underset{\phi=[\text{initial data}]}{\text{PDE SOL}}\,(x,t)$$

be the solution to the heat equation with initial data given by [initial data], i.e. it is the function v where v solves the PDE

$$\begin{cases} v_t - kv_{xx} = 0 \\ v\,|_{t=0} = [\text{initial data}] \end{cases}.$$

With this silly notation I think it will be easier to write down Duhamel's formula, which I will now derive.

6.3.1 Derivation of Duhamel

Let's say we want to figure out what $u(x,t)$ is. That is, we want to find out how much heat there is at the point x after t seconds. Since there is no initial data, all the heat contribution will come from the forcing term $f(x,t)$.

So, we need to "add up" all the heat contribution from $f(x,s)$ up to time t. Thus

$$u(x,t) = \int_0^t \Big(\text{Amount of heat that } [f \text{ at time } s] \text{ will contribute to } x\Big)\, ds.$$

How do we find out how much heat f at time s will contribute?

To do this, we need to look at f at time s and see what it does over time. For that, we just consider what f is at time s, and apply the heat equation to it! This is the function

$$\underset{\phi=(f \text{ at time } s)}{\text{PDE SOL}}\,.$$

(To be clear, "f at time s" is the function that takes y to $f(y,s)$, often denoted $f(\cdot,s)$.)

What values are we plugging into this quirky function? Well, I need to know how much heat it's adding to the point x, so we're plugging in x for the space variable. How about the

time variable? That's how long we're running the heat for until we reach time t. Since we're at time s, we are letting the heat spread for $t-s$ seconds. Therefore,

$$\Big(\text{Amount of heat that } [f \text{ at time } s] \text{ will contribute to } x\Big) = \underset{\phi=(f \text{ at time } s)}{\text{PDE SOL}}\,(x, t-s).$$

We have hence derived Duhamel's formula for the heat equation,

$$\boxed{u(x,t) = \int_0^t \underset{\phi=(f \text{ at time } s)}{\text{PDE SOL}}\,(x, t-s)\,ds}.$$

6.3.2 Duhamel (Ugly Version)

The above expression is a good way to remember the formula. For a more traditional way to write it, we can expand this expression further. We know that

$$\begin{aligned}\underset{\phi=(f \text{ at time } s)}{\text{PDE SOL}}\,(x,t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \cdot (f \text{ at time } s)(y)\,dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \cdot f(y,s)\,dy.\end{aligned}$$

So,

$$\underset{\phi=(f \text{ at time } s)}{\text{PDE SOL}}\,(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k (t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot f(y,s)\,dy.$$

Plugging this in to the formula we derived, we obtain the nasty formula,

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k (t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot f(y,s)\,dy\,ds.$$

Remember, this is just the solution to $\begin{cases} u_t - k u_{xx} = f \\ u\,|_{t=0} = 0 \end{cases}$. If we wanted the solution to $\begin{cases} u_t - k u_{xx} = f \\ u\,|_{t=0} = \phi \end{cases}$, we'd have to add the solution to the homogenous problem with initial condition ϕ, which we know how to solve. This gives the general solution

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k (t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot f(y,s)\,dy\,ds$$

to the heat equation with forcing term f and initial condition ϕ.

The ideas of Duhamel also work for other PDEs but the logic will be slightly different. Note that I haven't actually proven anything here, I'm just giving what I believe to be the most intuitive way to derive the formula on the spot.

7 Separation of Variables I: Fourier Series

Only like 3 people showed up today so let's talk about how to find a Fourier series.

IMPORTANT: In this chapter and the next, I'm going to be telling a lot of lies. This is because the underlying mathematics here is EXTREMELY advanced, and so it is impossible to teach this in any rigorous capacity without assuming years of experience in real analysis. Instead, it makes a lot more sense to teach this "only kinda correctly", trading truth for intuition. I will quietly point out every time I tell a big lie, but I won't elaborate very much on why I'm lying or how to remedy the lie.

7.1 The Punchline

We will ruin the joke by giving the punchline upfront so you know what the general goal is. The central idea is that **we want to express a function as a sum of waves**. To wit, the *Fourier series* of a function f defined on an interval $[0, L]$ can look like any of the following:

1. Fourier sine series: $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$ for some real numbers $a_1, a_2, \ldots$.
2. Fourier cosine series: $f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$ for some real numbers $a_0, a_1, \ldots$.
3. Fourier sine-cosine series: $f(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{2n\pi}{L}x\right) + \sum_{n=0}^{\infty} b_n \cos\left(\frac{2n\pi}{L}x\right)$ for some real numbers $a_1, a_2, \ldots, b_0, b_1, \ldots$.
4. Fourier exponential form: $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-2\pi i n x}$ for some real numbers $\ldots, a_{-1}, a_0, a_1, \ldots$.

We won't worry about the fourth one because, while I like it a lot, it's actually not going to be very helpful for the next chapter. So we'll stick with real numbers.

Anyways, there are a bunch of different Fourier series and actually remembering how to find the coefficients (the a_n's and b_n's) is very annoying. There is only one good way to find them, and it is using...

7.2 A Review of Linear Algebra

7.2.1 Basis

Let V be a vector space, like $\mathbb{R}^n$. Remember what a *basis* is? If V has dimension n, then there is a set of vectors $\{v_1, v_2, \cdots, v_n\}$ in V called a basis such that any vector $x \in V$ can be uniquely represented in the form

$$x = a_1 v_1 + \ldots + a_n v_n$$

for some real numbers $a_1, \ldots, a_n$.

The prototypical example for a basis is $\{(1,0,0),(0,1,0),(0,0,1)\}$, for $\mathbb{R}^3$.

7.2.2 Inner products

When V is nice enough, it has sometime called an *inner product*, which I shall write as $\langle v, w\rangle$. In $\mathbb{R}^n$, this is the "dot product", which can be used to measure the angle between the vectors.

Sadly, angle doesn't make sense in weirder vector spaces, so instead **let's think of an inner product $\langle v, w\rangle$ as measuring how much v "agrees" with w, or more concretely, how much of v is pointing in the direction of w.**

That last bolded bit is very important, but it only works well when the length of w is 1. Remember that the length or *norm* of a vector w, written as $\|w\|$, can be given by the identity $\|w\|^2 = \langle w, w\rangle$. [1]

This notion is needed to talk about what an orthonormal basis is.

7.2.3 Orthonormal basis

The nicest possible basis $\{v_1, \ldots, v_n\}$ is called an *orthonormal basis*. Let's break down that word:

- "Ortho" means perpendicular to each other. This means that $\langle v_i, v_j\rangle = 0$ for all v_i and v_j distinct.
- "normal" means length 1. This means that each v_i has length 1. In other words, $\langle v_i, v_i\rangle = 1$ for all i.

[1] Wowie look at me lying already! This is technically a definition, not an identity.

The prototypical example basis for $\mathbb{R}^3$, i.e. $\{(1,0,0),(0,1,0),(0,0,1)\}$, is an orthonormal basis.

The reason why this an orthonormal basis is so nice is because it makes it very easy to write a vector x in *terms of an orthonormal basis* $\{v_1,\ldots,v_n\}$. That is, it's easy to solve for the coefficients $a_1,a_2,\ldots,a_n$ to make the representation

$$x = a_1v_1 + a_2v_2 + \ldots + a_nv_n \qquad (*)$$

true.

Indeed, the coefficient a_i is just given by $\boxed{a_i = \langle x, v_i\rangle}$. That's it!

Intuitively this makes a ton of sense: a_i should be how much of x is pointing in the v_i direction. And, well, the inner product $\langle x, v_i\rangle$ indeed measures "how much v_i is in x"!

Brief Exercise: Check that my formula for a_i is indeed true by taking the equation $(*)$ and inner-product'ing each side with v_i. *(Remember that the inner product has a "distributive property". All the terms on the right side should get zero'd except for the ith one.)*

The bottom half of this page is intentionally blank. I hope you're having fun!

7.3 The vector space of functions

Let's apply linear algebra[2] to figure out what Fourier series are! The underlying vector space V in this case is the space of all functions on $[0, L]$. [3]

I want to find an orthonormal basis for this vector space. But before I can do that, I need to know if V has an inner product. It does![4] It's given by *integration*: For any two functions $f, g : [0, L] \to \mathbb{R}$, I define their inner product as follows.

$$\langle f, g \rangle := \int_0^L f(x)g(x)\,dx$$

If you haven't seen this before, **it's very important**[5].

This satisfies all the properties that an inner product needs to satisfy[6], letting us measure "how much the function g is 'part of' f". It also lets us measure the "length" of a function:

$$\|f\|^2 := \langle f, f \rangle = \int_0^L f(x)f(x)\,dx = \int_0^L f(x)^2\,dx.$$

This is called the L^2 *norm* of f[7].

Alright, let's find a basis for V.

7.4 One possible basis for this vector space

This vector space of functions V is infinite-dimensional, so our basis $\{v_1, v_2, v_3, \ldots\}$ will also be infinite. Many years ago, Fourier got super drunk and decided to try using sine functions to form a basis.

He chose the functions $\sin\left(\frac{\pi n}{L}x\right)$ for $n = 1, 2, 3, \ldots$. Essentially, these are all the frequencies for sine functions so that the sines will be 0 at both $x = 0$ and $x = L$. *You can think of this as all the simple waves you can make on a string if both of its ends stay fixed. You should plug in $x = 0$ and $x = L$ into these functions to make sure I'm not lying to you!*

[2] This is a lie, it's functional analysis

[3] Huge lie. This doesn't rigorously work.

[4] It doesn't. But it'll work out.

[5] Made you look! This is not a lie at all. Incredibly important notion here and pops up all the time in mathematics. For example, in probability and statistics you often see the expression $\mathbb{E}[XY]$ right? That's basically what this is.

[6] Alright this is the lie that explains why V is bad. If you're bored I encourage you to figure out why this is wrong, it's very subtle. Fortunately even if I'm lying here, the lie doesn't actually impact the core of the math so we're chilling.

[7] This is more lingo that's good to know.

Here's something interesting about this basis: The basis vectors are orthogonal to each other! I won't write it out here[8], but you can check that whenever $m \neq n$,

$$\int_0^L \sin\left(\frac{\pi m}{L}x\right)\sin\left(\frac{\pi n}{L}x\right)\,dx = 0.$$

Therefore the inner product is zero, i.e. $\left\langle \sin\left(\frac{\pi m}{L}x\right), \sin\left(\frac{\pi n}{L}x\right)\right\rangle = 0$.

Most takes on Fourier series will stop the linear algebra analogy here. I think this is insufficient and I will go one step further: Is this basis an *orthonormal basis*? We have the "ortho" part, but now we need the "normal" part, meaning we want these sine functions to all have "length 1".

Unfortunately this isn't true. Indeed, let's compute the the integral $\displaystyle\int_0^L \sin^2\left(\frac{\pi n}{L}x\right)\,dx$ to see why.

> Alright I'm going to do this really stupidly to minimize pain. Do you remember the double angle formula? Probably not. What I do want you to know, though, is that it basically says that $\sin^2$ is a sinusoid! It just oscillates between 0 and 1, thus having a mean value of $\frac{1}{2}$ (instead of 0 like sin does). Therefore,
>
> $$\int_0^L \sin^2\left(\frac{\pi n}{L}x\right)\,dx = L \cdot \text{Average value} = L \cdot \frac{1}{2} = \boxed{\frac{L}{2}}.$$

Fantastic. So, since the length (squared) of our basis vectors is

$$\int_0^L \sin^2\left(\frac{\pi n}{L}x\right)\,dx = \frac{L}{2},$$

and $\frac{L}{2}$ isn't necessarily equal to 1, our basis isn't orthonormal. **But we can fix this by scaling our basis vectors by a constant!** Indeed,

$$\int_0^L \left[\sqrt{\frac{2}{L}}\sin\left(\frac{\pi n}{L}x\right)\right]^2\,dx = 1.$$

We now have an orthonormal basis!

[8] This computation uses the *product-to-sum* formula from trigonometry. I only remember it because I was one of those math competition kids.

7.5 Sine Fourier Series

Our orthonormal basis is

$$\boxed{v_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L} x\right)}.$$

Since we now have an orthonormal basis, we can now write *any* function f in our vector space in terms of this basis! This means that we can write f uniquely as a linear combination

$$f(x) = a_1 v_1(x) + a_2 v_2(x) + a_3 v_3(x) + \dots,$$

for some correct choice of real numbers $a_1, a_2, \dots$.[9]

Figuring out what these a_i's need to be for this to work might sound hard, but *it's actually stupidly easy because we have an orthonormal basis!* Indeed, mimicking what we did when we were reviewing linear algebra, we immediately get the formula

$$\boxed{a_n = \langle f, v_n \rangle = \int_0^L f(x) v_n(x)\, dx}.$$

Congratulations, you now know Fourier series.

7.6 ...a slight omission

So, I've explained why the set of weird sinusoid functions $\{v_1, v_2, \cdots\}$ is orthonormal. Did you notice that I never actually explained why it's actually a basis?

This is because that's very hard to prove. So, uh, just trust me.[10]

[9] The equals sign here is a bit of a lie, I'll come back to this

[10] It is because of this that I sometimes have trouble figuring out whether I'm "using enough" sine/cosine functions to correctly form a Fourier series expansion for f. The next chapter of this notes gives some way to actually ensure that we have all the ones we actually need.

7.7 Example

Let's find the sine Fourier series of x on $[0,5]$. *(You have to specify the interval! You can't Fourier series something on the entire real line.)*

Step 1: Find the sine basis

For sine series, the pro-tip is to take all sine functions $\sin(kx)$ that vanish at the endpoints *(and also ignore $k<0$)*. In this case I want $\sin(k\cdot 0)=0$ (I get that for free) and $\sin(k\cdot 5)=0$. So $5k$ is a multiple of π. So $5k=n\pi$. So $k=n\pi/5$.

We take $\boxed{\sin(n\pi/5\cdot x)}$, where n runs through positive integers, as our basis. *(i encourage you to ponder why it makes sense that we ignore the sine functions we get for $n=0$ and $n<0$.)*

Step 2: Scale the sine basis so it becomes orthonormal

By either memorizing that the correcting factor is $\sqrt{2/L}$ (where $L=5$) or just running through the quick "silly computation" on the previous page, we take

$$\boxed{v_n(x) := \sqrt{\frac{2}{5}}\sin(n\pi/5\cdot x)}.$$

Personally I rederive it every time because I find memory to be very unreliable.

Step 3: Use inner products to find the coefficients

The sine Fourier series of x over $[0,5]$ is

$$x = a_1v_1(x) + a_2v_2(x) + \dots.$$

It remains to find the coefficients $\{a_n\}_n$. As I've emphasized, this is just given by the inner product

$$a_n = \langle x, v_n(x)\rangle.$$

By plugging in what v_n is, this inner product is

$$\langle x, v_n(x)\rangle = \int_0^5 x\cdot\sqrt{\frac{2}{5}}\sin(n\pi/5\cdot x)\,dx = \sqrt{\frac{2}{5}}\int_0^5 x\sin(n\pi/5\cdot x)\,dx.$$

Unfortunately the resulting integral is often **very annoying and easy to mess up**. Here we must integrate by parts.

$$= \sqrt{\frac{2}{5}}\left[x\cdot\frac{-\cos(n\pi/5\cdot x)}{n\pi/5}\,\Bigg|_{x=0}^{5} - \int_0^5 \frac{-\cos(n\pi/5\cdot x)}{n\pi/5}\,dx\right] = \sqrt{\frac{2}{5}}\left[x\cdot\frac{-\cos(n\pi/5\cdot x)}{n\pi/5}\,\Bigg|_{x=0}^{5}\right]$$

$$= -\sqrt{\frac{2}{5}} \cdot \frac{25\cos(n\pi)}{n\pi} - 0 = \boxed{\sqrt{\frac{2}{5}} \cdot \frac{25(-1)^{n+1}}{n\pi}}$$

(Note that $\cos(n\pi) = 1$ *when* n *is even, and* $\cos(n\pi) = -1$ *when* n *is odd. This observation can be written succinctly as* $\cos(n\pi) = (-1)^n$*.)*

Therefore the sine Fourier series of x over $[0, 5]$ is

$$x = \sum_{n=1}^{\infty} a_n v_n(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{5}} \frac{25(-1)^{n+1}}{n\pi} \cdot \sqrt{\frac{2}{5}} \sin(n\pi/5 \cdot x) = \boxed{\sum_{n=1}^{\infty} \frac{2}{5} \cdot \frac{25(-1)^{n+1}}{n\pi} \sin(n\pi/5 \cdot x)}.$$

If you are not a fan of the Step 2 normalization step, you can memorize that the correcting factor in this final answer is $2/L$. Personally I will absolutely forget that after I finish writing these notes though.

7.8 How to check your work

It is extremely easy to mess up. Therefore, I encourage you to use Desmos to check your work. Here is an example of me using Desmos to check that my answer for the sine Fourier series of x over $[0, 5]$ was correct: https://www.desmos.com/calculator/rk18pgjy39

In fact, while writing these notes I totally messed up the computation of a_n the first time. I only figured out I messed up by graphing my Fourier series and seeing that it didn't match up at all!

7.9 Other Fourier Series

Thus far we've been talking about sine Fourier series.

Definition 7.1 (Sine Fourier Series)

The **sine Fourier series** of $f : [0, L] \to \mathbb{R}$ is the representation of f in terms of the orthonormal basis $v_1, v_2, \ldots$, where

$$v_n(x) := \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L} x\right).$$

That is, it is the unique way to represent f in the form $f(x) = \sum_{n=1}^{\infty} a_n v_n(x)$ for some real numbers $a_1, a_2, \ldots$.

But we can also choose a different orthonormal basis, such as using cosines. Here, to figure out what cosines to use over $[0, L]$, we choose all the cosine functions $\cos(kx)$ that **have derivative zero** at the endpoints, i.e. there is a peak / valley at both $x = 0$ and $x = L$. Doing the math, we find that we need to use c for $n = 0, 1, 2, \ldots$. **This time we need to include** $n = 0$. (Whereas in the sine case, $\sin(0) = 0$ and zero can't be part of a basis.)

We still have $\int_0^L \cos^2(n\pi/L \cdot x)\, dx = \frac{L}{2}$, so $v_n(x) := \sqrt{\frac{2}{L}} \cos(n\pi/L \cdot x)$ will form an orthonormal basis, where n runs from 0 to ∞. What's tricky is that the computations change slightly for $n = 0$ which is why the normalization factor is different. *(This is exactly why I can't afford to memorize this — it's a total mess!)*

Definition 7.2 (Cosine Fourier Series)

The **cosine Fourier series** of $f : [0, L] \to \mathbb{R}$ is the representation of f in terms of the orthonormal basis $v_0, v_1, v_2, \ldots$, where

$$v_n(x) := \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n}{L} x\right) \text{ for all } n \geq 1, \text{ and } v_0(x) = \sqrt{\frac{1}{L}}.$$

That is, it is the unique way to represent f in the form $f(x) = \sum_{n=0}^{\infty} a_n v_n(x)$ for some real numbers $a_0, a_1, a_2, \ldots$.

(I'll update what the cosine series for x will be after the homework is due...)

There is also the sine-cosine series, which uses **both** sine and cosines. But the frequencies we use is different from before: Here we use all the sines and cosines that **do a positive integer number of periods in** $[0, L]$. This means we're using $\sin(2\pi n/L \cdot x)$ for $n = 1, 2, \ldots$ and $\cos(2\pi n/L \cdot x)$ for $n = 0, 1, 2, \ldots$.

Definition 7.3 (Sine-Cosine Fourier Series)

The **sine-cosine Fourier series** of $f : [0, L] \to \mathbb{R}$ is the representation of f in terms of the orthonormal basis $v_1, v_2, \ldots, w_0, w_1, w_2, \ldots$, where

$$v_m(x) := \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n}{L} x\right) \text{ for all } m,$$

$$w_n(x) := \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n}{L} x\right), \qquad n \geq 1$$

$$w_0(x) = \sqrt{\frac{1}{L}}.$$

That is, it is the unique way to represent f in the form

$$f(x) = \sum_{m=1}^{\infty} a_m v_m(x) + \sum_{n=0}^{\infty} b_n w_n(x)$$

for some real numbers $a_1, a_2, \ldots, b_0, b_1, b_2, \ldots$.

It goes without saying that I think the resulting formulas for the coefficients here are practically impossible to memorize, so I choose to rederive them every time. I know these notes are long, but that's all just setup for the concepts. The actual computations for the derivation are quite short.

It turns out that in the sine-cosine Fourier series for x over $[0, 5]$ pretty much all the cosine terms die out except for the constant term (i.e. b_0), and so the sine-cosine series for x over $[0, 5]$ is

$$x = \frac{5}{2} + \sum_{n=1}^{\infty} \frac{-5}{\pi n} \sin\left(\frac{2n\pi}{5} x\right).$$

7.10 (Optional?) Convergence of Fourier Series

Just real quick, I should address a lie I've been telling throughout: The Fourier series does not always "work"! That is, the "equality"

$$f(x) = \sum_{n=1}^{\infty} a_n v_n(x)$$

where a_n and v_n are chosen according to, e.g. the sine Fourier series, is not necessarily for all x.

Typically though, the only problematic x-values are where f is doing something weird at x, like a jump discontinuity.

In general, if f is differentiable at x, you can be rest assured that the Fourier series of f, when evaluated at x, will converge correctly to $f(x)$.

I will not prove any of this because it is hard.

8 Separation of Variables II: Defeating the Laplacian

In both of the two time-dependent PDEs we've studied thus far, there has been an evil annoying triangle:

- The wave equation, $u_{tt} - \Delta u = 0$.
- The heat equation, $u_t - \Delta u = 0$.

This is the Laplace operator,

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n} \partial_{x_i}^2 u = \sum_{i=1}^{n} u_{ii},$$

and it is evil. **Our main enemy here is the Laplacian**. The point of separation of variables is to defeat the Laplacian.

note: i am presenting a different perspective on this topic than what the professor does, you may or may not find it more intuitive. there nevertheless is some overlap in philosophy and the resulting computations are going to be the same.

if you're pressed for time, just read the examples. most of what i write here is for intuition's sake and justifying what's happening.

8.1 (Optional) The real enemy is the negative Laplacian

Pretty much always, our real target is going to be $-\Delta$ rather than Δ. In short, this is because $-\Delta$ is a more natural operator than Δ. I don't want to distract too much from the main topic so see the first answer in https://math.stackexchange.com/questions/4884068/the-positive-laplacian-is-indeed-the-negative-laplacian.

8.2 More linear algebra review

Symmetric matrices have a surprising incredible property, that you hopefully learned in your linear algebra class.

Theorem 8.1 (Spectral Theorem)

Let A be an $n \times n$ symmetric matrix. Then there is an **orthonormal basis** $v_1, \ldots, v_n$ of $\mathbb{R}^n$ such that under this basis, A becomes a diagonal matrix.
To be precise, if we take any vector $x \in \mathbb{R}^n$ and represent it in terms of this basis as $x = a_1v_1 + a_2v_2 + \ldots + a_nv_n$ for some real numbers $\{a_i\}_i$, then

$$Ax = \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 + \ldots + \lambda_n a_n v_n$$

where λ_i is the eigenvalue associated with eigenvector v_i.

Some ways to think of this theorem are:

- All symmetric matrices are *basically* diagonal.
- All symmetric matrices can be diagonalized using an orthonormal basis.
- Symmetric matrices just choose n orthogonal vectors and stretch space in those n directions.

8.3 "Separation of variables" in linear algebra

Everything I'm doing here is basically separation of variables but for $\mathbb{R}^n$ instead of the infinite-dimensional vector spaces we deal with for PDEs. This analogy should help you process what's going on.

Let A be symmetric and $y \in \mathbb{R}^n$. How can we solve $Ax = y$? Here, y is a given vector and we are solving for x. We follow these simple steps:

1. Diagonalize A. That is, we first find the orthonormal eigenvectors $v_1, \ldots, v_n$ and associated eigenvalues $\lambda_1, \ldots, \lambda_n$ for A. They're guaranteed to exist because A is symmetric.
2. Write knowns and unknowns in terms of the basis $\{v_i\}_i$. For x, we write

   $$x = a_1v_1 + \ldots + a_nv_n$$

   where the $\{a_i\}$ are unknown real numbers we must solve for. We win once we figure out what they are.

   For y, we write

   $$y = b_1v_1 + \ldots + b_nv_n$$

   where the $\{b_i\}$ are **known** values that can be found in terms of y, via $b_i = \langle y, v_i \rangle$.

3. Plug in x and y into $Ax = y$, and use the fact that the $\{v_i\}$ are eigenvectors to simplify. We have
$$A(a_1v_1 + \ldots + a_nv_n) = b_1v_1 + \ldots + b_nv_n.$$
But now the left side simplifies, using $Av_i = \lambda_i v_i$, giving us
$$\lambda_1 a_1 v_1 + \ldots + \lambda_n a_n v_n = b_1v_1 + \ldots + b_nv_n.$$

4. Compare coefficients to solve for the unknowns. The coefficient of v_i on each side needs to be equal, so
$$\lambda_i a_i = b_i$$
for every i. Since we're solving for a_i, we find that $\boxed{a_i = b_i/\lambda_i}$. *(of course, we should assume that all eigenvalues are nonzero for this expression to be valid.)*

5. Clean up. Plugging in the value we got for a_i into the equation for x, we have
$$\begin{aligned} x &= a_1v_1 + \ldots + a_nv_n \\ &= \frac{b_1}{\lambda_1} \cdot v_1 + \ldots + \frac{b_n}{\lambda_n} v_n \\ &= \boxed{\frac{\langle y, v_1\rangle}{\lambda_1} \cdot v_1 + \ldots + \frac{\langle y, v_n\rangle}{\lambda_n} v_n}. \end{aligned}$$

This is basically what separation of variables is. As it pertains to PDE, the only step that arguably gets more complicated is solving for the coefficients.

8.4 Hold on, what is a "symmetric matrix" when doing PDE stuff?

Great question! The "matrix" we're trying to tame is the operator $-\Delta$ (which in 1D is just $-\partial_{xx}$). [1] Is this symmetric? What does that even mean?

To do this we need one more linear algebra analogy. Recall that for any $n \times n$ matrix A, we have that
$$\langle Ax, y\rangle = \langle x, A^T y\rangle.$$
So, a way to characterize A being symmetric is that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all x and y. We can play the same game for $-\Delta$.

[1] Although I am thinking of $-\Delta$ as if it were a matrix, you can't really represent it as a matrix because we're in infinite dimensions.

Definition 8.1 (When is the Laplacian "symmetric"?)

We can say that $-\Delta$ is *symmetric* on a vector space of functions V if $\langle -\Delta f, g\rangle = \langle f, -\Delta g\rangle$ for all functions f and g in this vector space. That is,

$$\int (-\Delta f)g = \int f(-\Delta g).$$

In 1D, this equates to verifying that

$$\int f_{xx}g = \int f g_{xx}.$$

(i've tossed the negative sign for temporary ease.)

Let's check whether $-\Delta$ is symmetric in general. I'll stick with 1D but it's the same in higher dimensions. For any functions f and g over some vector space V of functions on $[a, b]$, we can spam integration by parts to get:

$$\begin{aligned}
\int_a^b f_{xx}(x)g(x)\,dx &= f_x(x)g(x)|_{x=a}^b - \int_a^b f_x(x)g_x(x)\,dx \\
&= f_x(b)g(b) - f_x(a)g(a) - \left(\int_a^b f_x(x)g_x(x)\,dx\right) \\
&= f_x(b)g(b) - f_x(a)g(a) - \left(f(x)g_x(x)\,|_{x=a}^b - \int_a^b f(x)g_{xx}(x)\,dx\right) \\
&= f_x(b)g(b) - f_x(a)g(a) - f(b)g_x(b) + f(a)g_x(a) + \int_a^b f(x)g_{xx}(x)\,dx
\end{aligned}$$

Yikes, this isn't *quite* the equality $\int f_{xx}g = \int f g_{xx}$ that we wanted. There's a bunch of trash left over from the boundary terms! Boooooo.

This brings me to my key point, which might seem like annoying nit-picking but it's **essential** and, in my opinion, **it's the only way to tell when separation of variables actually works!**

CRUCIAL POINT: $-\Delta$ can only be symmetric if the boundary terms become 0. In other words, we need the integration by parts formula $\int f_{xx}g = \int f g_{xx}$ to be true without the trash boundary terms.

This entire theory FAILS if this doesn't happen.

8.5 When the theory will work: Examples

The boundary terms can vanish if the vector space V of functions that we're looking at is nice enough. V is mainly going to be determined by the PDE that we're looking at. Here are a few examples and non-examples.

Example 1: Consider this heat equation with forcing on $[0, 1]$ and Dirichlet boundary conditions,

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = \phi(x) := e^{e^{\sin x}} + 999 \\ u(0, t) = 0 \\ u(1, t) = 0 \end{cases}.$$

(note: dont worry about the u_t derivative, that problem will solve itself. separation of variables just focuses on killing the spatial u_{xx} derivative.)

The boundary conditions here *(they're just the last two conditions)* give us the vector space V that we need to consider: It's the vector space of functions that vanish at 0 and 1. This is great for integration by parts!

Indeed, for all f, g with $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$, we can get $\int_0^1 f''g\,dx = \int_0^1 fg''\,dx$, with no garbage boundary terms. *(you should check that i'm not lying to you.)*

Conclusion: Separation of variables will work for this PDE.

Example 2: Consider this wave equation with forcing on $[42, 9001]$ and von Neumann boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x, 0) = \phi(x) := 7777777 + \log x \\ u_t(x, 0) = \psi(x) := \log(7777777) + x \\ u_x(42, t) = 0 \\ u_x(9001, t) = 0 \end{cases}.$$

Here the boundary conditions are $u_x(42, t) = 0$ and $u_x(9001, t) = 0$, and they're called "von Neumann" because of the x-derivative.

So this time, our vector space V consists of functions f whose *derivative* vanishes at the boundary. That is, $f'(42) = 0$ and $f'(9001) = 0$. It turns out that this is *also* good for integration by parts, and we will still get $\int_0^1 f''g\,dx = \int_0^1 fg''\,dx$ with no garbage boundary terms, for all f and g in this vector space. *(you definitely should check that i'm not lying to you here!)*

Conclusion: Separation of variables will work for this PDE.

Example 3: Consider this heat equation on all of space,

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi(x) := \sin x \end{cases}.$$

We already know that this can be solved. But can we solve it differently using separation of variables? That depends on the boundary conditions.

...wait a second, there are no boundary conditions since this is the heat equation on the whole real line. With no assurance that the functions we're considering will vanish at $\pm\infty$ in any way, we're screwed.

Conclusion: Separation of variables will **not** work for this PDE.

Example 4: Consider this heat equation on $[0,1]$ with periodic boundary conditions,

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi(x) := x \\ u(0,t) = u(1,t) \\ u_x(0,t) = u_x(1,t) \end{cases}.$$

It turns out that *periodicity*, i.e. the condition that f and f' are the same at the endpoints of $[0,1]$, is great for integration by parts. So $-\partial_{xx}$ will be symmetric on a space of such functions. *(you should check that i'm not lying to you.)*

Conclusion: Separation of variables will work!

Example 5: Consider the wave equation on $[0,1]$ with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = \phi(x) := x \\ u_t(x,0) = \psi(x) := x^2 \\ u(0,t) = 100 \\ u(1,t) = 200 \end{cases}.$$

Uh-oh, the boundary conditions are no longer *zero* boundary conditions! So integration by parts on $\int_0^1 f''g$ will give us garbage boundary terms. Yuck!

Initial conclusion: Separation of variables will not work...

...but in this case we can actually fix the issue! We just need to **transform** the PDE so that the boundary conditions become 0. I won't elaborate too much for now on how you figure this out, but just as an example, here you substitute $v(x,t) := u(x,t) - 100(1+x)$. Then v will satisfy the PDE

$$\begin{cases} v_{tt} - v_{xx} = 0 \\ v(x,0) = \phi_1(x) := x - 100(1+x) \\ v_t(x,0) = \psi_1(x) := x^2 \\ v(0,t) = 0 \\ v(1,t) = 0 \end{cases},$$

and now we're golden.

Conclusion: Separation of variables DOES work, **but only after some fidgeting.**

(note: in all these situations i've ignored the initial conditions ϕ and ψ. do the boundary values for the ϕ and ψ need to satisfy the boundary conditions we set forth? it turns out that they do not, for reasons that i won't discuss.)

(more footnotes) [2] [3]

8.6 Completing the analogy bridge from linear algebra to PDE

Let's review the whole analogy we've been building up so far:

- Just as a matrix A is a linear operator, so is $-\Delta$.
- An eigenvector of A is v such that $Av = \lambda v$. An eigenvector of $-\Delta$ is a function v such that $-\Delta v = \lambda v$. In both cases λ is the associated eigenvalue.
- The identity $\langle Ax, y\rangle = \langle x, Ay\rangle$ characterizes A being symmetric. The identity $\int(-\Delta f)g = \int f(-\Delta g)$ characterizes $-\Delta$ being symmetric. In 1D this is the identity $\int f''g = \int fg''$.
- An incomplete list of boundary conditions that will ensure that $-\Delta$ is symmetric:
  - Zero Dirichlet boundary conditions
  - Zero von Neumann boundary conditions
  - Periodic boundary conditions

[2] I've told SO many lies. $-\Delta$ isn't even a valid operator because the only realistic space that you can consider it an endomorphism on is like C^∞, since otherwise the loss of regularity screws things up, like $-\Delta : C^2 \to C^2$ is just not true because the codomain will contain things that aren't C^2.

[3] The previous footnote is itself a lie because we're actually working in a Sobolev space and all derivatives are weak derivatives.

To complete the analogy, I will now translate the "separation of variables in linear algebra" protocol into what it actually looks like in PDE. Looking back, this protocol for linear algebra was:

1. Diagonalize A using a basis of orthonormal eigenvectors $v_1, \ldots, v_n$.
2. Write knowns and unknowns in terms of this basis.
3. Plug that into the equation we're solving.
4. Compare coefficients.
5. Win.

For PDE's separation of variables, this looks like the following.

1. Diagonalize $-\Delta$ using a basis of orthonormal eigenvectors *(...eigenfunctions)* $v_1(x), v_2(x), v_3(x), \ldots$. *(this time there are infinitely many.)* They are guaranteed to exist if we've ensured that $-\Delta$ is symmetric. [4]
2. Write the unknown function we're solving for, u, with respect to this basis. That is, we write
$$u = a_1 v_1 + a_2 v_2 + \ldots$$
and we must solve for the $\{a_i\}$'s. If there is a time, variable, then **the coefficients may depend on time.**
$$u(x,t) = a_1(t)v_1(x) + a_2(t)v_2(x) + \ldots$$
Analogously we need to write any other known functions in terms of this basis as well, such as the initial conditions and the forcing terms.
By the way, we call this a **Fourier series**! And the product $a_i(t)v_i(x)$ is the **separation of variables**, separating time and space. [5]
3. Plug all that into the PDE, as well as the PDE's initial condtions if there are any. Since we expanded in terms of nice functions $\{v_i(x)\}_i$, $-\Delta$ will act nicely on them and simplify a ton.
4. Compare coefficients. If the coefficients depend on time then this will unfortunately result in some differential equations that you must solve.

[4] This is a lie. Rigorously speaking we actually need to diagonalize the *inverse Laplacian*. Then, the reason why this theory works is because $-\Delta^{-1}$ is a *compact operator*. The extremely advanced reader should read Evans as a reference.

[5] A lot of authors *start* with this assumption of separation, and then add a bunch of such solutions to obtain the final answer. I don't like this because it doesn't give much insight as to when this methodology actually works.

5. We win!

That's it! It's honestly not that much more complicated than the linear algebra version, at least conceptually. The computations, however, might be kinda annoying.

8.7 Example 1

Example 8.1: Solve the forced heat equation on $[0,3]$ with Dirichlet boundary conditions,

$$\begin{cases} u_t - u_{xx} = xt \\ u(x,0) = \phi(x) := 1 \\ u(0,t) = 0 \\ u(3,t) = 0 \end{cases}$$

Solution. **Step 0: Make sure separation of variables actually works**

The boundary conditions are 0, so we are safe to proceed.

Step 1: Find the orthonormal basis

We seek the eigenfunctions of the $-\partial_{xx}$ operator over the space of functions that vanish on the boundary of $[0,3]$. So we to find the functions $v(x)$ and associated eigenvalues λ that satisfy the following ODE:

$$\begin{cases} -v''(x) = \lambda v(x) \\ v(0) = 0 \\ v(3) = 0 \end{cases}$$

You might have learned to solve this by finding the roots of the polynomial $x^2 + \lambda$. I'm a grad student so I'm going to find the solution by **guessing**: By experience, I know all solutions to $-v''(x) = \lambda v(x)$ are going to look like the following form:

$$v(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Now we solve for A, B, and λ. Plugging in $x = 0$ gives

$$0 = A + 0$$

so $A = 0$. So $v(x) = B\sin(\sqrt{\lambda}x)$. Plugging in $x = 3$ now gives

$$0 = B\sin(3\sqrt{\lambda}).$$

Eigenvectors can't be zero, so $B \neq 0$. Other than that, B can be anything, so this equation is actually a constraint on λ. Indeed, $\sin(3\sqrt{\lambda}) = 0$ iff $3\sqrt{\lambda} = n\pi$ iff $\boxed{\lambda = \frac{n^2\pi^2}{9}}$ for a positive

integer n. *(...and why do i ignore the case $n = 0$? why do i ignore $n < 0$?)* So

$$v(x) = B \sin\left(\frac{n\pi}{3}x\right).$$

Finally we pick B so that v has length 1, i.e. $\int_0^3 v(x)^2\,dx = 1$. *(remember, we want an orthonormal basis.)* Using my stupidity from the previous chapter, we know that $\int_0^3 \sin^2\left(\frac{n\pi}{3}x\right)\,dx = 3/2$. Therefore we pick $B = \sqrt{\frac{2}{3}}$. This gives the eigenvector, $\boxed{v(x) = \sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right)}$.

Conclusion: The eigenvectors are $\boxed{v_n(x) := \sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right)}$ for each positive integer n. The associated eigenvalue is $\boxed{\lambda_n := \frac{n^2\pi^2}{9}}$.

Step 2: Write everything in terms of this orthonormal basis.

Since the eigenvectors $\{v_n\}_n$ are sine functions, it just so happens that the expansions we write here are all going to be *sine Fourier series*.

We write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) v_n(x)$$

where the unknown coefficients $a_n(t)$ depend on time.

To write xt in terms of the $\{v_n\}_n$ basis, we need to expand x as a sine Fourier series (the t is just a constant). The intuition from the previous chapter makes that pretty easy: Since we have an orthonormal basis, the coefficient of v_n will be given by the inner product

$$\langle x, v_n(x)\rangle = \int_0^3 x\sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right)\,dx.$$

I plugged this into Wolfram Alpha and got $\frac{-3\sqrt{6}(-1)^n}{\pi n}$. Thus

$$xt = \sum_{n=1}^{\infty} \frac{-3\sqrt{6}(-1)^n t}{\pi n} v_n(x).$$

Lastly there's the initial condition $\phi(x) = 1$. We do the same thing: The v_n coefficient for the series for ϕ is

$$\langle 1, v_n(x)\rangle = \int_0^3 1 \cdot \sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right)\,dx.$$

I plugged this into Wolfram Alpha and got $\frac{-\sqrt{6}((-1)^n-1)}{\pi n}$. So

$$1 = \sum_{n=1}^{\infty} \frac{-\sqrt{6}((-1)^n - 1)}{\pi n} v_n(x).$$

Step 3: Plug everything into the PDE

Plugging into $u_t - u_{xx} = xt$, we get

$$\sum_{n=1}^{\infty} a_n'(t)v_n(x) + \sum_{n=1}^{\infty} a_n(t)(-v_n''(x)) = \sum_{n=1}^{\infty} \frac{-3\sqrt{6}(-1)^n t}{\pi n} v_n(x),$$

and since $-v_n'' = \lambda_n v_n$ *(this is the whole point of separation of variables!!!)* this simplifies to

$$\sum_{n=1}^{\infty} a_n'(t)v_n(x) + \sum_{n=1}^{\infty} \lambda_n a_n(t)v_n(x) = \sum_{n=1}^{\infty} \frac{-3\sqrt{6}(-1)^n t}{\pi n} v_n(x).$$

Merging the two series on the left gives

$$\sum_{n=1}^{\infty} [a_n'(t) + \lambda_n a_n(t)]v_n(x) = \sum_{n=1}^{\infty} \frac{-3\sqrt{6}(-1)^n t}{\pi n} v_n(x). \qquad (*)$$

Setting this aside, we now plug into the initial condition $u(x,0) = 1$. This gives

$$\sum_{n=1}^{\infty} a_n(0)v_n(x) = \sum_{n=1}^{\infty} \frac{-\sqrt{6}((-1)^n - 1)}{\pi n} v_n(x). \qquad (**)$$

Now we play with these two equations.

Step 4: Compare coefficients

In equation $(*)$, comparing the nth coefficients gives

$$a_n'(t) + \lambda_n a_n(t) = \frac{-3\sqrt{6}(-1)^n t}{\pi n}.$$

This is a differential equation for $a_n(t)$! To solve it properly we need an initial condition. This is given by comparing the nth coefficients in equation $(**)$, which gives

$$a_n(0) = \frac{-\sqrt{6}((-1)^n - 1)}{\pi n}.$$

If this looks revolting to solve, that's because it kinda is. In your scratchwork you should replace the ugly expressions with a variable, such as $b_n := \frac{-3\sqrt{6}(-1)^n}{\pi n}$ and $c_n := \frac{-\sqrt{6}((-1)^n-1)}{\pi n}$. Then we have the ODE,

$$\begin{cases} a_n'(t) + \lambda_n a_n(t) = b_n t \\ a_n(0) = c_n \end{cases}.$$

To solve, I multiply by $e^{\lambda_n t}$, giving

$$a_n'(t)e^{\lambda_n t} + \lambda_n a_n(t)e^{\lambda_n t} = b_n t e^{\lambda_n t}.$$

The left side simplifies by the product rule,

$$\left(a_n(t)e^{\lambda_n t}\right)' = b_n t e^{\lambda_n t},$$

and now we may integrate both sides from 0 to t to get

$$a_n(t)e^{\lambda_n t} - a_n(0) = \int_0^t b_n s e^{\lambda_n s}\, ds = \frac{b_n(e^{\lambda_n t}(\lambda_n t - 1) + 1)}{\lambda_n^2}.$$

(yeah i plugged that into lil' Wolfy, i can't be bothered to compute that.)

Rearranging,

$$a_n(t) = c_n(t)e^{-\lambda_n t} + \frac{b_n}{\lambda_n^2} \cdot \left(\lambda_n t - 1 + e^{-\lambda_n t}\right).$$

Plugging in the values of b_n, c_n, and λ_n gives

$$a_n(t) = \frac{-\sqrt{6}((-1)^n - 1)}{\pi n} e^{-\frac{n^2\pi^2}{9}t} + \frac{-3\sqrt{6}(-1)^n}{\pi n} \cdot \frac{81}{n^4\pi^4} \cdot \left(\frac{n^2\pi^2}{9}t - 1 + e^{-\frac{n^2\pi^2}{9}t}\right).$$

oh god, where's the vomit bucket, i — i gotta puke — BLEGHHHHHHHH.

Step 5: We win

That's it, the unknowns have been solved for. Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)v_n(x)$$

$$= \boxed{\sum_{n=1}^{\infty} \left[\frac{-\sqrt{6}((-1)^n - 1)}{\pi n} e^{-\frac{n^2\pi^2}{9}t} + \frac{-3\sqrt{6}(-1)^n}{\pi n} \cdot \frac{81}{n^4\pi^4} \cdot \left(\frac{n^2\pi^2}{9}t - 1 + e^{-\frac{n^2\pi^2}{9}t}\right)\right] \cdot \sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right)}$$

■

Ugh, that was disgusting! We should check our work to make sure we didn't mess up. So, I graphed the solution we got in Desmos: https://www.desmos.com/calculator/k0zzmr10mf It seems to check out! That means the solution is probably correct.

8.8 Example 2

Example 8.2: Solve the wave equation on $[0, \pi]$ with von Neumann boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = \phi(x) := 1 \\ u_t(x,0) = \psi(x) := \cos(3x) \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \end{cases}$$

I don't have much free time so I'll only do some of the steps.

Solution. **Step 0: Make sure separation of variables actually works**

Zero von-Neumann boundary conditions is one of the criteria that makes sure everything works, so we may proceed.

Step 1: Find the orthonormal basis

So we want $-v''(x) = \lambda v(x)$ and for v to satisfy the boundary conditions given in the PDE, which are $v'(0) = 0$ and $v'(\pi) = 0$.

All solutions to $-v'' = \lambda v$ take the form

$$v(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Differentiating gives

$$v'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

Plugging in $x = 0$ gives

$$0 = B\sqrt{\lambda},$$

so either $B = 0$ or $\lambda = 0$. Quickly checking if $\lambda = 0$ is actually an eigenvalue, we find that this corresponds to the eigenvector $v(x) = 1$, or when scaled, $\boxed{v_0(x) := 1/\sqrt{\pi}}$. *(we didn't get this solution last time because the 0 function is not an eigenvector.)*

If otherwise $\lambda \neq 0$ then we can conclude $B = 0$. So

$$v'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x),$$

and plugging in $x = \pi$ we find that $\sin(\sqrt{\lambda}\pi) = 0$. This happens exactly when $\sqrt{\lambda}$ is an integer, so $\lambda = n^2$ for n a positive integer n.

This eigenvalue corresponds to the eigenvector $v(x) = A\cos(nx)$. It remains to choose A so that $v(x)$ has length 1. We choose $A = \sqrt{2/\pi}$. Thus our eigenvectors are

$$\boxed{v_0(x) := \frac{1}{\sqrt{\pi}}, \qquad v_n(x) := \sqrt{\frac{2}{\pi}}\cos(nx)}$$

for positive integers n, and the corresponding eigenvalues are

$$\boxed{\lambda_0 := 0, \qquad \lambda_n := n^2}.$$

Step 2: Write everything in terms of this orthonormal basis

For this problem I intentionally chose ϕ and ψ to be super easy to expand in this series (which, by the way, will be *Fourier cosine series*!). There's also no forcing term, so that's one less thing you have to expand in this series.

Step 3: Plug everything into the PDE

It's gonna be the same deal as in the first example. But remember that we have two initial conditions, one with the ϕ and one with the ψ. You have to plug into both of these as well, otherwise you don't have enough info to do the next step.

Step 4: Compare coefficients

Here you will get a second-order ODE for the unknown coefficients $a_n(t)$. Fortunately the lack of a forcing term means it's not too complicated, so the general solution should be guessable.

If you're rusty on second-order ODE, brush up by Googling around, I'm sure it's covered quite well by many a mathtuber.

Step 5: Win

It's not so bad! ■

8.9 Example 3

Example 8.3: Solve the heat equation on $[0, \pi]$ with periodic boundary conditions,

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = \phi(x) := \sin x \\ u_t(x,0) = \psi(x) := \cos(3x) \\ u(0,t) = u(\pi,t) \\ u_x(0,t) = u_x(\pi,t) \end{cases}$$

Solution. **Step 0: Make sure separation of variables actually works**

I mentioned before that periodic boundary conditions is an example of a condition that makes $-\partial_{xx}$ symmetric, so we are safe to proceed.

Step 1: Find the orthonormal basis

So we want $-v''(x) = \lambda v(x)$ and for v to satisfy the boundary conditions given in the PDE, which are $v(0) = v(\pi)$ and $v'(0) = v'(\pi)$.

All solutions to $-v'' = \lambda v$ take the form

$$v(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Plugging in $x = 0$ and $x = \pi$ gives

$$A = A\cos(\sqrt{\lambda}\pi) + B\sin(\sqrt{\lambda}\pi). \qquad (*)$$

Well, that's not enough information, so now let's find $v'(x)$. Differentiating gives

$$v'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

Plugging in $x = 0$ and $x = \pi$ now gives

$$B\sqrt{\lambda} = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi).$$

$\lambda = 0$ will give an eigenvector *(check that it's nonzero!)*, so this is an eigenvalue. Now assume $\lambda \neq 0$. Then, we can divide to get

$$B = -A\sin(\sqrt{\lambda}\pi) + B\cos(\sqrt{\lambda}\pi). \qquad (**)$$

The equations $(*)$ and $(**)$ form a system of equations in the unknowns $\sin(\sqrt{\lambda}\pi)$ and $\cos(\sqrt{\lambda}\pi)$. If you solve this you will find that $\sin(\sqrt{\lambda}\pi) = 0$ and $\cos(\sqrt{\lambda}\pi) = 1$. This is satisfies exactly when $\sqrt{\lambda}$ is an even integer, so $\boxed{\lambda_n := 4n^2}$ for positive integers n.

Small issue: What are A and B? There is literally no more information so we can't solve for them. That's actually fine: This means that the eigenvalue $\lambda_n = 4n^2$ has *a multiplicity of 2*, meaning that now it corresponds to two eigenvectors. One is $\sin(2n\pi)$ and the other is $\cos(2n\pi)$. *(sadly we technically have to check that these are orthogonal since this isn't guaranteed by anything we've discussed... but i mean, they look orthogonal enough, so...)* Thus our orthonormal basis (before scaling) will consist of these functions, and the constant function 1.

Step 2: Write everything in terms of this orthonormal basis

Same as usual, this time we're now doing a *sine-cosine Fourier series*! See the previous chapter. Your unknowns are now going to be a bunch of functions $a_n(t)$ and a bunch of functions $b_n(t)$.

Step 3: Plug everything into the PDE

Same as usual.

Step 4: Compare coefficients

Same as usual.

Step 5: Win

It's really not so bad! ■

In your homework you will encounter different boundary conditions than the ones I've discussed here. They won't correspond perfectly to any of the Fourier series we've covered in this notes, but it's still "a" Fourier series, and it's guaranteed to work because the boundary conditions should ensure that $-\partial_{xx}$ is symmetric *(you should verify this!)*, causing all the theory to work down the line. Just follow the protocol.

9 More Fourier examples and convergence

9.1 An Example

Let's try to find the sine Fourier series of the function $f(x) = x$ over $(-\pi, \pi)$.

If you don't like the fact that the interval isn't of the form $(0, L)$, you can just shift everything over by π.

Ok, so for sine series I need to find all the sines that vanish at the endpoints. Since the interval starts at $-\pi$ my sines should look like $\sin(k(x+\pi))$.

The first frequency that vanishes at the endpoints is going to be the first k for which $\sin(k(\pi+\pi)) = 0$, and that's $k = 1/2$. Now all the other frequencies will be just that but with an n tacked in front:

$$\sin\left(\frac{n}{2}(x+\pi)\right)$$

Now I scale this so that v_n has length 1.

$$v_n(x) = \frac{1}{\sqrt{\pi}}\sin\left(\frac{n}{2}(x+\pi)\right)$$

So our Fourier series takes the form

$$f(x) = \sum_{n=1}^{\infty} a_n v_n(x).$$

The coefficient a_n is given by the inner product $\langle f, v_n\rangle$, which is

$$a_n = \int_{-\pi}^{\pi} f(x)v_n(x)\,dx = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{\pi}}\sin\left(\frac{n}{2}(x+\pi)\right).$$

After integrating by parts once, we end up with

$$a_n = \frac{2\sqrt{\pi}}{n} \cdot (-1 - \cos(n\pi)).$$

Thus our Fourier series is

$$\boxed{x = \sum_{n=1}^{\infty} \frac{2}{n}(-1-\cos(n\pi))\sin\left(\frac{n}{2}(x+\pi)\right).}$$

We also found the cosine series after that.

9.2 Plugging stuff in?

Food for thought: If we were to plug something into the equation we just got, like $x = 1$, to get

$$1 \stackrel{?}{=} \sum_{n=1}^{\infty} \frac{2}{n} (-1 - \cos(n\pi)) \sin\left(\frac{n}{2}(1+\pi)\right),$$

can we say that this new equality is true?

You might think, duh it has to be. But it's not that simple. In the Fourier series we got, which is

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1 - \cos(n\pi)) \sin\left(\frac{n}{2}(x+\pi)\right),$$

you should think of the equals sign here as being only "mostly" true, and potentially dishonest. That's because we think of the infinite sum here as summing a bunch of *functions*, with the infinite sum *converging* to another function in some sense that isn't necessarily a pointwise convergence.

In short, the infinite sum is weird and so the "equality" we get from Fourier series might not be a "true equality" at every value of x.

That's why we care about various modes of convergence for Fourier series, in particular pointwise convergence, uniform convergence, and L^2 convergence. *(people also care about L^p convergence and some weirder ones but that's out of the scope of this class.*

Here, is we want to know whether we can plug in $x = 1$ to get a true equality of numbers, we need to know if the Fourier sine series converges pointwise. So we must use one of the convergence criteria discussed in class.

For example, we know that if the function f we're Fourier'ing is differentiable at a point $x = a$, then the Fourier series of f evaluated at $x = a$ will converge to $f(a)$.

That's a condition for convergence at a single point. More globally speaking: If f is a differentiable function, then its Fourier series will converge *pointwise* to f.

In our case, the f is $f(x) = x$ over $(-\pi, \pi)$. This is definitely differentiable at $x = 1$, so the equality

$$1 = \sum_{n=1}^{\infty} \frac{2}{n} (-1 - \cos(n\pi)) \sin\left(\frac{n}{2}(1+\pi)\right)$$

will be true. In fact, since x is differentiable everywhere in $(-\pi, \pi)$, we have that

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1 - \cos(n\pi)) \sin\left(\frac{n}{2}(x+\pi)\right)$$

will be a true equation for every value of x we plug in that's (strictly) between $-\pi$ and π.

10 More separation of variables...?

We pretty much just reviewed the separation of variables protocol as described in the previous sections.

One thing I'd like to review, though, is how one solves a second-order ODE such as

$$f''(x) - 3f'(x) + 2f(x) = 10,$$

which may come up in solving problems. First, you let D be the "differentiation operator", so that the above can be written as

$$(D^2 - 3D + 2)f = 10,$$

and now you "factor" the "D polynomial" to get

$$(D - 2)(D - 1)f = 10$$

(which is hand-wavy and weird, but can be made entirely rigorous!).

Now we can let $g = (D - 1)f$, i.e. $g(x) = f'(x) - f(x)$, and now this gives

$$(D - 2)g = 10$$

i.e. $g'(x) - 2g(x) = 10$.

So we've turned the 2nd-order ODE into two 1st-order ODEs which are both very solvable. First we solve the one in terms of g.

$$g'(x) - 2g(x) = 10$$

$$g'(X)e^{-2x} - 2g(x)e^{-2x} = 10e^{-2x}$$

$$\frac{d}{dx}\left(g(x)e^{-2x}\right) = 10e^{-2x}$$

$$g(x)e^{-2x} - g(0) = \int_0^x 10e^{-2t}\,dt = 1 - 5e^{-2x}$$

$$g(x) = C_1e^{2x} - 5$$

Here I let $C_1 = g(0) + 1$.

Now we solve for f

$$g(x) = f'(x) - f(x)$$

$$C_1e^{2x} - 5 = f'(x) - f(x)$$

$$C_1e^{x} - 5e^{-x} = e^{-x}f'(x) - e^{-x}f(x)$$

$$C_1 e^x - 5e^{-x} = \frac{d}{ddx}\left(f(x)e^{-x}\right)$$

$$\int_0^x C_1 e^t - 5e^{-t}\, dt = f(x)e^{-x} - f(0)$$

$$C_1 e^x + 5e^{-x} + C_2 = f(x)e^{-x}$$

$$f(x) = C_1 e^{2x} + C_2 e^x + 5$$

Yay.

11 The Fourier Transform

The way I define the Fourier transform is different. In particular I use the definition

$$\hat{f}(k) := \int_{\mathbb{R}} f(x)e^{-2\pi ixk}\,dx.$$

However, pretty much all the properties are the same, sometimes differing only by a constant factor from the properties covered in lecture.

The Fourier transform may also be expressed as an *operator*. Often you see $\mathcal{F}$. But I like feeling fancy and morally superior to everyone, so I like to use $\mathscr{F}$, so

$$\mathscr{F}(f)(k) := \hat{f}(k).$$

(a bunch of pure math people like me also like using ξ as the frequency variable instead of k, so i'm already taking 200 psychic damage with this concession)

See question 33 of my survey (https://cims.nyu.edu/~tjl8195/survey/results.html#q33) to see just how much disagreement there is on what the Fourier transform is. Morally, though, they are pretty much all the same. *(mine is totally better though ;))*

11.1 Properties

11.1.1 Linearity

The Fourier transform is linear, with

$$\widehat{f+g} = \hat{f} + \hat{g}.$$

Similarly $\widehat{(af)} = a\hat{f}$ where a is a constant.

11.1.2 Differentiation turns into multiplication by ik

We have

$$\widehat{f'}(k) = 2\pi ik\hat{f}(k).$$

(any extra factors of 2π that you didn't see in lecture are because we define the fourier transform differently.)

This is a fantastic property: It is the main reason why Fourier analysis is so useful for studying PDE. It makes derivatives go away! That's fantastic.

Of course, this means that even *multiple derivatives* will go away:

$$\widehat{f''}(k) = (2\pi ik)^2 \hat{f}(k) = -4\pi^2 k^2 \hat{f}(k)$$

11.1.3 Multiplication turns into convolution, and convolution turns into multiplication

Recall that the *convolution* is

$$(f \star g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy \left(= \int_{\mathbb{R}} f(y)g(x-y)\,dy \right).$$

literally everyone else uses the asterisk, i.e. $f * g$, to denote convolution. i'm in the minority this time! see Q91 of my dumb survey.

The way Fourier transform reacts to a *product* is by turning it into a convolution:

$$\widehat{fg} = \hat{f} \star \hat{g}$$

Typically this is not very nice, which is why things can get a bit complicated when you want to Fourier a product. But such difficulties are inevitable.

Conversely, sometimes you might want to do the opposite, i.e. Fourier'ing a *convolution*. In this case, this is lovely: Convolution turns into products!

$$\widehat{f \star g} = \hat{f}\hat{g}$$

This situation doesn't happen that often, but when it does, it feels *amazing*.

11.1.4 Transformations

Given that we know that the Fourier transform of f is $\hat{f}$, we often are interested in finding the Fourier transform of related expressions such as $f(x-a)$ and $f(x/b)$. In other words, if we "shift" time and/or "dilate" time, what does this do to the *frequencies* of f?

Translation Property:

$$\widehat{f(x-a)} = e^{-2\pi ika}\hat{f}(k).$$

I write it like this for visual appeal, but it's not very mathematically proper. To be more precise, I should write it like this: If $g(x) = f(x-a)$, then

$$\hat{g}(k) = e^{-2\pi ika}\hat{f}(k).$$

(note: in lecture it might look like e^{-ika} instead.)

The proof of this property is to essentially plug g into the Fourier transform and use a u-substitution to massage it into something in terms of $\hat{f}$.

$$\begin{aligned}
\hat{g}(k) &= \int_{\mathbb{R}} g(x)e^{-2\pi ikx}\,dx \\
&= \int_{\mathbb{R}} f(x-a)e^{-2\pi ikx}\,dx \\
&= \int \mathbb{R}f(u)e^{-2\pi ik(u+a)\,du}\,du && (u = x-a) \\
&= e^{-2\pi ika}\int_{\mathbb{R}} f(u)e^{-2\pi iku}\,du \\
&= e^{-2\pi ika}\int_{\mathbb{R}} f(x)e^{-2\pi ikx}\,dx && \text{(Rename } u \text{ to } x \text{ for visual appeal)} \\
&= e^{-2\pi ika}\hat{f}(k)
\end{aligned}$$

Scaling Property: If $g(x) = f(x/b)$, where $b > 0$ is a constant, then

$$\hat{g}(k) = b\hat{f}(bk).$$

In other words, if I dilate time by slowing it down by a factor of b, then in the frequency world, we actually *speed up* by a factor of b, **and** moreover we *blow up* vertically by a factor of b.

The proof of this property is very similar to the previous proof: Plug g into the Fourier transform and do a u-substitution.

$$\begin{aligned}
\hat{g}(k) &= \int_{\mathbb{R}} g(x)e^{-2\pi ikx}\,dx \\
&= \int_{\mathbb{R}} f(x/b)e^{-2\pi ikx}\,dx \\
&= \int_{\mathbb{R}} f(u)e^{-2\pi ik(bu)\cdot b}\,du && (u = x/b,\ du = \tfrac{1}{b}dx) \\
&= b\int_{\mathbb{R}} f(u)e^{-2\pi i(bk)u}\,du \\
&= b\int_{\mathbb{R}} f(x)e^{-2\pi i(bk)x}\,dx \\
&= b\hat{f}(bk)
\end{aligned}$$

11.1.5 The Inverse Fourier Transform

I define the *Inverse Fourier Transform* as

$$\mathscr{F}^{-1}(f)(x) =: \check{f}(x) = \int_{\mathbb{R}} f(k)e^{2\pi ikx}\,dk.$$

I want you to notice that this *basically* looks like the Fourier transform. Therefore, most of the properties should carry over, there just might be a difference in sign. For example:

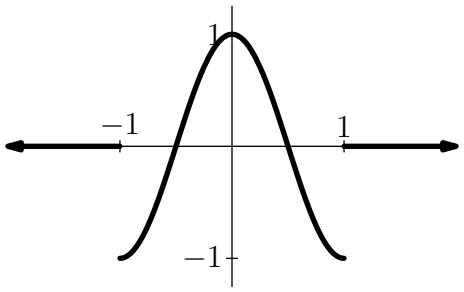
- $\widetilde{(f+g)} = \check{f} + \check{g}$
- $\widetilde{(f')}(x) = -2\pi i x \check{f}(x)$
- $\widetilde{(f \star g)} = \check{f}\check{g}$ and $\widetilde{(fg)} = \check{f} \star \check{g}$

11.2 Example 1: Cosine wave but cut off

You saw in lecture that the Fourier of a cosine or sine function is weird, entering the mysterious territory of *tempered distributions*. We will try our best not to enter this territory unless we have to, since it's impossible to talk about it rigorously with the tools we currently have.

Ok, but what if we take only *part* of a cosine wave, such as

$$f(x) := \cos(2\pi \cdot \tfrac{1}{2}x) \cdot 1_{[-1,1]}(x)?$$



That is, we only take the part of this cosine wave between -1 and 1, and make it 0 everywhere else. Then, this is *integrable* enough to have a Fourier transform.

Let's compute it. We have

$$\hat{f}(k) = \int_{\mathbb{R}} \cos(\pi x) 1_{[-1,1]}(x) e^{-2\pi i k x}\, dx.$$

The indicator function simply changes the bounds of the integration, so this is

$$= \int_{-1}^{1} \cos(\pi x) e^{-2\pi i k x}\, dx.$$

There are several ways to evaluate this. You can try integrating by parts twice, and you can also try using Euler's formula on the exponential term and reduce to playing with integrals of products of trig functions. But then you'd have to be good at trig identities. Instead, let's try using Euler's formulae in *reverse*! Given that

$$e^{ix} = \cos x + i \sin x,$$

we can find the formulas,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

which are *super* nice when we don't want to deal with trig functions, which is honestly like 90% of the time.

Here let's use the cosine formula. This gives

$$= \int_{-1}^{1} \frac{e^{i\pi x} + e^{-i\pi x}}{2} \cdot e^{-2\pi i k x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{(1-2k)\pi i x}\, dx + \frac{1}{2}\int_{-1}^{1} e^{(-1-2k)\pi i x}\, dx$$

$$= \frac{1}{2}\left(\frac{1}{(1-2k)\pi i} e^{(1-2k)\pi i x}\Big|_{x=-1}^{1} \right) + \frac{1}{2}\left(\frac{1}{(-1-2k)\pi i} e^{(-1-2k)\pi i x}\Big|_{x=-1}^{1} \right)$$

$$= \frac{e^{(1-2k)\pi i} - e^{-(1-2k)\pi i}}{2\pi(1-2k)i} + \frac{e^{(-1-2k)\pi i} - e^{(1+2k)\pi i}}{2\pi(1-2k)i}.$$

Reapplying Euler's formula again, we find that this is

$$\frac{\sin((1-2k)\pi)}{(1-2k)\pi} + \frac{\sin((-1-2k)\pi)}{(-1-2k)\pi}$$

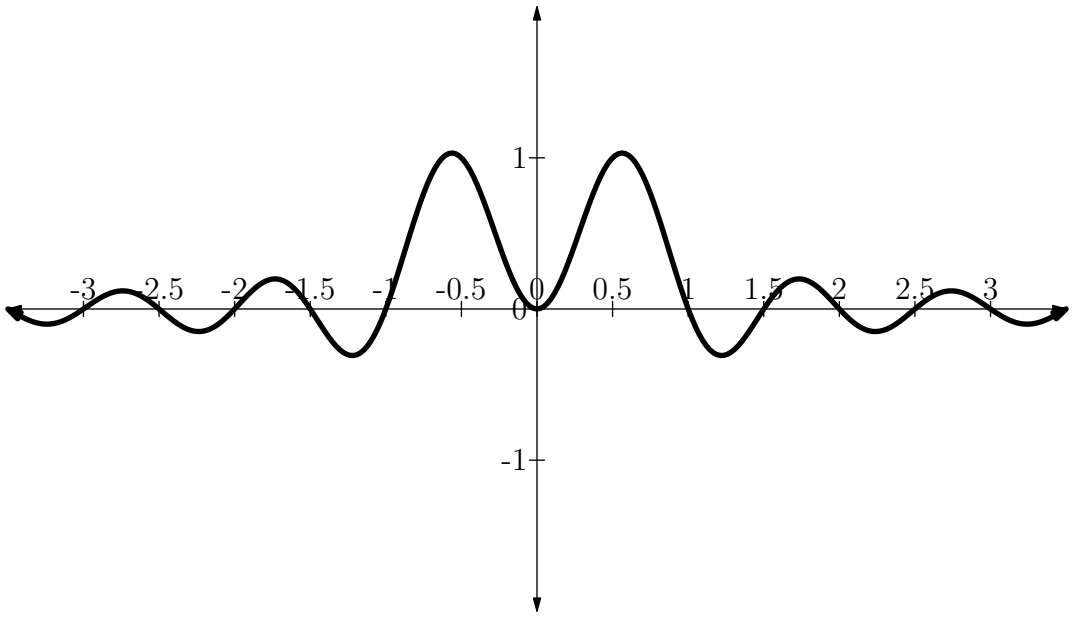
or

$$\boxed{\frac{\sin((1-2k)\pi)}{(1-2k)\pi} + \frac{\sin((1+2k)\pi)}{(1+2k)\pi}}.$$

Using the "sinc" function, which is just $\operatorname{sinc}(x) := \frac{\sin x}{x}$, this is often expressed as

$$\operatorname{sinc}((1-2k)\pi) + \operatorname{sinc}((1+2k)\pi).$$

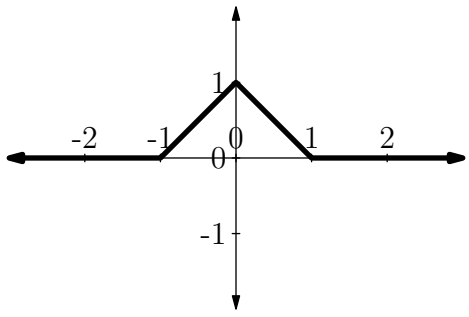
Let's plot what that looks like.



Notice that the two "peaks" here are at around $k = \pm\frac{1}{2}$, meaning that the Fourier Transform has detected that the original function $f(x)$ looks quite a bit like a wave with frequency $2\pi \cdot \frac{1}{2}$. And... that's definitely true! So the answer we got for $\hat{f}(k)$ matches the intuition for what the Fourier Transform is doing.

11.3 Example 2: Triangular Bump

Next, let's Fourier the function $f(x) := \begin{cases} 1+x, & -1 < x \leq 0 \\ 1-x, & 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$, which is nasty when written like that so stare at this picture instead.



You *could* just Fourier this manually and do it carefully, integrating by parts some number of times and getting a nasty expression to work with, *or* you can be a bit clever and use the following:

Hint: It turns out that $f = 1_{[-1/2,1/2]} \star 1_{[-1/2,1/2]}$.

(no, i don't expect you to come up with such magical observations)

If you didn't come to recitation, I'll let you figure out on your own why this is true *(there's a very intuitive reason if you truly understand convolution).* Now, since Fourier turns convolution into multiplication,

$$\hat{f} = \mathscr{F}(1_{[-1/2,1/2]} \star 1_{[-1/2,1/2]}) = \mathscr{F}(1_{[-1/2,1/2]}) \cdot \mathscr{F}(1_{[-1/2,1/2]}) = \mathscr{F}(1_{[-1/2,1/2]})^2.$$

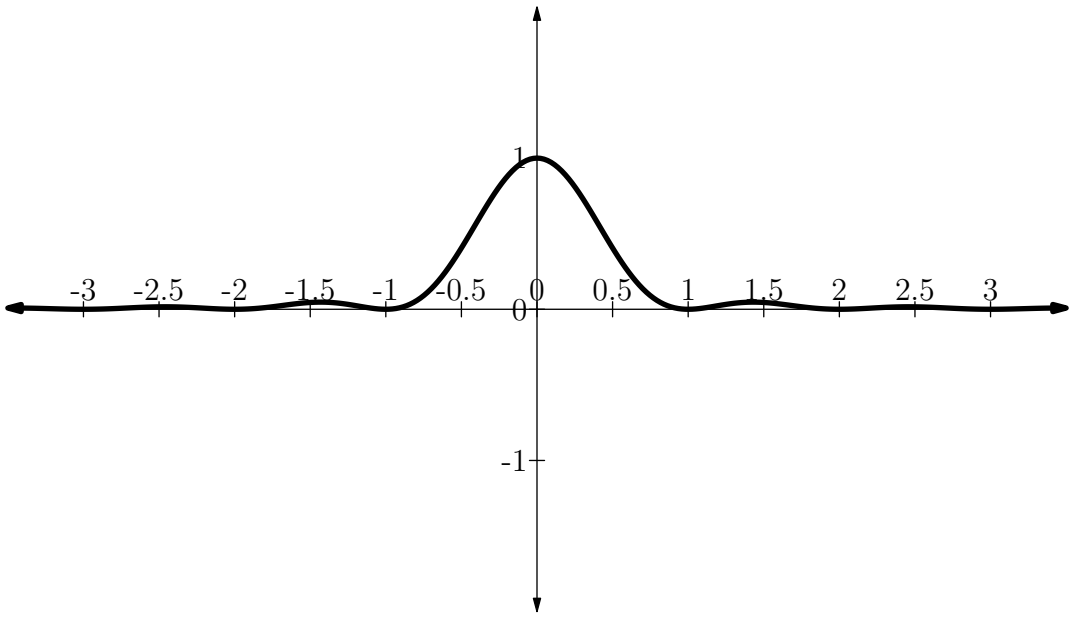
So we just have to Fourier $1_{[-1/2,1/2]}$, which is a "rectangular bump". You did something like this in class. So, if you trust me, you'll just get something like

$$\mathscr{F}(1_{[-1/2,1/2]})(k) = \operatorname{sinc}(\pi k).$$

Thus

$$\hat{f}(k) = \boxed{\operatorname{sinc}^2(\pi k)}.$$

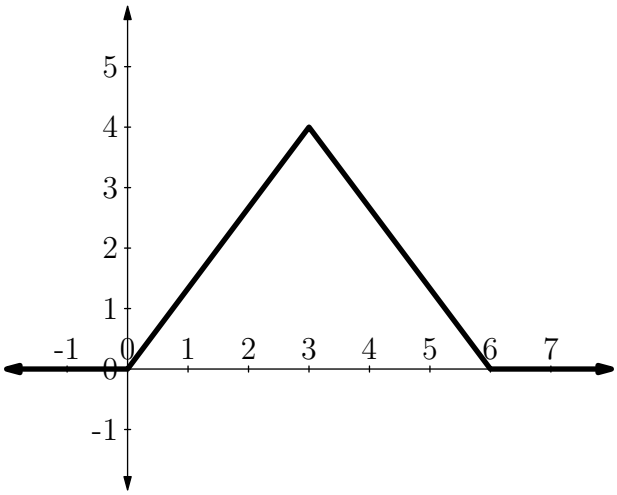
I'll plot what that looks like because I already wrote the code to generate these diagrams which I don't want to waste, but I don't think it's necessarily that enlightening.



I will note, though, that so far we've been getting pretty "lucky": All the Fourier transforms so far ended up being real-valued functions, which has made them pretty easy to draw. In general they won't be so nice. *(can you figure out why these two examples both ended up giving real-valued answers? it's actually not luck at all...)*

11.4 Example 3: Triangular Bump but transformed a bit

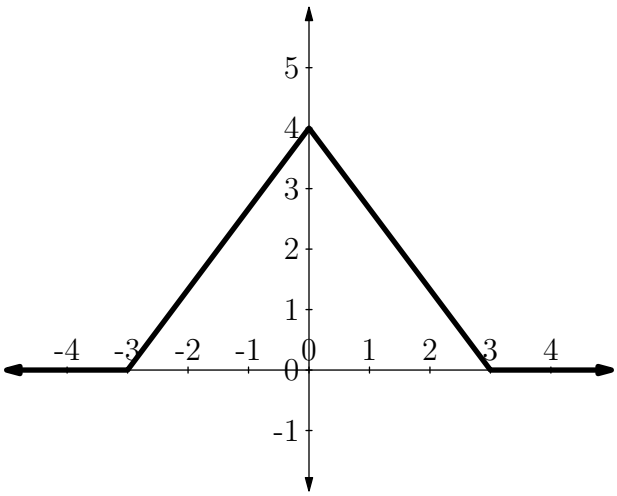
Now I want to Fourier this guy, which I'll call $h(x)$.



This looks a lot like the previous example, it's just translated and scaled. How can we use the answer to the previous example to quickly obtain $\hat{h}$?

Let's do this in two steps: First the scaling, then the translation. First, let's scale the $f(x)$ from the previous example to match the proportions of the new triangle.

We'll call the following graph $g(x)$.



$g(x)$ is just $f(x)$ scaled horizontally by a factor of 3, and scaled vertically by a factor of 4. So we can write g in terms of f as

$$g(x) = 4f(x/3).$$

Now

$$\hat{g}(k) = 4 \cdot \widehat{f(x/3)},$$

and by the "dilation" property discussed earlier, $\widehat{f(x/3)} = 3\hat{f}(3k)$. *(this notation is abusive for readability.)* thus

$$\hat{g}(k) = 12\hat{f}(3k).$$

Now let's get $\hat{h}$ in terms of $\hat{g}$. h is just g shifted to the right by 3 units. That is, $h(x) = g(x-3)$. Therefore, by the translation property discussed earlier,

$$\hat{h}(k) = e^{-2\pi ik\cdot 3}\hat{g}(k),$$

and plugging in $\hat{g}$ in terms of $\hat{f}$ gives

$$\hat{h}(k) = e^{-6\pi ik} \cdot 12\hat{f}(3k) = \boxed{e^{-6\pi ik} \cdot 12\,\mathrm{sinc}^2(3\pi k)}.$$

This time the answer is not a real-valued function so I won't try to graph it.

11.5 Solving the Heat Equation using the Fourier Transform

Let's solve our old friend,

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

using Fourier analysis.

As a rule of thumb, your Fourier transforms in these situations are *always going to be in the spatial variable*. In this case, we take the Fourier transform in the x variable. Let's Fourier each side of the PDE.

$$\mathscr{F}(u_t - u_{xx}) = 0$$
$$\mathscr{F}(u_t) - \mathscr{F}(u_{xx}) = 0$$

First Term: The derivative on u is a *time derivative*, whereas the Fourier Transform is in *space*, so these two guys actually don't interact at all. So one can actually "move the time derivative out", to see that $\mathscr{F}(u_t)$ is simply $\hat{u}_t$, the time derivative of $\hat{u}$.

Second Term: This time the derivatives on u are spatial, so they will interact with the Fourier Transform in accordance to the derivative property: Differentiation becomes multiplication by (a multiple of) ik.

$$\mathscr{F}(u_{xx}) = (2\pi ik)^2\hat{u} = -4\pi^2k^2\hat{u}.$$

In sum, the heat equation when converted to Fourier land is

$$\hat{u}_t + 4\pi^2k^2\hat{u} = 0.$$

This is just an ODE in the t variable, where we are solving for $\hat{u}$, and as far as t is concerned, the $4\pi^2k^2$ term is just a constant. So, by your favorite method, you can find that

$$\hat{u} = Ae^{-4\pi^2k^2t}$$

for a constant A. We should be careful though — A is a constant from t's perspective, but it definitely can depend on k. So, written more precisely,

$$\hat{u}(k,t) = A(k)e^{-4\pi^2k^2t}.$$

We can actually figure out what $A(k)$ is. Plugging in $t = 0$ gives $\hat{u}(k,0) = A(k)$. But, $\hat{u}(k,0) = \hat{u_0}(k)$ by Fourier'ing each side of the initial condition $u(x,0) = u_0(x)$. So $A(k) = \hat{u_0}(k)$.

$$\hat{u}(k,t) = \hat{u_0}(k)e^{-4\pi^2k^2t}$$

Finally, to get $u(x,t)$, we take the inverse Fourier Transform of both sides. Inverse Fourier, like Fourier, turns multiplication into convolution, thus

$$u(x,t) = \widetilde{\left(\hat{u_0}(k)e^{-4\pi^2k^2t}\right)} = \check{u_0} \star \left(\widetilde{e^{-4\pi^2k^2t}}\right) = u_0 \star \left(\widetilde{e^{-4\pi^2k^2t}}\right).$$

(any time im writing an equation that has both an x and a k, i'm kinda abusing notation. you should think of $\left(\widetilde{e^{-4\pi^2k^2t}}\right)$ as $\check{g}(x)$ where $g(k) = e^{-4\pi^2k^2t}$. a more technically correct way to write this in one single expression is as $\left(\widetilde{e^{-4\pi^2(\cdot)^2t}}\right)(x)$, which is an eyesore!)

It remains to figure out what $\left(\widetilde{e^{-4\pi^2k^2t}}\right)$ is.

To do this, let's take a little break and talk about Gaussians. In lecture, you found the Fourier transform of a Gaussian, which basically looks like e^{-x^2}, and you found (using some sly tricks that are quite nontrivial) that it's another Gaussian. Since *my* Fourier transform is defined differently, the key result that I'm used to is slightly different: It's that $e^{-\pi x^2}$ is its own Fourier transform,

$$\widehat{(e^{-\pi x^2})} = e^{-\pi k^2}.$$

Based on this, what is $\widehat{(ae^{-\pi bx^2})}$? *(assuming $b > 0$.)*

Written more properly, let $\phi(x) = e^{-\pi x^2}$. Let $\psi(x) = ae^{-\pi bx^2}$. Then since $\psi(x) = a\phi(x\sqrt{b})$, we have by using the various scaling properties that

$$\hat{\psi}(k) = \frac{a}{\sqrt{b}}\hat{\phi}(k/\sqrt{b}) = \frac{a}{\sqrt{b}}\phi(k/\sqrt{b}) = \frac{a}{\sqrt{b}}e^{\frac{-\pi}{b}k^2}.$$

In conclusion (written informally),

$$\widehat{(ae^{-\pi bx^2})} = \frac{a}{\sqrt{b}}e^{\frac{-\pi}{b}k^2}.$$

In fact, all properties used here work for the inverse Fourier transform as well, so we can also conclude that

$$\widetilde{\left(ae^{-\pi bk^2}\right)} = \frac{a}{\sqrt{b}} e^{\frac{-\pi}{b}x^2}.$$

Now let's go back to determining $\widetilde{\left(e^{-4\pi^2 k^2 t}\right)}$. Ah, well, we can just use the previous equation with $a = 1$ and $b = 4\pi t$. Thus

$$\widetilde{\left(e^{-4\pi^2 k^2 t}\right)} = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}.$$

(waiiiit a minute, this looks familiar...)

Finally finally, we have solved the heat equation,

$$u(x,t) = \left(u_0 \star \frac{1}{\sqrt{4\pi t}} e^{\frac{-(\cdot)^2}{4t}}\right)(x) = \int_{\mathbb{R}} u_0(y) \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-y)^2}{4t}}\, dy.$$

Holy cow, that's the heat solution we know and love!

12 The Laplace Equation

12.1 Integration by parts in multiple dimensions

Recall that

$$\int_a^b uv'\,dx = uv\,|_{x=a}^b - \int_a^b u'v\,dx.$$

You should think of this as the following:

1. Pick up a derivative from v, and evaluate the resulting integrand on the boundary.
2. Take that derivative you've been holding on to, give it to u, and integrate. Also, subtract this term.

For two functions $F : \Omega \to \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$, we can try the same thing for an integral such as

$$\int_\Omega F \cdot \nabla u\,dx.$$

Naively, if we were to steal the derivative from u, then we'd get Fu, which we "evaluate" on the boundary via a surface integral over ∂U. Then we give that derivative to F. So we expect something like

$$\int_\Omega F \cdot \nabla u\,dx \stackrel{??}{=} \int_{\partial\Omega} Fu\,dS - \int_\Omega (\text{derivative of } F)u\,dx.$$

As-is this doesn't make sense. To fix this:

- The integrand of the $\int_{\partial\Omega}$ is Fu which is a *vector*, not a number. To make it a number, recall that surface integrals often involve the unit outward normal $\vec{n}$. So a natural way to turn this vector into a number is to dot product with $\vec{n}$. The correct integrand indeed turns out to be $(F \cdot \vec{n})u$.
- We need to figure out the correct way to differentiate F. Since u is a number, and F is a vector, we want to find the type of derivative that takes a vector to a number. The *divergence* is perfect for that.

So the correct formulation of integration by parts in multiple dimensions is

$$\boxed{\int_\Omega F \cdot \nabla u\,dx = \int_{\partial\Omega} (F \cdot \vec{n})u\,dS - \int_\Omega (\operatorname{div} F)u\,dx}.$$

When the functions involved vanish on the boundary then we get something like

$$\int_\Omega F \cdot \nabla u \, dx = -\int_\Omega (\operatorname{div} F) u \, dx,$$

which you can think of as the only sensible way to move a derivative from u to F.

Examples

Assume all functions involved here vanish at the boundary term. Boundary terms are ugly.

When we steal a gradient and lend it to a function that already has a gradient, it becomes a *Laplacian*.

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega (-\Delta u) v \, dx$$

If we do this twice we get like:

$$\int_\Omega (\Delta u) v \, dx = \int_\Omega u (\Delta v) \, dx$$

(Notice no minus sign because we did it twice.)

12.2 Motivations for the Laplace equation

- Solutions to $-\Delta u = 0$ are **stable heat distributions**: If we recall that the heat equation is $u_t - \Delta u = 0$ then the stable (unchanging) solutions are those with $u_t = 0$, and these satisfy $-\Delta u = 0$.
- There is also a strong probabilistic connection: Let Ω be a domain, g a function on $\partial\Omega$ (boundary data). Pick a point $x \in \Omega$.

  Question: Start at x and "move around randomly" (this is *Brownian motion*) until you hit the boundary at a point y. What is the expected value of $g(y)$?

  Answer: The expected value is $u(x)$, where u satisfies the PDE

  $$\begin{cases} -\Delta u = 0 \\ u\,|_{\partial\Omega} = g \end{cases}.$$

  Thus we can think of a solution to the Laplace equation at some point as a weighted average of the boundary.

These motivations are very helpful for picturing what harmonic functions (solutions to $-\Delta u = 0$) look like.

12.3 Energy

In nature, things tend to take the path of least resistance, i.e. things want to minimize energy. This is yet another reason why the Laplace equation comes up a lot.

Theorem 12.1

Let Ω be a domain and g a function on $\partial\Omega$.
The following statements are equivalent:

1. u minimizes the energy
$$E[u] := \int_\Omega |\nabla u|^2 \, dx$$
over all functions $u \in C^2$ with $u = g$ on $\partial\Omega$.

2. u solves
$$\begin{cases} -\Delta u = 0 \\ u\,|_{\partial\Omega} = g \end{cases}.$$

Proof. **(1 implies 2)**

So we're assuming u minimizes the energy. Now let's kick u a little: Consider a "test function" v, which vanishes at the boundary, and consider the function $u + tv$.

Then the function
$$f(t) := E[u + tv] = \int_\Omega |\nabla(u + tv)|^2 \, dx$$
is minimizes when $t = 0$. So $f'(0) = 0$. Computing:

$$\begin{aligned} 0 = f'(0) &= \frac{d}{dt} \int_\Omega |\nabla(u + tv)|^2 \, dx \Big|_{t=0} \\ &= \frac{d}{dt} \int_\Omega |\nabla u|^2 + 2t\nabla u \cdot \nabla v + t^2 |\nabla v|^2 \, dx \Big|_{t=0} \\ &= \int_\Omega 2\nabla u \cdot \nabla v + 2t|\nabla v|^2 \, dx \Big|_{t=0} \\ &= \int_\Omega 2\nabla u \cdot \nabla v \, dx \end{aligned}$$

So, $\int_\Omega \nabla u \cdot \nabla v \, dx = 0$ for *all* v. Integrating by parts,

$$0 = \int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega (\Delta u) v \, dx$$

where the boundary term $\int_{\partial\Omega} (\nabla u \cdot \vec{n}) v \, dx$ vanishes because v vanishes at the boundary.

This can only be true for all v if $-\Delta u = 0$. So u solves the Laplace equation.

(2 implies 1)

Now we're assuming u solves the Laplace equation. We want to show that

$$\int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |\nabla w|^2 \, dx$$

for all C^2 functions w that $= g$ on the boundary. But I think it's easier to think about this as "taking u and kicking it", so I prefer to show the equivalent statement that

$$\int_\Omega |\nabla u|^2 \, dx \overset{?}{\leq} \int_\Omega |\nabla (u+v)|^2 \, dx$$

where $u + v$ is $= g$ on the boundary, i.e. v is $= 0$ on the boundary.

Expanding the right side, we want to show

$$\int_\Omega |\nabla u|^2 \, dx \overset{?}{\leq} \int_\Omega |\nabla u|^2 \ + 2\nabla u \cdot \nabla v + |\nabla v|^2 \, dx$$

or

$$0 \overset{?}{\leq} \int_\Omega 2\nabla u \cdot \nabla v + |\nabla v|^2 \, dx.$$

I haven't actually used the fact that $-\Delta u = 0$ yet, so let's use it now by screwing with that $\nabla u \cdot \nabla v$ term. We'll do that by integrating by parts.

$$\int_\Omega \nabla u \nabla v \, dx = \int_{\partial\Omega} (\nabla u \cdot \vec{n}) v \, dS - \int_\Omega (\Delta u) v \, dx,$$

and this is really nice — v vanishes on the boundary so $\int_{\partial\Omega} (\nabla u \cdot \vec{n}) v \, dS = 0$, and u satisfies the Laplace equation so $\int_\Omega (\Delta u) v \, dx = 0$. So this whole term is just zero. Now we just have to show

$$0 \overset{?}{\leq} \int_\Omega |\nabla v|^2 \, dx,$$

which is... kinda obvious. □

13 Just a bit more on energy methods

We reviewed the following:

- Energy methods (which I'll talk about here)
- Duhamel's Principle (which I did not explain well, so refer to Chapter 6 of these notes for a far-more organized overview)
- Method of characteristics, particularly of the form $au_t + bu_x + cu = 0$ (I'd like to stress that if you're doing it my way, the cu term does *not* change the methodology whatsoever! It just makes your answer more complicated because you'll find that u is no longer necessarily constant over the characteristics. Anyways I've written about this in detail in Chapter 1 of these notes, so do take a look!)

In Chapter 3 of these notes, I've discussed how one generally finds "energy results", and how one uses them to deduce things. What I didn't talk about in that chapter is how one uses them in the context of the Laplace equation. It's not all too different.

13.1 Using energy to obtain uniqueness to Poisson

Recall that the *Poisson equation* over a bounded domain Ω is given by

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u\,|_{\partial\Omega} = g \end{cases}$$

for a function f on Ω and a function g on $\partial\Omega$. (The specific case $f = 0$ is the Laplace equation.)

It turns out that if a solution to this exists, then it must be unique. To see this, take two solutions u and v. We want to show $u = v$. Equivalently, we want to show $u - v = 0$. But $u - v$ solves the "stupid PDE",

$$\begin{cases} -\Delta(u - v) = 0 & \text{in } \Omega \\ (u - v)\,|_{\partial\Omega} = 0 \end{cases}.$$

So now our job is to prove that the only solution w of the stupid PDE

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w\,|_{\partial\Omega} = 0 \end{cases}$$

is $w = 0$.

This is where energy comes in. Taking $-\Delta w = 0$, we multiply by w on each side to get

$$-w\Delta w = 0,$$

and now we integrate to get

$$\int_\Omega -w\Delta w \, dx = 0.$$

By integration by parts, $\int_\Omega -w\Delta w \, dx = \int_\Omega \nabla w \cdot \nabla w \, dx = \int_\Omega |\nabla w|^2 \, dx$ (and what allows us to do this is actually the fact that $w\,|_{\partial\Omega} = 0$.). So we actually have

$$\int_\Omega |\nabla w|^2 \, dx = 0.$$

Huh, it's very hard for the integral of |Something|^2 to be 0... unless the Something is zero. So it must be true that

$$\nabla w = 0.$$

In other words, the derivatives of w are zero, so w must be constant. Which constant? Well, I know $w = 0$ on the boundary, so the only possible constant function that w could be equal to is the zero function. Thus $w = 0$ everywhere.

That's what we wanted to show. Yay.

13.2 Using energy to reason about eigenvalues

Recall that an *eigenvalue* for the (negative) Laplacian on a domain Ω is a real number λ such that there is non-zero solution v to

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ v\,|_{\partial\Omega} = 0 \end{cases}.$$

We call v the *eigenfunction* associated with λ, which you can really think of as just being an eigenvector of $-\Delta$ because that's literally true.

(the rationale for requiring that these "eigenvectors" vanish on the boundary is subtle and i can't go too deep into it. the short explanation is that math generally sucks if you have non-zero boundary conditions.)

It turns out that eigenvalues must always be ≥ 0. To see this, we do the usual energy trick of multiplying by v to get

$$-v\Delta v = \lambda |v|^2,$$

and then integrating to get

$$\int_\Omega -v\Delta v \, dx = \lambda \int_\Omega |v|^2 \, dx.$$

The same trick as usual now — we integrate by parts (legal because v vanishes on the boundary!) to manipulate the left side, getting us that $\int_\Omega -v\Delta v\,dx = \int_\Omega \nabla v \cdot \nabla v\,dx$ which is just $\int_\Omega |\nabla v|^2\,dx$. So

$$\int_\Omega |\nabla v|^2\,dx = \lambda \int_\Omega |v|^2\,dx.$$

If you stare at this, the path to the proof becomes clearer: This *essentially* says that

$$\text{(Something non-negative)} = \lambda \cdot \text{(Something non-negative)},$$

which "should" tell us that λ has to be ≥ 0 because positive times positive equals positive.

That's basically the argument. There's just an annoying problem. If it happened to be the case that both of those intergrals came out to 0 then we actually get

$$0 = \lambda \cdot 0$$

meaning that in theory λ could be $= -100$ which would be bad. So we just have to rule out that this situation can't happen.

This situation happening means both sides of the $\int_\Omega |\nabla v|^2\,dx = \lambda \int_\Omega |v|^2\,dx$ equation are zero, in particular this means that $\int_\Omega |v|^2\,dx = 0$ *(the astute reader should check this logical leap carefully to make sure i'm not skipping any steps — i may or may not be)*, and that's only possible if $v = 0$.

So, this case is indeed ruled out because, if you remember linear algebra, we never consider 0 to be an eigenvector. Tada, we're done.

13.3 aight that's it from me

Good luck on the final!