Effect of Boundary Conditions on Second-Order Singularly-Perturbed Phase Transition Models on $\mathbb{R}$

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Abstract. The second-order singularly-perturbed problem concerns the integral functional $\int_\Omega [\varepsilon_n^{-1}W(u) + \varepsilon_n^3\|\nabla^2 u\|^2]\ dx$ for a bounded open set $\Omega \subseteq \mathbb{R}^N$, a sequence $\varepsilon_n \to 0^+$ of positive reals, and a function $W : \mathbb{R} \to [0, \infty)$ with exactly two distinct zeroes. This functional is of interest since it models the behavior of phase transitions, and its Gamma limit as $n \to \infty$ was studied by Fonseca and Mantegazza. In this paper, we study an instance of the problem for $N = 1$. We find a different form for the Gamma limit, and study the Gamma limit under the addition of boundary data.

Master's Thesis

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Chapter 1

Introduction

For a bounded open set $\Omega \subseteq \mathbb{R}^N$, we may imagine Ω as a container for a liquid whose density is given by $u : \Omega \to \mathbb{R}$. The potential energy of the liquid can be measured by the integral functional $u \mapsto \int_\Omega W(u)\, dx$ where $W : \mathbb{R} \to [0, \infty)$ is the energy per unit volume.

Suppose that W is a *two-well potential*, so that W has exactly two distinct zeroes z_1, z_2 with $z_1 < z_2$, which are the *phases* of the liquid. Then the liquid will tend to take on the two densities z_1 and z_2, and in particular its density will simply take the form $u = z_1 \cdot 1_E + z_2 \cdot 1_{\Omega \setminus E}$ in order to minimize the potential $\int_\Omega W(u)\, dx$.

Such rapid changes in density induce high *interfacial energy* between phases. To account for this, the Van der Waals-Cahn-Hilliard theory of phase transitions [11] [12] [14] [22] models the potential energy via an integral functional of the form

$$J_\varepsilon(u) := \int_\Omega \left[W(u) + \varepsilon^2 \|\nabla u\|^2 \right] \, dx, \qquad u \in W^{1,2}(\Omega). \tag{1.1}$$

The problem of interest is to minimize $J_\varepsilon(u)$ subject to a mass constraint $\int_\Omega u\, dx = m$. A minimizing function u_ε represents a stable density distribution that the liquid would likely conform to energetically.

What sort of stable density distribution is approached as $\varepsilon \to 0^+$? Specifically, suppose that we have a sequence $\varepsilon_n \to 0^+$ such that the sequence of minimizers u_{ε_n} converges to a function u in some reasonable sense. What properties must be satisfied by u?

Using Gamma convergence (described in Section 2.5), Modica [17] and Sternberg [21] independently proved that such a u satisfies $u \in \{z_1, z_2\}$ almost everywhere, and minimizes the perimeter in Ω of $u^{-1}(z_1)$. In particular, they prove that for each sequence of positive reals ε_n with $\varepsilon_n \to 0^+$, we have that

$$\Gamma\text{-}\lim_{n\to\infty}(J_{\varepsilon_n}/\varepsilon_n)(u) = \begin{cases} 2\displaystyle\int_{z_1}^{z_2} \sqrt{W(x)}\,dx\,\mathrm{Per}_\Omega(u^{-1}(z_1)), & u \in BV(\Omega;\{z_1,z_2\}) \\ +\infty, & \text{otherwise} \end{cases}$$

under $L^1(\Omega)$ convergence, where $\mathrm{Per}_\Omega(E)$ denotes the *perimeter* of a set $E \subseteq \Omega$, defined as

$$\mathrm{Per}_\Omega(E) := \sup\left\{\int_E \mathrm{div}\,\varphi : \varphi \in C_0^\infty(\mathbb{R}^N), |\varphi| \le 1\right\}.$$

See also [1], [2], [3], [10], [13], [19].

Owen, Rubinstein and Sternberg [20] studied the family of functionals J_ε under boundary conditions instead of a mass constraint. They prove that if Ω has C^2 boundary, $h_\varepsilon \in L^p(\partial\Omega) \cap L^\infty(\partial\Omega)$ is the trace of a function in $W^{1,2}(\Omega)$ for each $\varepsilon > 0$, $h_\varepsilon \to h \in L^1(\partial\Omega) \cap L^\infty(\partial\Omega)$ in $L^1(\partial\Omega)$ as $\varepsilon \to 0$, $\int_{\partial\Omega}\left|\frac{\partial h_\varepsilon}{\partial\sigma}\right|$ is bounded in ε and $\left\|\frac{\partial h_\varepsilon}{\partial\sigma}\right\|_{L^\infty(\partial\Omega)} \le C\varepsilon^{-1/4}$ for a constant $C > 0$, where σ is a surface parameter on $\partial\Omega$, and $K_\varepsilon : L^1(\Omega) \to \overline{\mathbb{R}}$ is defined as

$$K_\varepsilon := \begin{cases} \int_\Omega \left[\varepsilon^{-1}W(u) + \varepsilon\|\nabla u\|^2\right]\,dx, & u \in W^{1,2}(\Omega) \text{ and } \mathrm{Tr}\,u = h_\varepsilon \\ +\infty, & \text{otherwise} \end{cases}, \tag{1.2}$$

where $\overline{\mathbb{R}} := [-\infty, \infty]$, then

$$\begin{aligned}\Gamma\text{-}\lim_{n\to\infty} K_{\varepsilon_n}(u) \\ &= \begin{cases} \int_\Omega |\nabla\chi(u)| + \int_{\partial\Omega} |\chi(h(x)) - \chi(\mathrm{Tr}(u)(x))|\,d\mathcal{H}^{N-1}(x), & u \in BV(\Omega;\{z_1,z_2\}) \\ +\infty, & \text{otherwise} \end{cases},\end{aligned}$$

where

$$\chi(t) := 2\int_{z_1}^{t} \sqrt{W(z)}\,dz$$

and $\varepsilon_n \to 0^+$. See also [18].

Fonseca and Mantegazza [9] consider a second-order derivative. To be precise, they define

$$H_\varepsilon(u) := \int_\Omega \left[\varepsilon^{-1}W(u) + \varepsilon^3\|\nabla^2 u\|^2\right]\,dx \tag{1.3}$$

and proved that for every sequence of positive reals $\varepsilon_n \to 0^+$, we have

$$\Gamma\text{-}\lim_{n\to\infty} H_{\varepsilon_n}(u) = \begin{cases} c\,\mathrm{Per}_\Omega(u^{-1}(z_1)), & u \in BV(\Omega,\{z_1,z_2\}) \\ +\infty, & \text{otherwise} \end{cases}$$

under $L^1(\Omega)$ convergence, where

$$c := \min\left\{\int_{\mathbb{R}} \left[W(u) + |u''|^2\right]\,dt : u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}), \lim_{t\to-\infty} u(t) = z_1, \lim_{t\to\infty} u(t) = z_2\right\}. \tag{1.4}$$

For a treatment of more general functionals, see [5], [6], and [7].

Fonseca and Mantegazza's proofs appeal to rather sophisticated constructions and tools such as Young measures. In this paper, we consider a 1-dimensional instance of the problem solved by Fonseca and Mantegazza, which will allow for alternative and more elementary methodologies.

Our goal will ultimately be to combine the efforts of Owen, Rubinstein, and Sternberg with those of Fonseca and Mantegazza by considering the addition of boundary conditions as in (1.2) to the second-order problem as in (1.3), which will be possible due to our alternative methodologies.

Define

$$\Phi(u) := \left(\int_0^1 (u(y)^2-1)^2\,dy\right)^{3/4}\left(\int_0^1 |u''(y)|^2\,dy\right)^{1/4} \tag{1.5}$$

for all $u \in W^{2,2}(0,1)$. Define the families

$$\mathscr{J} := \{u \in W^{2,2}(0,1) : u(0_+) = -1, u(1_-) = 1, u'(0_+) = u'(1_-) = 0\} \tag{1.6}$$

and

$$\mathscr{J}_1(t) := \{u \in W^{2,2}(0,1) : u(0_+) = -1, u(1_-) = t, u'(0_+) = 0\} \tag{1.7}$$

for each $t \in \mathbb{R}$. Let

$$\alpha := \frac{2}{3^{3/4}} \inf_{u \in \mathscr{J}} \Phi(u) \tag{1.8}$$

and

$$\beta(t) := \frac{4}{3^{3/4}} \inf_{u \in \mathscr{J}_1(t)} \Phi(u) \tag{1.9}$$

for all $t \in \mathbb{R}$. Our main result is the following theorem.

Theorem 1.1 *Let $\Omega = (a,b)$. Let $a_0, b_0 \in \mathbb{R}$, and for each $\varepsilon > 0$, let $a_\varepsilon, b_\varepsilon \in \mathbb{R}$ be such that $a_\varepsilon \to a_0$ and $b_\varepsilon \to b_0$ as $\varepsilon \to 0^+$. For each $\varepsilon > 0$ define a functional $G_\varepsilon(u) : L^2(\Omega) \to \overline{\mathbb{R}}$ via*

$$G_\varepsilon(u) := \begin{cases} \int_\Omega \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right] dx, & u \in W^{2,2}(\Omega), u(a_+) = a_\varepsilon, u(b_-) = b_\varepsilon \\ +\infty, & \text{otherwise} \end{cases}.$$

Let $\varepsilon_n \to 0^+$ be a sequence of positive reals. Then, under strong $L^2(\Omega)$ convergence, we have that

$$\Gamma\text{-}\lim_{n\to\infty} G_{\varepsilon_n}(u)$$
$$= \begin{cases} \alpha \operatorname{essVar}_\Omega u + \beta(-a_0 \operatorname{sgn} u(a_+)) + \beta(-b_0 \operatorname{sgn} u(b_-)), & u \in BPV(\Omega; \{-1,1\}) \\ +\infty, & \text{otherwise} \end{cases}.$$

Here, essVar denotes *essential variation* and $BPV(\Omega)$ denotes the space of functions with bounded pointwise variation, both of which are described in Section 2.2. The addition of boundary conditions is of particular interest here because they can ensure at least one phase transition by preventing the existence of trivial minimizers.

The Gamma convergence result with respect to the metric in $L^2(\Omega)$ is justified by the following compactness result.

Theorem 1.2 (Compactness for Second Order Problem) *Let* $\Omega := (a,b)$. *For each* $\varepsilon > 0$, *define the functional* $F_\varepsilon : L^2(\Omega) \to \overline{\mathbb{R}}$ *via*

$$F_\varepsilon(u) := \begin{cases} \int_\Omega \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right] dx, & u \in W^{2,2}(\Omega) \\ +\infty, & \textit{otherwise} \end{cases}. \tag{1.10}$$

Let $\{\varepsilon_n\}_n$ *be a sequence of positive reals with* $\varepsilon_n \to 0^+$. *If* $u_n \in L^2(\Omega)$ *are such that* $\sup_{n\in\mathbb{N}} F_{\varepsilon_n}(u_n) < \infty$, *then there exists a subsequence* $\{u_{n_k}\}_k$ *for which* $u_{n_k} \to u$ *in* $L^2(\Omega)$ *for some* $u \in BPV(\Omega; \{-1,1\})$.

Proofs of this result are given in [6] and [9]. We will provide yet another proof.

The structure of this thesis is as follows. In Section 2, we review L^p spaces, pointwise variation, and results in Sobolev spaces. We then define Gamma convergence and motivate its study.

In Section 3, we prove Theorem 1.2, and then give a more elementary proof for Fonseca and Mantegazza's results in the one-dimensional case. Specifically, let $\Omega := (a,b)$ and define the integral functional $F_\varepsilon : L^2(\Omega) \to \overline{\mathbb{R}}$ as in (1.10). Let $\{\varepsilon_n\}_n$ be a sequence of positive reals with $\varepsilon_n \to 0^+$. We first prove that if $\{u_n\}_n \subset L^2(\Omega)$ is a sequence with $\sup_{n\in\mathbb{N}} F_{\varepsilon_n}(u_n) < \infty$, then we can find a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \to u$ in $L^2(\Omega)$ for some $u \in BPV(\Omega; \{-1,1\})$. Then, we prove that

$$\Gamma\text{-}\lim_{n\to+\infty} F_{\varepsilon_n}(u) = \begin{cases} \alpha \operatorname{essVar}_\Omega u, & u \in BPV(\Omega; \{-1,1\}) \\ +\infty, & \text{otherwise} \end{cases}$$

under $L^2(\Omega)$ convergence, where α is defined as in (1.8).

Lastly, in Section 4 we build off of the work done in Section 3 to prove our main result.

The current work, and a *slicing* methodology, will be used to extend to the N-dimensional case.

Chapter 2

Preliminaries

2.1 L^p spaces

We denote the Lebesgue measure on $\mathbb{R}$ by $\mathcal{L}^1$.

For $1 \leq p \leq \infty$, $L^p(E)$ denotes the Lebesgue space for p-th powers, i.e. the space of measurable functions $u : E \to \mathbb{R}$ for which $\int_E |u|^p \, dx < \infty$. $\|u\|_{L^p(E)} := \left(\int_E |u|^p \, dx\right)^{1/p}$ is the norm on this space, and for a sequence $\{u_n\}_n \subset L^p(E)$ we say that $u_n \to u$ in $L^p(E)$ if $\|u_n - u\|_{L^p(E)} \to 0$. We may refer to this as *strong* $L^p(E)$ convergence in order to distinguish it from a related convergence called *weak* $L^p(E)$ convergence, which we will later define.

One useful property of $L^p(E)$ convergence is that if $u_n \to u$ in $L^p(E)$, then there exists a subsequence u_{n_k} such that $u_{n_k} \to u$ almost everywhere.

We will require a generalization of the Lebesgue Dominated Convergence Theorem, based on *equi integrability.*

Definition 2.1 (Equi-Integrability on $\mathbb{R}$) *Let $E \subseteq \mathbb{R}$ be measurable with $\mathcal{L}^1(E) < \infty$, and let $u_n \in L^1(E)$ be a sequence. Then $\{u_n\}_n$ is equi-integrable if*

$$\lim_{M \to +\infty} \sup_{n \in \mathbb{N}} \int_{\{|u_n| \geq M\}} |u_n| \, dx = 0.$$

The convergence theorem we need is as follows.

Theorem 2.2 (Vitali's Convergence Theorem on $\mathbb{R}$) *Let $1 \leq p < \infty$. Let $E \subseteq \mathbb{R}$ be measurable with $\mathcal{L}^1(E) < \infty$, and let $\{u_n\}_n \subset L^p(E)$. Then, for $u \in L^p(E)$, we have that $u_n \to u$ in $L^p(E)$ if and only if the following two conditions holds:*

1. $u_n \to u$ *in measure,*

2. $\{|u_n|^p\}_n$ *is equi-integrable.*

We will also make use of an important integral inequality. For $1 \le p < \infty$ and functions $f_1, f_2, \cdots, f_n \in L^p(E)$ for some measurable E, recall that

$$\sum_{i=1}^{n} \left(\int_E |f_i|^p \, dx \right)^{1/p} \ge \left(\int_E \left| \sum_{i=1}^{n} f_i \right|^p \, dx \right)^{1/p}.$$

This follows from the standard Minkowski's inequality. There exists a continuous analogue of this inequality in the sense that, in a way, we may replace the sums with integrals.

Theorem 2.3 (Minkowski's Inequality for Integrals on $\mathbb{R}$) *Let* $1 \le p < \infty$. *Let* $E, F \subseteq \mathbb{R}$ *be measurable and let* $f \in L^p(E \times F)$. *Then:*

$$\int_F \left(\int_E |f(x,y)|^p \, dx \right)^{1/p} dy \ge \left(\int_E \left(\int_F |f(x,y)| \, dy \right)^p dx \right)^{1/p}.$$

2.2 Pointwise Variation and Essential Variation

For a function $u : I \to \mathbb{R}$, where I is an interval of $\mathbb{R}$, we may define the *pointwise variation* of u as

$$\operatorname*{Var}_I u := \sup \left\{ \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| : x_i \in I, x_0 < x_1 < \ldots < x_n \right\}.$$

If $\operatorname{Var}_I u < \infty$ then we write $u \in BPV(I)$. A useful property is that if u is absolutely continuous, then $\operatorname{Var}_I u = \int_I |u'| \, dx$

A family of functions having uniformly bounded pointwise variation is a powerful property.

Theorem 2.4 (Helly's Selection Theorem) *Let* I *be an interval of* $\mathbb{R}$ *and let* $\mathcal{F} \subseteq BPV(I)$ *be an infinite family of functions* $u : I \to \mathbb{R}$ *such that* $\sup_{u \in \mathcal{F}} \operatorname{Var}_I u \le C$ *for a constant* $C > 0$. *Assume moreover that there exists* $x_0 \in I$ *such that the set* $\{u(x_0) : u \in \mathcal{F}\}$ *is bounded. Then there exists a sequence* $\{u_n\}_n \in \mathcal{F}$ *that converges pointwise to some* $u \in BPV(I)$.

This is given as Theorem 2.44 in [16], where a proof can be found.

In the case that such a pointwise convergence is obtained, we can moreover obtain a bound on $\operatorname{Var}_I u$.

Theorem 2.5 *If $u_n, u \in BPV(I)$ and $u_n \to u$ pointwise, then*

$$\liminf_{n\to\infty} \operatorname*{Var}_I u_n \geq \operatorname*{Var}_I u.$$

See Proposition 2.47 in [16] for a proof.

Now suppose, say, $u \in L^p(I)$. Then there is no sense in speaking of $\operatorname{Var}_I u$ because the values of u are not well-defined pointwise. The workaround is to define the *essential pointwise variation* of u.

Definition 2.6 (Essential Pointwise Variation) *Let I be an interval and $u \in L^1_{loc}(I)$. Then the* essential pointwise variation *of u over I is given by*

$$\operatorname*{essVar}_I u := \inf\left\{\operatorname*{Var}_I \tilde{u} : \tilde{u} \textit{ is a representative of } u\right\}.$$

A nice property is that the infimum in the definition for essVar_I is obtained, i.e., there is always a representative $\tilde{u}$ for u such that $\operatorname{essVar}_I u = \operatorname{Var}_I \tilde{u}$. This follows from Theorem 2.4 and Theorem 2.5.

For $u \in L^p(I)$, we write $u \in BPV(I)$ if u has a representative $\tilde{u}$ with $\tilde{u} \in BPV(I)$. By the property from above, we see that $u \in BPV(I)$ if and only if $\operatorname{essVar}_I u < \infty$.

2.3 Sobolev Spaces

We begin by defining weak differentiation.

Definition 2.7 (Weak Derivative) *Let $\Omega \subseteq \mathbb{R}$ be an open set and $1 \leq p \leq \infty$. For $u \in L^p(\Omega)$, we say that u admits a* weak derivative of order $k \in \mathbb{N}$ *if there exists $v \in L^p(\Omega)$ satisfying*

$$\int_\Omega u\varphi^{(k)}\,dx = (-1)^k \int_\Omega v\varphi\,dx$$

for all $\varphi \in C_c^\infty(\Omega)$.

It is not too difficult to verify that the weak derivative is unique up to almost-everywhere equivalence. If u has a differentiable representative, we call its derivative (in the traditional sense) the *strong* derivative, so as to distinguish the two notions of derivative.

The weak derivative is also notated in the same way as the strong derivative. For example, if $\Omega := (-1, 1)$ and we take $u(x) = |x|$, then u admits a weak derivative given by $u'(x) = \operatorname{sgn}(x)$.

Sobolev spaces consist of L^p functions that admit weak derivatives.

Definition 2.8 (Sobolev Space) *Let $\Omega \subseteq \mathbb{R}$ be open, $1 \leq p \leq \infty$, and $k \in \mathbb{N}$. Then the* Sobolev space $W^{k,p}(\Omega)$ *is the normed space of all $u \in L^p(\Omega)$ that admit weak derivatives up to order k, such that $u^{(l)} \in L^p(\Omega)$ for all $1 \leq l \leq k$. We may endow $W^{k,p}(\Omega)$ with the following norm:*

$$\|u\|_{W^{k,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{l=1}^{k} \|u^{(l)}\|_{L^p(\Omega)}.$$

This definition may be unwieldy for showing that a function belongs to a Sobolev space, so we often work with the following equivalent condition.

Theorem 2.9 (ACL Condition for $k = 1$ on $\mathbb{R}$) *Let $\Omega \subseteq \mathbb{R}$ be an open set and $1 \leq p < \infty$. Then $u \in W^{1,p}(\Omega)$ if and only if u is absolutely continuous and $u, u' \in L^p(\Omega)$. Moreover, if $u \in W^{1,p}(\Omega)$ then the strong and weak derivatives of u agree.*

Absolute continuity of L^p functions is discussed in the sense that there exists an absolutely continuous representative, and similarly for differentiability. A proof of this condition may be found in [16].

An analogue of this condition exists for $k = 2$.

Theorem 2.10 (ACL Condition for $k = 2$ on $\mathbb{R}$) *Let $\Omega \subseteq \mathbb{R}$ be an open set and $1 \leq p < \infty$. Then $u \in W^{2,p}(\Omega)$ if and only if $u \in C^1(\Omega)$, u' is absolutely continuous, and $u, u', u'' \in L^p(\Omega)$. Moreover, if $u \in W^{2,p}(\Omega)$ then the first and second-order strong derivatives of u agree with their weak analogues.*

A proof of this may be found in [15]. One particular corollary of this condition is that if $u \in W^{1,1}(I)$ then u has an absolutely continuous representative $\tilde{u}$ and we can show that $\operatorname{essVar}_I u = \int_I |\tilde{u}'| \, dx$.

Various algebraic properties satisfied by strong derivatives have analogues for weak derivatives. For instance, we have the following chain rule.

Theorem 2.11 (Chain Rule) *Let $\Omega \subseteq \mathbb{R}$ be an open, bounded interval and let $1 \leq p \leq \infty$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $u \in W^{1,p}(\Omega)$. Then $f(u) \in W^{1,p}(\Omega)$ and $\frac{d}{dx}f(u) = f'(\tilde{u})u'$, where $\tilde{u}$ is the absolutely continuous representative of u, and we take $f'(\tilde{u}(x))u'(x)$ to be 0 whenever $u'(x) = 0$.*

This is a consequence of the chain rule for absolutely continuous functions (See Theorem 3.24 in [16]).

It is useful to obtain a bound for $\int_a^b |u'|^2\,dx$ in terms of $\int_a^b |u|^2\,dx$ and $\int_a^b |u''|^2\,dx$ for $u \in W^{2,p}(a,b)$. The following two results are special cases of Lemma 7.38 and Theorem 7.37 in [16].

Lemma 2.12 *Let $I := (a,b)$. Suppose $u \in W^{2,2}(I)$ such that u' has at least one zero in $[a,b]$. Then there exists a universal constant $c > 0$ such that*

$$\int_a^b |u'|^2\,dx \leq c \left(\int_a^b u^2\,dx\right)^{1/2} \left(\int_a^b |u''|^2\,dx\right)^{1/2}.$$

Theorem 2.13 *For an open interval I and $u \in W^{2,2}(I)$, there exists a universal constant $c > 0$ such that*

$$\left(\int_I |u'|^2\,dx\right)^{1/2} \leq cl^{-1}\left(\int_I u^2\,dx\right)^{1/2} + cl\left(\int_I |u''|^2\,dx\right)^{1/2}$$

for every l with $0 < l < \mathcal{L}^1(I)$.

We note that by applying the inequality $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$ for $a, b \in \mathbb{R}$ to the above theorem, we may obtain the inequality

$$\int_I |u'|^2\,dx \leq c'l^{-2}\int_I u^2\,dx + c'l^2\int_I |u''|^2\,dx$$

under the same conditions but for a different universal constant $c' > 0$.

Lastly, Sobolev spaces admit useful embedding results.

Theorem 2.14 (Morrey's Embedding on $\mathbb{R}$) *Let $\Omega \subseteq \mathbb{R}$ be an open set. Then we have the continuous inclusion $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{1}{p}}(\Omega)$. That is, $W^{1,p}(\Omega) \subseteq C^{0,1-\frac{1}{p}}(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|u\|_{C^{0,1-\frac{1}{p}}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

A statement and proof for larger dimensions may be found in [16]. When over $\mathbb{R}$ as above, a simpler proof consists of applying Hölder's Inequality.

2.4 Weak Convergence

Definition 2.15 (Weak Convergence in L^p) *Let $1 \leq p < \infty$. Let $E \subseteq \mathbb{R}$ be measurable, and let $u_n \in L^p(E)$ for all $n \in \mathbb{N}$. For $u \in L^p(E)$, we say that $\{u_n\}_n$* converges weakly *to u in $L^p(E)$ if*

$$\lim_{n\to\infty} \int_E u_n v \, dx = \int_E uv \, dx$$

for all $v \in L^{p'}(E)$, where $p' \in [1, \infty]$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, and we write $u_n \rightharpoonup u$ in $L^p(E)$.

Definition 2.16 (Weak Convergence in $W^{k,p}$) *For $\Omega \subseteq \mathbb{R}$ open, $k \in \mathbb{N}$, $1 \leq p < \infty$, and $u_n, u \in W^{k,p}(\Omega)$, we say that $\{u_n\}_n$* converges weakly *in $W^{k,p}(\Omega)$ if $u_n^{(l)} \rightharpoonup u^{(l)}$ in $L^p(\Omega)$ for every $0 \leq l \leq k$, and we write $u_n \rightharpoonup u$ in $W^{k,p}(\Omega)$.*

It can be shown that weak limits are unique up to almost-everywhere equivalence. Moreover, Hölder's inequality implies that strong convergence in $L^p(E)$ implies weak convergence in $L^p(E)$, and consequently strong convergence in $W^{k,p}(\Omega)$ implies weak convergence in $W^{k,p}(\Omega)$.

Proofs of the theorems that follow may be found in [8].

Weak convergence enables us to consider a notion of *weak compactness*.

Theorem 2.17 (Weak Compactness in $W^{k,p}$) *Let $\Omega \subseteq \mathbb{R}$ be open, $k \in \mathbb{N}$, $1 < p < \infty$, and suppose that $u_n \in W^{k,p}(\Omega)$ is such that $\{u_n\}_n$ is uniformly bounded on $W^{k,p}(\Omega)$, that is, $\sup_{n\in\mathbb{N}} \|u_n\|_{W^{k,p}(\Omega)} < \infty$. Then there exist a subsequence $\{u_{n_l}\}_l$ and $u \in W^{k,p}(\Omega)$ such that $u_{n_l} \rightharpoonup u$ in $W^{k,p}(\Omega)$.*

Lastly, weak convergence is useful for obtaining a weak form of semi-continuity for certain functions. A particular result we shall use is the following.

Theorem 2.18 *Let $1 \leq p < \infty$ and let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex. Then the integral functional $F : L^p(\Omega) \to \overline{\mathbb{R}}$ defined by*

$$F(u) := \int_\Omega f(u) \, dx$$

is sequentially lower semi-continuous with respect to weak $L^p_{loc}(\Omega)$ convergence. That is, we have

$$F(u) \leq \liminf_{n\to\infty} F(u_n)$$

whenever $u_n, u \in L^p(\Omega)$ such that $u_n \rightharpoonup u$ in $L^p_{loc}(\Omega)$.

For a proof, see Theorem 5.14 in [8].

2.5 Gamma Limits

The Gamma limit is a notion of function convergence that is of interest because it preserves minima.

Definition 2.19 *For a metric space (X, d) and a sequence $\{f_n\}_n$, $f_n : X \to \overline{\mathbb{R}}$, we say that $\{f_n\}_n$* Gamma converges *to a function $f : X \to \overline{\mathbb{R}}$ if for all $x_0 \in X$:*

1. $\liminf_{n\to\infty} f_n(x_n) \geq f(x_0)$ *for* ***all*** *sequences x_n with $x_n \xrightarrow{d} x_0$, and*
2. $\limsup_{n\to\infty} f_n(x_n) \leq f(x_0)$ *for* ***some*** *sequence x_n with $x_n \xrightarrow{d} x_0$,*

and we write $f = \Gamma\text{-}\lim_{n\to\infty} f_n$.

The two inequality conditions are referred to as the *liminf inequality* and *limsup inequality* respectively.

The following properties are proven in Chapter 1 of [4].

- The Gamma limit, if it exists, is unique pointwise. Moreover, it is given precisely by
$$\left(\Gamma\text{-}\lim_{n\to\infty} f_n\right)(x) = \inf\left\{\liminf_{n\to\infty} f_n(x_n) : x_n \xrightarrow{d} x\right\}.$$
- It is not necessarily true that the Gamma limit coincides with the pointwise limit, if both exist.
- If $f = \Gamma\text{-}\lim f_n$ exists and x_n is a minimizer for f_n, then every accumulation point of the sequence $\{x_n\}_n$ is a minimizer for f.

As an example, let us define $\Omega := (0, 1)$ and consider the functional $F_\varepsilon : W^{1,2}(\Omega) \to \mathbb{R}$ defined by
$$F_\varepsilon(u) := \begin{cases} \int_0^1 \left[(u^2-1)^2 + \varepsilon^2|u'|^2\right] dx, & \int_\Omega u\, dx = 0 \\ +\infty, & \text{otherwise} \end{cases}$$
for $\varepsilon > 0$. Although it is quite non-trivial, it can be shown that F_ε has a minimizer.

The $\varepsilon^2|u'|^2$ term is a complication. Can it be removed without much change to the minimizers? As before, let $\{\varepsilon_n\}_n$ be a sequence with $\varepsilon_n \to 0^+$. We can show that
$$\Gamma\text{-}\lim_{n\to\infty} F_{\varepsilon_n}(u) = \int_0^1 (u^2-1)^2\, dx$$

under $W^{1,2}(\Omega)$ convergence. Now let us attempt to repeat the previous logic. If we take u_n to be a minimizer of F_{ε_n}, then any subsequence of u_n converging in $W^{1,2}(\Omega)$ must converge to a minimizer of $\int_0^1 (u^2-1)^2\,dx$. However, we can show that

$$\inf\left\{\int_0^1 (u^2-1)^2\,dx : \int_\Omega u\,dx = 0 \text{ and } u \in W^{2,1}(\Omega)\right\} = 0,$$

hence a minimizer $u_{\min}$ would have to take the form $u_{\min} = 1\cdot 1_E + (-1)\cdot 1_{\Omega\setminus E}$ for some E with $\mathcal{L}^1(E) = \frac{1}{2}$, which cannot be consistent with the requirement that $u \in W^{2,1}(\Omega)$. Hence, there is no such minimizer!

Our failure to deduce anything meaningful from this Gamma convergence result is a consequence of missing a compactness result that guarantees the existence of converging subsequences of $\{u_n\}_n$. Fortunately, without computing the exact minimizers, we can still recover a reasonable Gamma limit result by weakening the convergence. Specifically, we may sacrifice the strong $W^{1,2}(\Omega)$ convergence for weak convergence in $L^4(\Omega)$. The advantage of this weakening is that if we take u_n to be a minimizer of F_{ε_n}, then we can use weak compactness to prove that there is a subsequence $\{u_{n_k}\}_k$ and some $u \in L^4(\Omega)$ for which $u_{n_k} \rightharpoonup u$ in $L^4(\Omega)$, which is precisely the sort of compactness result we seek. Under weak $L^4(\Omega)$ convergence, the Gamma limit changes to

$$\Gamma\text{-}\lim_{n\to\infty} F_{\varepsilon_n}(u) = \int_0^1 f^{**}(u)\,dx,$$

where $f^{**}(z) := \begin{cases} (z^2-1)^2, & |z| > 1 \\ 0, & |z| \le 1 \end{cases}$ denotes the *convex envelope* of $f(z) := (z^2-1)^2$. Due to the existence of a compactness result, we are guaranteed the existence of a subsequence $\{u_{n_k}\}_k$ converging weakly in $L^2(\Omega)$ to a minimizer of $\int_\Omega f^{**}(u)\,dx$, and indeed there are many minimizers of this functional.

This demonstrates the importance of choosing the correct metric of convergence for the Gamma limit. If the convergence is too strong, there may be no compactness result. On the other hand, the weaker the convergence, the weaker the result we end up proving.

Chapter 3

The Second-Order Singularly-Perturbed Problem

Our first goal is to prove Theorem 1.2. As a steppingstone, we first prove a similar compactness result in the context of a first order singularly-perturbed problem, which is of interest in itself (see [17], [19], [21] for a different proof).

Theorem 3.1 (Compactness for First Order Problem) *Let $\Omega := (a, b)$ be a non-empty open interval, and let $\{\varepsilon_n\}_n$ be a sequence of positive reals with $\varepsilon_n \to 0^+$.*

(i) If we have a sequence $\{u_n\}_n \subset W^{1,2}(\Omega)$ with

$$C := \sup_{n\in\mathbb{N}} \int_\Omega \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n |u_n'|^2\right] dx < \infty,$$

then there exists a subsequence $\{u_{n_k}\}_k$ for which $u_{n_k} \to u$ in $L^2(\Omega)$ for some $u \in BPV(\Omega; \{-1, 1\})$.

(ii) Moreover, such a u must satisfy $\operatorname{essVar}_\Omega u \le \frac{3}{4}C$.

Proof. Since $\int_\Omega (u_n^2 - 1)^2\, dx \le C\varepsilon_n \to 0$, we see that $u_n^2 - 1 \to 0$ in $L^2(\Omega)$. So, by extraction of a subsequence, we may assume that $u_n^2 - 1 \to 0$ almost everywhere. In particular, we get that $|u_n| \to 1$ almost everywhere.

Applying the AM-GM inequality inside the integral gives

$$C \ge \int_\Omega \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n |u_n'|^2\right] dx \ge \int_\Omega 2|u_n^2 - 1| \cdot |u_n'|\, dx = \int_\Omega |U_n'|\, dx$$

for all n, where $U_n := 2u_n - \frac{2}{3}u_n^3$. Taking $\tilde{U}_n$ to be the absolutely continuous representative of U_n, we deduce that $\mathrm{Var}_\Omega \tilde{U}_n \le C$ for all n. Moreover, since $|u_n| \to 1$ almost everywhere, we have that $|U_n| \to 2 - \frac{2}{3} = \frac{4}{3}$ almost everywhere by continuity of $z \mapsto 2z - \frac{2}{3}z^3$, and so, in particular, there is some $x \in \Omega$ for which $\tilde{U}_n(x)$ converges. From this, we may now apply Helly's Selection Theorem (Theorem 2.4) to find a subsequence $\tilde{U}_{n_k}$ that converges pointwise to some $\tilde{U} \in BPV(\Omega)$, and $C \ge \liminf_{n\to\infty} \mathrm{Var}_\Omega \tilde{U}_n \ge \mathrm{Var}\,\tilde{U}$ by Theorem 2.5.

It follows that $\{U_{n_k}\}_k$ converges to some U almost everywhere that has representative $\tilde{U}$, with $U = \pm\frac{4}{3}$ almost everywhere, and moreover $\mathrm{essVar}_\Omega U \le \mathrm{Var}\,\tilde{U} \le C < \infty$. Hence, by taking $u := \frac{3}{4}U$, we have that

$$\underset{\Omega}{\mathrm{essVar}}\, u \le \frac{3}{4}C < \infty, \tag{3.1}$$

and particularly u has a representative $\tilde{u} \in BPV(\Omega; \{-1, 1\})$.

To see that $u_{n_k} \to u$ almost everywhere, recall that $|u_{n_k}| \to 1$ almost everywhere. Taking representatives $\tilde{u}_{n_k}$, we have that almost every $x_0 \in \Omega$ satisfies $|\tilde{u}_{n_k}(x_0)| \to 1$, $2\tilde{u}_{n_k} - \frac{2}{3}\tilde{u}_{n_k}^3 = \tilde{U}_{n_k}$ for all k, $\tilde{U}_{n_k}(x_0) \to \tilde{U}(x_0)$, and $\tilde{U}(x_0) = \frac{4}{3}\tilde{u}(x_0)$.

For every such x_0 we have that $2\tilde{u}_{n_k}(x_0) - \frac{2}{3}u_{n_k}(x_0)^3 \to \frac{4}{3}\tilde{u}(x_0)$. If $\tilde{u}(x_0) = 1$, then $2\tilde{u}_{n_k}(x_0) - \frac{2}{3}\tilde{u}_{n_k}(x_0)^3 - \frac{4}{3} \to 0$, and by factoring we obtain $(\tilde{u}_{n_k}(x_0) - 1)^2(\tilde{u}_{n_k}(x_0) + 2) \to 0$. But $|\tilde{u}_{n_k}(x_0)| \to 1$, so $\liminf_{k\to\infty} |\tilde{u}_{n_k}(x_0) + 2| \ge 1$, and hence it must follow that $(\tilde{u}_{n_k}(x_0) - 1)^2 \to 0$, i.e., $\tilde{u}_{n_k}(x_0) \to 1 = \tilde{u}(x_0)$. By a symmetrical argument we see that if $\tilde{u}(x_0) = -1$ then $\tilde{u}_{n_k}(x_0) \to -1 = \tilde{u}(x_0)$. This shows that $u_{n_k} \to u$ almost everywhere.

Lastly, to obtain $L^2(\Omega)$ convergence, we may use the fact that $\mathrm{essVar}_\Omega U_{n_k} \le C$ and that $\{U_{n_k}\}_k$ converges almost everywhere, to deduce that $\{U_{n_k}\}_k$ is uniformly bounded by some constant C'. Then the inverse image of $[-C, C]$ under $z \mapsto 2z - \frac{2}{3}z^3$ is compact by continuity, thus $|U_{n_k}| = |2u_{n_k} - \frac{2}{3}u_{n_k}^3| \le C'$ implies that $|u_{n_k}| \le C''$ for some constant C''. In particular, we see that $\{u_{n_k}\}_k$ is uniformly bounded, so we may apply dominated convergence to obtain the desired $L^2(\Omega)$ convergence. This proves item (i), and the bound claimed in item (ii) was acquired in (3.1). □

Next, we will prove an essential interpolation inequality.

Theorem 3.2 *Let $\Omega := (a, b)$ be an interval in $\mathbb{R}$, and for each $0 < \varepsilon < \frac{b-a}{2}$ let $\Omega_\varepsilon := (a + \varepsilon, b - \varepsilon)$. Then there is a universal constant $C > 0$ such that*

$$\int_{\Omega_\varepsilon} \varepsilon |u'|^2\, dx \le C \int_\Omega \left[\varepsilon^{-1}(u^2 - 1)^2 + \varepsilon^3 |u''|^2\right]\, dx$$

for all $u \in W^{2,2}(\Omega)$ and $0 < \varepsilon < \frac{b-a}{2}$.

We first prove a small lemma.

Lemma 3.3 *Let Ω be an open interval in $\mathbb{R}$ and $u \in W^{2,2}(\Omega)$. Then the set $Z := \{x \in \Omega : u(x) = 0, u'(x) \neq 0\}$ is countable.*

Proof. Let $x_0 \in Z$. It suffices to show that x_0 is an isolated point of Z. We may assume without loss of generality that $u'(x_0) > 0$.

Since $u \in W^{2,2}(\Omega)$ we know that u' is continuous (See Theorem 2.10). Thus, there exists $\delta > 0$ such that $u'(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. We conclude that there is no $x \in (x_0 - \delta, x_0 + \delta)$ for which $u(x) = 0$ (so that in particular, $x \notin Z$), otherwise we obtain a contradiction from Rolle's Theorem. □

Although it is not worth stating as a lemma, we will be using the inequality $(|z| - 1)^2 \leq (z^2 - 1)^2$ quite liberally, which follows from factoring $(z^2 - 1)^2$ and taking cases on the sign of z.

We may now prove Theorem 3.2.

Proof. We first claim that

$$\int_{\{u=0\}} |u'|^2 \, dx = 0, \tag{3.2}$$

or equivalently,

$$\int_{\{y:u(y)=0,u'(y)\neq 0\}} |u'|^2 \, dx = 0.$$

By Lemma 3.3, the domain over which we integrate here is countable, so the integral is indeed 0.

It follows that $\int_\Omega |u'|^2 \, dx = \int_{u^{-1}((-\infty,-0)\cup(0,+\infty))} |u'|^2 \, dx$, so it remains to examine the set $u^{-1}((-\infty, -0) \cup (0, +\infty))$. This set is open by continuity, so it is the at-most countable union of disjoint open intervals. Let those intervals that do not have either a or b as an endpoint be enumerated as $\{(a_i, b_i)\}_{i\in\mathcal{I}}$ for a countable, possibly empty index set $\mathcal{I}$. Let the union of these intervals be V, so V is where $u \neq 0$ except possibly near the endpoints of (a, b).

Then for all $i \in \mathcal{I}$ we have that $u(a_i) = u(b_i)$ by a continuity argument. Thus by Rolle's Theorem there is $c_i \in (a_i, b_i)$ for which $u'(c_i) = 0$. Moreover, $u \neq 0$ over (a_i, b_i), thus $v := |u| - 1$ is differentiable over (a_i, b_i). If we assume, without loss of generality, that $u > 0$ in (a_i, b_i), then $v = u - 1$, so $v \in W^{2,2}(a_i, b_i)$, with $v' = u'$ and $v'' = u''$. By Lemma 2.12, it follows that

$$\begin{aligned}\int_{a_i}^{b_i} |u'|^2 \, dx &\leq c_1 \left(\int_{a_i}^{b_i} (|u| - 1)^2 \, dx \right)^{1/2} \left(\int_{a_i}^{b_i} |u''|^2 \, dx \right)^{1/2} \\ &\leq c_1 \left(\int_{a_i}^{b_i} (u^2 - 1)^2 \, dx \right)^{1/2} \left(\int_{a_i}^{b_i} |u''|^2 \, dx \right)^{1/2}\end{aligned}$$

for a constant $c_1 > 0$. Multiplying by ε gives

$$\int_{a_i}^{b_i} \varepsilon|u'|^2\,dx \le c_1 \left(\int_{a_i}^{b_i} \varepsilon^{-1}(u^2-1)^2\,dx\right)^{1/2} \left(\int_{a_i}^{b_i} \varepsilon^3|u''|^2\,dx\right)^{1/2},$$

then applying the AM-GM inequality gives

$$\int_{a_i}^{b_i} \varepsilon|u'|^2\,dx \le \frac{c_1}{2}\int_{a_i}^{b_i} \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right]\,dx.$$

Summing this inequality over all $i \in \mathcal{I}$, we conclude that

$$\int_V \varepsilon|u'|^2\,dx \le \frac{c_1}{2}\int_V \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right]\,dx. \tag{3.3}$$

We are almost done. Let $S := \inf\{x \in \Omega : u(x) = 0\}$ and $T := \sup\{x \in \Omega : u(x) = 0\}$. If $\{x \in \Omega : u(x) = 0\}$ is empty then we take $S = a$, and similarly for T. The intervals (a, S) and (T, b), if distinct and non-empty, are the two intervals that we omitted from $\{x \in \Omega : u(x) \neq 0\}$ to obtain V, so that $V \cup (a, S) \cup (T, b) = \{x \in \Omega : u(x) \neq 0\}$. It remains to handle these two intervals.

If $S - a \le \varepsilon$, then clearly $\int_{(a,S)\cap\Omega_\varepsilon} \varepsilon|u'|^2\,dx = 0$. If otherwise $S - a > \varepsilon$, then we may apply Theorem 2.13 to $|u| - 1 \in W^{2,2}(a, S)$ to get

$$\begin{aligned}\int_a^S |u'|^2\,dx &\le c_2\varepsilon^{-2}\int_a^S (|u|-1)^2\,dx + c_1\varepsilon^2\int_a^S |u''|^2\,dx \\ &\le c_2\varepsilon^{-2}\int_a^S (u^2-1)^2\,dx + c_1\varepsilon^2\int_a^S |u''|^2\,dx,\end{aligned}$$

for a constant $c_2 > 0$. Multiplying by ε gives

$$\int_{(a,S)\cap\Omega_\varepsilon} \varepsilon|u'|^2\,dx \le \int_a^S \varepsilon|u'|^2\,dx \le c_2\int_a^S \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right]\,dx. \tag{3.4}$$

Thus the inequality (3.4) holds in either the cases $S - a \le \varepsilon$ and $S - a > \varepsilon$. Applying the same arguments above to the interval (T, b), we see that in either of the cases $b - T \le \varepsilon$ and $b - T > \varepsilon$ we obtain the inequality

$$\int_{(T,b)\cap\Omega_\varepsilon} \varepsilon|u'|^2\,dx \le c_2\int_T^b \varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\,dx. \tag{3.5}$$

By summing (3.3), (3.4), and (3.5), we get the bound

$$\begin{aligned}\int_{\{u\neq 0\}\cap\Omega_\varepsilon} \varepsilon|u'|^2\,dx &\le \int_{E\cup((a,S)\cap\Omega_\varepsilon)\cup((T,b)\cap\Omega_\varepsilon)} \varepsilon|u'|^2\,dx \\ &\le \max(c_1/2, c_2)\int_{E\cup(a,S)\cup(T,b)} \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right]\,dx \\ &= \max(c_1/2, c_2)\int_{|u|>\alpha} \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3|u''|^2\right]\,dx. \end{aligned}\tag{3.6}$$

Finally, once we combine (3.2), and (3.6), we arrive at the inequality

$$\int_{\Omega_\varepsilon} \varepsilon |u'|^2\,dx \le c_3 \int_\Omega \left[\varepsilon^{-1}(u^2-1)^2 + \varepsilon^3 |u''|^2\right]\,dx,$$

where $c_3 := \max(c_1/2, c_2)$. □

With this interpolation inequality proven, we need one more lemma before we prove Theorem 1.2.

Lemma 3.4 *Let Ω be an open interval of $\mathbb{R}$ and $u_n \in L^2(\Omega)$ be such that $\lim_{n\to\infty} \int_\Omega (u_n^2 - 1)^2\,dx = 0$. Then $\{|u_n|^2\}_n$ is equi-integrable.*

Proof. Find $c > 0$ so small such that $cz^2 \le (z^2-1)^2$ for all z large enough, say, $z \ge M_0$. Now

$$\int_\Omega |u_n|^2\,dx = \int_{\{|u_n|<M_0\}} |u_n|^2\,dx + \int_{\{|u_n|\ge M_0\}} |u_n|^2\,dx \le M_0^2 + \int_{\{|u_n|\ge M_0\}} \frac{1}{c}(u_n^2-1)^2\,dx < \infty,$$

so $|u_n|^2$ is integrable for all n.

Fix $\eta > 0$. Find N_η such that $\int_\Omega (u_n^2-1)^2\,dx \le c\eta$ for all $n > N_\eta$. By integrability of $|u_n|^2$ for $n = 1, 2, \cdots, N_\eta$, we choose $M_\eta \ge M_0$ so large that $\int_{|u_n|\ge M_\eta} |u_n|^2\,dx < \eta$ for all such n. We deduce that for all $n \ge N_\eta$ we have

$$\int_{|u_n|\ge M_\eta} |u_n|^2\,dx \le \frac{1}{c}\int_{|u_n|\ge M_\eta} (u_n^2-1)^2\,dx \le \eta.$$

□

We now turn to the proof of Theorem 1.2.

Proof. Let $C := \sup_{n\in\mathbb{N}} F_{\varepsilon_n}(u_n)$. We first claim that for each $\delta > 0$ there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ such that $u_{n_k} \to v$ in $L^2(a+\delta, b-\delta)$ for some function $v \in BPV((a+\delta, b-\delta); \{-1, 1\})$, with $\operatorname{essVar}_{(a+\delta,b-\delta)} v \le C_3$ for a constant C_3 not depending on δ.

To see this, we note that $(a+\varepsilon_n, b-\varepsilon_n) \subseteq (a+\delta, b-\delta)$ for all large enough n, so that we may apply Theorem 3.2 to obtain the bound

$$\int_{a+\delta}^{b-\delta} \varepsilon_n |u_n'|^2\,dx \le \int_{a+\varepsilon_n}^{b-\varepsilon_n} \varepsilon_n |u_n'|^2\,dx \le C' \int_\Omega \left[\varepsilon_n^{-1}(u_n^2-1)^2 + \varepsilon_n^3 |u_n''|^2\right]\,dx \le CC'$$

for all large enough n and for a universal constant C'. It follows that

$$\int_{a+\delta}^{b-\delta} \varepsilon_n^{-1}(u_n^2-1)^2 + \varepsilon_n |u_n'|^2\,dx \le C + CC'$$

for all such n, and so by item (i) of Theorem 3.1 applied to the interval $(a+\delta, b-\delta)$, we may find a subsequence $\{u_{n_k}\}_k$ with $u_{n_k} \to v$ in $L^2(a+\delta, b-\delta)$ for some $v \in BPV((a+\delta, b-\delta); \{-1,1\})$, and moreover by item (ii) of Theorem 3.1 we must have that $\operatorname{essVar}_{(a+\delta,b-\delta)} v \leq \frac{3}{4}(C+CC')$. Hence the claim is proven with $C_3 := \frac{3}{4}(C+CC')$.

We finish with a diagonalization argument. Let $\Omega_m := (a+\delta_m, b-\delta_m)$ for a sequence $\delta_m \to 0^+$. For each m we define a subsequence $\{u_{m,n}\}_m$ of $\{u_n\}_n$ recursively as follows:

- $u_{1,n} := u_n$ for all n.
- For $m \geq 2$, we take $\{u_{m,n}\}_n$ to be a subsequence of $\{u_{m-1,n}\}_n$ that converges in $L^2(\Omega_m)$ to some $v_m : \Omega_m \to \{-1,1\}$ with $\operatorname{essVar}_{\Omega_m} v_m \leq C_3$, which exists by the claim.

Let $w_n := u_{n,n}$. We claim that for some $u \in L^2(\Omega; \{-1,1\})$ we have that $w_n \to u$ in $L^2(\Omega_m)$ for every m, so that, in particular, we have $w_n \to u$ in $L^2_{\text{loc}}(\Omega)$.

To see this, consider m_1, m_2 with $m_2 > m_1$, so that $\{u_{m_2,n}\}_n$ is a subsequence of $\{u_{m_1,n}\}_n$. Then we write $u_{m_2,k} := u_{m_1,n_k}$ for some sequence n_k, and so

$$\|v_{m_2} - v_{m_1}\|_{L^2(\Omega_{m_1})} \leq \|v_{m_2} - u_{m_2,k}\|_{L^2(\Omega_{m_1})} + \|u_{m_1,n_k} - v_{m_1}\|_{L^2(\Omega_{m_1})}.$$

Sending $k \to +\infty$ we deduce that $v_{m_2} = v_{m_1}$ almost everywhere over Ω_{m_1}, so in general v_{m_2} is an extension of v_{m_1} for all m_2 and m_1 with $m_2 > m_1$. Take u to be the maximal such extension, defined over all of Ω. Then for any m, we have that

$$\|w_n - u\|_{L^2(\Omega_m)} = \|u_{n,n} - v_m\|_{L^2(\Omega_m)} \xrightarrow{n\to+\infty} 0$$

because $\{u_{n,n}\}_{n=m}^{\infty}$ is a subsequence of $\{u_{m,n}\}_n$, which converges to v_m in $L^2(\Omega_m)$.

This proves the claim. In particular, we see that $w_n \to u$ in measure. Moreover the family $\{|w_n|^2\}_n$ is equi-integrable by Lemma 3.4. Thus by Vitali's Convergence Theorem, we get that in fact, $w_n \to u$ in $L^2(\Omega)$.

Lastly, to see that $u \in BPV(\Omega)$, note that

$$\operatorname*{essVar}_{\Omega} u = \lim_{m\to\infty} \operatorname*{essVar}_{\Omega_m} u = \lim_{m\to\infty} \operatorname*{essVar}_{\Omega_m} v_m \leq C_3,$$

and this concludes the proof. □

We now find the Gamma limit under $L^2(\Omega)$ convergence.

Theorem 3.5 *Let $\Omega := (a,b)$ be an open interval of $\mathbb{R}$. For each $\varepsilon > 0$, define a functional $F_\varepsilon : L^2(\Omega) \to \overline{\mathbb{R}}$ as in* (1.10). *Let $\{\varepsilon_n\}_n$ be a sequence of positive reals with $\varepsilon_n \to 0^+$. For all $u \in L^2(\Omega)$, we have*

$$\Gamma\text{-}\lim_{n\to+\infty} F_{\varepsilon_n}(u) = \begin{cases} \alpha \operatorname{essVar}_\Omega u, & u \in BPV(\Omega; \{-1,1\}) \\ +\infty, & \textit{otherwise} \end{cases}$$

under $L^2(\Omega)$ *convergence, where* α *is defined as in* (1.8).

We begin with a useful interpolation result.

Lemma 3.6 *Let* $T > 0$ *and* $A, m \in \mathbb{R}$ *with* $m \neq 0$. *Then there exists* $f : [0, T] \to \mathbb{R}$ *satisfying the following properties:*

(i) $f \in C^\infty([0, T])$,

(ii) $f(0) = A$, $f'(0) = m$, $f(T) = 0$, *and* $f'(T) = 0$,

(iii) $\int_0^T |f^{(k)}(x)|^2 \, dx \leq C_k(A^2 + m^2T^2)T^{1-2k}$ *for a constant* C_k *depending only on* k,

(iv) $\sup_{[0,T]} |f| \leq |A| + \frac{1}{2}|m||T|$.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ as

$$g(x) := \begin{cases} mx + A, & -T/2 < x < T/2 \\ 0, & \text{otherwise} \end{cases}.$$

Let $\varphi \in C_c^\infty((-1, 1))$ be a non-negative and symmetric mollifier, so that $\varphi(x) = \varphi(-x)$ for all $x \in (-1, 1)$. For each $\varepsilon > 0$ we define $\varphi_\varepsilon(x) := \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ and the mollification

$$g_\varepsilon(x) := \int_{x-\varepsilon}^{x+\varepsilon} g(y)\varphi_\varepsilon(x - y) \, dy.$$

If $\varepsilon < T/2$ then we recover the properties $g_\varepsilon(0) = A$, $g'_\varepsilon(0) = m$, $g_\varepsilon(T) = 0$, and $g'_\varepsilon(T) = 0$, because by symmetry of φ_ε we have that $g_\varepsilon(x) = mx + A$ for x near 0 and $g_\varepsilon(x) = 0$ for x near T. Note that for each k, we have that $|g(y)\varphi_\varepsilon^{(k)}(x - y)| \leq \varepsilon^{-k}\|\varphi^{(k)}\|_\infty |g(y)|$ for all $x, y \in \mathbb{R}$. As g is integrable, we may use dominated convergence to obtain

$$\begin{aligned} g_\varepsilon^{(k)}(x) &= \frac{d^k}{dx^k}\int_{x-\varepsilon}^{x+\varepsilon} g(y)\varphi_\varepsilon(x - y) \, dy = \frac{d^k}{dx^k}\int_{\mathbb{R}} g(y)\varphi_\varepsilon(x - y) \, dy \\ &= \int_{\mathbb{R}} g(y)\varphi_\varepsilon^{(k)}(x - y) \, dy = \int_{x-\varepsilon}^{x+\varepsilon} g(y)\varphi_\varepsilon^{(k)}(x - y) \, dy. \end{aligned}$$

Now we integrate in x and apply Minkowski's inequality for integrals (Theorem 2.3) to get

$$\begin{aligned}\int_0^T g_\varepsilon^{(k)}(x)^2\,dx &= \int_0^T \left(\int_{x-\varepsilon}^{x+\varepsilon} g(y)\varphi_\varepsilon^{(k)}(x-y)\,dy\right)^2 dx \\ &\leq \left(\int_{-\varepsilon}^{T+\varepsilon}\left(\int_{(y-\varepsilon,y+\varepsilon)\cap[0,T]} |g(y)|^2|\varphi_\varepsilon^{(k)}(x-y)|^2\,dx\right)^{1/2} dy\right)^2 \\ &\leq \left(\int_{-\varepsilon}^{T+\varepsilon} |g(y)|\left(\int_{y-\varepsilon}^{y+\varepsilon} |\varphi_\varepsilon^{(k)}(x-y)|^2\,dx\right)^{1/2} dy\right)^2 \\ &= \left(\int_{-\varepsilon}^{T+\varepsilon} |g(y)|\,dy\right)^2 \int_{-\varepsilon}^{\varepsilon} |\varphi_\varepsilon^{(k)}(z)|^2\,dz.\end{aligned}$$

The first integral is bounded as

$$\int_{-\varepsilon}^{T+\varepsilon} |g(y)|\,dy \leq \int_{-\varepsilon}^{T+\varepsilon} (|m||y|+|A|)\,dy = \frac{|m|}{2}\left((T+\varepsilon)^2+\varepsilon^2\right) + (T+2\varepsilon)|A|.$$

As for the second integral, we write $\varphi_\varepsilon^{(k)}(x) = \frac{d^k}{dx^k}\frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon^{k+1}}\varphi^{(k)}\left(\frac{x}{\varepsilon}\right)$, so that

$$\int_{-\varepsilon}^{\varepsilon} |\varphi_\varepsilon^{(k)}(z)|^2\,dz = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\varepsilon^{2k+2}}|\varphi^{(k)}(z/\varepsilon)|^2\,dz = \frac{1}{\varepsilon^{2k+1}}\int_{-1}^{1} |\varphi^{(k)}(z)|^2\,dz.$$

Altogether, we get that

$$\int_0^T g_\varepsilon^{(k)}(x)^2\,dx \leq \frac{\left(\frac{|m|}{2}\left((T+\varepsilon)^2+\varepsilon^2\right)+(T+2\varepsilon)|A|\right)^2}{\varepsilon^{2k+1}}\|\varphi^{(k)}\|_{L^2(\mathbb{R})}^2.$$

Choosing $\varepsilon := T/4$, and applying the inequality $(a+b)^2 \leq 2a^2+2b^2$, we see that

$$\int_0^T g_{T/4}^{(k)}(x)^2\,dx \leq \frac{|m|^2T^4+|A|^2T^2}{T^{2k+1}}C_k = C_k(|A|^2+|m|^2T^2)T^{1-2k}$$

for a constant C_k depending only on k.

Finally, since $|g(x)| \leq |m||x|+|A| \leq |m|\frac{|T|}{2}+|A|$ for all x, we must have

$$|g_{T/4}(x)| \leq \int_{x-\varepsilon}^{x+\varepsilon} |g(y)|\varphi_\varepsilon(x-y)\,dy \leq \left(|m|\frac{|T|}{2}+|A|\right)\int_{x-\varepsilon}^{x+\varepsilon}\varphi_\varepsilon(x-y)\,dy = |m|\frac{|T|}{2}+|A|$$

for all x, so that in particular $\sup_{[0,T]}|g_{T/4}(x)| \leq |m|\frac{|T|}{2}+|A|$. We conclude that $f := g_{T/4}$ satisfies the required properties. □

Now we prove the Theorem 3.5.

Proof. First, if $u \in L^2(\Omega; \{-1,1\}) \setminus BPV(\Omega; \{-1,1\})$, we claim that for any sequence $\{u_n\}_n$ with $u_n \to u$ in $L^2(\Omega)$, we have $\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) = +\infty$. If not, then $\sup_{n\in\mathbb{N}} F_{\varepsilon_n}(u_n) < \infty$ and so by Theorem 1.2 a subsequence $\{u_{n_k}\}_k$ converges to some $v \in BPV(\Omega; \{-1,1\})$ in $L^2(\Omega)$. But since $u = v$, we have $u \in BPV(\Omega; \{-1,1\})$, contradiction. Hence we need only consider the case $u \in BPV(\Omega; \{-1,1\})$.

Step 1. If $u \notin L^2(\Omega; \{-1,1\})$ then for some $\delta > 0$ the set $E := \{x \in \Omega : ||u(x)| - 1| \geq \delta\}$ has positive measure. We may now use Minkowski's inequality to write

$$\begin{aligned}\int_E \varepsilon_n^{-1}(u_n^2 - 1)^2\,dx &\geq \int_E \varepsilon_n^{-1}(|u_n| - 1)^2\,dx \\ &\geq \varepsilon_n^{-1}\left(\left(\int_E (|u| - 1)^2\,dx\right)^{1/2} - \left(\int_E \big||u| - |u_n|\big|^2\,dx\right)^{1/2}\right)^2 \\ &\geq \varepsilon_n^{-1}\left(\delta\mathcal{L}^1(E)^{1/2} - \left(\int_E |u - u_n|^2\,dx\right)^{1/2}\right)^2.\end{aligned}$$

It is clear that $F_{\varepsilon_n}(u_n) \to +\infty$ because $\|u - u_n\|_{L^2(E)} \to 0$ and $\varepsilon_n^{-1} \to +\infty$. Hence $\Gamma\text{-}\lim_{n\to\infty} F_{\varepsilon_n}(u) = +\infty$.

Step 2. We now consider the case $u \in BPV(\Omega; \{-1,1\})$.

Suppose u jumps between -1 and 1 (or vice versa) at the J points $x_1, x_2, \cdots, x_J \in \Omega$. Find $\delta_1, \cdots, \delta_J > 0$ so small that the intervals (a_i, b_i) are disjoint, where $(a_i, b_i) := (x_i - \delta_i, x_i + \delta_i)$.

Let $u_n \to u$ in $L^2(\Omega)$. Define $L := \liminf\limits_{n\to\infty} F_{\varepsilon_n}(u_n)$. For the liminf inequality, we need to prove that $L \geq \alpha \operatorname{essVar} u$.

By extraction of a subsequence, assume that $L = \lim\limits_{n\to\infty} F_{\varepsilon_n}(u_n)$. Fix an arbitrary $\eta \in (0,1)$, and find N_η so large that

$$L + \eta \geq F_{\varepsilon_n}(u_n)$$

for all $n \geq N_\eta$.

Fix some x_i with u jumping from -1 to 1 at x_i.

The key claim that we shall prove in this step is that, for all n large enough, we may modify u_n in (a_i, b_i) by "anchoring" it to -1 at a_i and to 1 at b_i with a derivative of 0 at both points, without spending more than $C\eta$ potential energy for a constant C not depending on η. To be precise,, there exists $v_{n,i} : [a_i, b_i] \to \mathbb{R}$ such that:

- $v_{n,i} \in W^{2,2}(a_i, b_i)$,
- $v_{n,i}(a_i) = -1$, $v_{n,i}(b_i) = 1$,

- $v'_{n,i}(a_i) = v'_{n,i}(b_i) = 0$,
- $\displaystyle\int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v''_{n,i}|^2\right] dx \le \int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2\right] dx + C\eta.$

To do this, we consider only $n \ge N_\eta$, and it suffices to modify u_n in (x_i, b_i) by affixing to 1 in the manner described. First, observe that since $u_n \to u$ in $L^2(\Omega)$, we have that $u_n \to 1$ in $L^2(x_i, b_i)$, so $\|u_n - 1\|_{L^2(x_i,b_i)} \to 0$ as $n \to +\infty$. Putting this aside, note that we also have $\int_{x_i}^{b_i} \frac{|u_n-1|^2}{\|u_n-1\|^2_{L^2(x_i,b_i)}}\, dx = 1$. Combining this with $F_{\varepsilon_n}(u_n) \le L + \eta$, it follows that

$$\int_{x_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2 + \frac{|u_n - 1|^2}{\|u_n - 1\|^2_{L^2(x_i,b_i)}}\right] dx \le L + \eta + 1.$$

Now let $K_n := \lceil \varepsilon_n^{-1} \rceil$ and subdivide the interval (x_i, b_i) into K_n same-length intervals (y_{j-1}, y_j) so that

$$x_i = y_0 < y_1 < \cdots < y_{K_n} = b_i.$$

By a “pigeonhole principle-like argument", there exists $1 \le j \le K_n$ such that

$$\int_{y_{j-1}}^{y_j} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2 + \frac{|u_n - 1|^2}{\|u_n - 1\|^2_{L^2(x_i,b_i)}}\right] dx \le \frac{L + \eta + 1}{K_n} \le \varepsilon_n(L + \eta + 1). \quad (3.7)$$

A particular consequence is the bound $\int_{y_{j-1}}^{y_j} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2\right] dx \le \frac{L+\eta+1}{K_n} \le \varepsilon_n(L + \eta + 1)$, which implies by Theorem 3.2 (applied with the intervals in the inclusion $(\frac{2}{3}y_{j-1} + \frac{1}{3}y_j, \frac{1}{3}y_{j-1} + \frac{2}{3}y_j) \Subset (y_{j-1}, y_j)$) that

$$\int_{\frac{2}{3}y_{j-1}+\frac{1}{3}y_j}^{\frac{1}{3}y_{j-1}+\frac{2}{3}y_j} \varepsilon_n |u'_n|^2\, dx \le c \int_{y_{j-1}}^{y_j} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2\right] dx \le \varepsilon_n c(L + \eta + 1)$$

for all sufficiently large n, for a universal constant c. Combining this with (3.7), we obtain

$$\int_{\frac{2}{3}y_{j-1}+\frac{1}{3}y_j}^{\frac{1}{3}y_{j-1}+\frac{2}{3}y_j} \left[\varepsilon_n |u'_n|^2 + \frac{|u_n - 1|^2}{\|u_n - 1\|^2_{L^2(x_i,b_i)}}\right] dx \le \varepsilon_n(c + 1)(L + \eta + 1).$$

Let $H := \varepsilon_n |u'_n|^2 + \frac{|u_n-1|^2}{\|u_n-1\|^2_{L^2(x_i,b_i)}}$. If there does not exist $x \in (\frac{2}{3}y_{j-1} + \frac{1}{3}y_j, \frac{1}{3}y_{j-1} + \frac{2}{3}y_j)$ for which $H(x) \le \frac{6K_n\varepsilon_n(c+1)(L+\eta+1)}{b_i - x_i}$, then

$$\varepsilon_n(c + 1)(L + \eta + 1) \ge \int_{\frac{2}{3}y_{j-1}+\frac{1}{3}y_j}^{\frac{1}{3}y_{j-1}+\frac{2}{3}y_j} H\, dx \ge \left(\frac{6K_n\varepsilon_n(c + 1)(L + \eta + 1)}{b_i - x_i}\right)\left(\frac{y_j - y_{j-1}}{3}\right)$$

$$= \left(\frac{6K_n\varepsilon_n(c + 1)(L + \eta + 1)}{b_i - x_i}\right)\left(\frac{b_i - x_i}{3K_n}\right) = 2\varepsilon_n(c + 1)(L + \eta + 1),$$

which is a contradiction. We conclude that for all large enough n, there exists $x_0 \in (\frac{2}{3}y_{j-1} + \frac{1}{3}y_j, \frac{1}{3}y_{j-1} + \frac{2}{3}y_j)$, depending on n, such that $H(x_0) \leq (C_1/2)K_n\varepsilon_n \leq (C_1/2)\left(\frac{\varepsilon_n+1}{\varepsilon_n}\right)\varepsilon_n = (C_1/2)(\varepsilon_n + 1) \leq C_1$ for a constant C_1 that does not depend on n. In particular, we now know that such an x_0 satisfies

1. $\varepsilon_n^2|u_n'(x_0)|^2 \leq \varepsilon_n C_1$ and
2. $|u_n(x_0) - 1| \leq C_1^{1/2}\|u_n - 1\|_{L^2(x_i,b_i)}$.

These properties, combined with the fact that $u_n \to 1$ in $L^2(x_i, b_i)$, imply that, for all large enough n, we have $\varepsilon_n^2|u_n'(x_0)|^2 + |u_n(x_0) - 1|^2 \leq \eta$.

Let $A := u_n(x_0) - 1$, $m := u_n'(x_0)$, and $T := y_j - x_0$. Using these constants, we may find a smooth $f : [0, T] \to \mathbb{R}$ as described in Lemma 3.6. We are now ready to define $v_{n,i}$ over (x_i, b_i) as

$$v_{n,i}(x) := \begin{cases} u_n(x), & x_i < x \leq x_0 \\ f(x - x_0) + 1, & x_0 < x \leq y_j \\ 1, & y_j < x < b_i \end{cases}.$$

By virtue of f being a smooth connector, we must have $v_{n,i} \in W^{2,2}(x_i, b_i)$. Moreover, by property (iii) in Lemma 3.6, we have the bounds

$$\int_{x_i}^{b_i} (v_{n,i} - 1)^2\, dx = \int_{x_0}^{y_j} |f(x - x_0)|^2\, dx \leq C_0(A^2 + m^2T^2)T \tag{3.8}$$

and

$$\int_{x_i}^{b_i} |v_{n,i}''|^2\, dx = \int_{x_0}^{y_j} |f''(x - x_0)|^2\, dx \leq C_2(A^2 + m^2T^2)T^{-3} \tag{3.9}$$

for universal constants $C_0, C_2 > 0$.

Since (3.8) is not quite a bound on the integral of $(v_{n,i}^2 - 1)^2$, we will need to prove the inequality

$$\int_{x_i}^{b_i} (v_{n,i}^2 - 1)^2\, dx \leq C' \int_{x_i}^{b_i} (v_{n,i} - 1)^2\, dx \tag{3.10}$$

for all n large enough, for some constant $C' > 0$. Indeed, observe that for all $x \in (x_i, b_i)$, we have by property (iv) of Lemma 3.6 that

$$|v_{n,i}(x) + 1| \leq 2 + |v_{n,i} - 1| \leq 2 + \sup_{[0,T]} |f| \leq 2 + |A| + \frac{1}{2}|m| \cdot |T|.$$

Now, $T \leq \varepsilon_n$, and from $\varepsilon_n^2|u_n'(x_0)|^2 + |u_n(x_0) - 1|^2 \leq \eta \leq 1$ we have that $|A| = |u_n(x_0) - 1| \leq 1$ and $|m||T| = |u_n'(x_0)|(y_j - x_0) \leq \varepsilon_n|u_n'(x_0)| \leq 1$. We hence obtain $|v_{n,i}(x) + 1| \leq 4$ for all $x \in (x_i, b_i)$, and so taking $C' = 4^2$ we get

$$\int_{x_i}^{b_i} (v_{n,i}^2 - 1)^2\, dx = \int_{x_i}^{b_i} (v_{n,i} - 1)^2(v_{n,i} + 1)^2\, dx \leq C' \int_{x_i}^{b_i} (v_{n,i} - 1)^2,$$

as we wanted.

We now add (3.8) to (3.9) and apply (3.10) to obtain

$$\begin{aligned}
&\int_{x_0}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \\
&\leq \max(C_0, C_2) \max(1, C') \left[\frac{T}{\varepsilon_n}(A^2 + m^2T^2) + \frac{\varepsilon_n^3}{T^3}(A^2 + m^2T^2)\right] \\
&= C_3(A^2 + m^2T^2)\left(\frac{T}{\varepsilon_n} + \frac{\varepsilon_n^3}{T^3}\right)
\end{aligned}$$

for some constant $C_3 > 0$.

Since $\frac{2}{3}y_{j-1} + \frac{1}{3}y_j < x_0 < \frac{1}{3}y_{j-1} + \frac{2}{3}y_j$, we have that $(b_i - x_i)\varepsilon_n \geq \frac{b_i - x_i}{K_n} = y_j - y_{j-1} > y_j - x_0 = T > y_j - (\frac{1}{3}y_{j-1} + \frac{2}{3}y_j) = \frac{1}{3}(y_j - y_{j-1}) = \frac{b_i - x_i}{3K_n} \geq \frac{b_i - x_i}{3} \cdot \frac{\varepsilon_n}{1+\varepsilon_n}$. We use this to write

$$\begin{aligned}
&\int_{x_0}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \\
&\leq C_3((u_n(x_0) - 1)^2 + |u_n'(x_0)|^2 T^2)\left(\frac{T}{\varepsilon_n} + \frac{\varepsilon_n^3}{T^3}\right) \\
&\leq C_3((u_n(x_0) - 1)^2 + |u_n'(x_0)|^2 (b_i - x_i)^2 \varepsilon_n^2)\left(\frac{(b_i - x_i)\varepsilon_n}{\varepsilon_n} + \varepsilon_n^3 \frac{27(1+\varepsilon_n)^3}{(b_i - x_i)^3 \varepsilon_n^3}\right) \\
&\leq C_3((u_n(x_0) - 1)^2 + |u_n'(x_0)|^2 (b_i - x_i)^2 \varepsilon_n^2)\left(b_i - x_i + \frac{27 \cdot 8}{(b_i - x_i)^3}\right) \\
&\leq C_4((u_n(x_0) - 1)^2 + |u_n'(x_0)|^2 \varepsilon_n^2) \leq C_4 \eta,
\end{aligned}$$

Where $C_4 > 0$ is a constant with no dependence on n, and we have applied our choice of x_0. Finally, it follows that

$$\begin{aligned}
&\int_{x_i}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \\
&= \int_{x_i}^{x_0} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx + \int_{x_0}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \\
&= \int_{x_i}^{x_0} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u_n''|^2\right] dx + \int_{x_0}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \\
&= \int_{x_i}^{x_0} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u_n''|^2\right] dx + C_4 \eta \\
&\leq \int_{x_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u_n''|^2\right] dx + C_4 \eta.
\end{aligned}$$

Arguing similarly for the interval (a_i, x_i), we obtain

$$\int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}^2 - 1)^2 + \varepsilon_n^3 |v_{n,i}''|^2\right] dx \leq \int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u_n''|^2\right] dx + 2C_4 \eta. \tag{3.11}$$

Thus the key claim has been proven.

Step 3. We now complete the liminf argument. We recall the definitions of the family $\mathscr{J}$ and the constant α from (1.6) and (1.8) respectively.

Let us first write

$$L+\eta \geq F_{\varepsilon_n}(u_n) \geq \sum_{i=1}^{J} \int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2-1)^2+\varepsilon_n^3|u_n''|^2\right]\,dx$$

for all large enough n. We apply the key claim (3.11) to every interval (a_i,b_i) to get

$$L+\eta \geq -2C_4J\eta + \sum_{i=1}^{J} \int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(v_{n,i}(x)^2-1)^2+\varepsilon_n^3|v_{n,i}''(x)|^2\right]\,dx.$$

Consider the change of variables $y := \frac{x-a_i}{b_i-a_i}$. Defining $w_{n,i} : [0,1] \to \mathbb{R}$ via $w_{n,i}(y) = v_{n,i}\left((b_i-a_i)y+a_i\right)$, we have that $w_{n,i}''(y) = (b_i-a_i)^2 v_{n,i}''\left((b_i-a_i)y+a_i\right)$, so that we get

$$\begin{aligned} L+\eta &\geq -2C_4J\eta + \sum_{i=1}^{J} (b_i-a_i) \int_0^1 \left[\varepsilon_n^{-1}(w_{n,i}(y)^2-1)^2 + \frac{\varepsilon_n^3}{(b_i-a_i)^4}|w_{n,i}''(y)|^2\right] dy \\ &= -2C_4J\eta + \sum_{i=1}^{J} \frac{b_i-a_i}{\varepsilon_n} \int_0^1 (w_{n,i}(y)^2-1)^2\,dy + \frac{\varepsilon_n^3}{(b_i-a_i)^3}\int_0^1 |w_{n,i}''(y)|^2\,dy, \end{aligned}$$

and by applying the weighted AM-GM inequality we may go down again to obtain

$$\begin{aligned} L+\eta &\geq -2C_4J\eta + \sum_{i=1}^{J} \frac{4}{3^{3/4}} \left(\int_0^1 (w_{n,i}(y)^2-1)^2\,dy\right)^{3/4} \left(\int_0^1 |w_{n,i}''(y)|^2\,dy\right)^{1/4} \\ &= -2C_4J\eta + \sum_{i=1}^{J} \frac{4}{3^{3/4}}\Phi(w_{n,i}), \end{aligned}$$

where Φ is defined in (1.5). Finally, as $w_{n,i} \in \mathscr{J}$ for all i, we have that $\frac{4}{3^{3/4}}\Phi(w_{n,i}) \geq \frac{4}{3^{3/4}} \inf_{w\in\mathscr{J}} \Phi(w) = 2\alpha$, thus

$$\begin{aligned} L+\eta &\geq -2C_4J\eta + \sum_{i=1}^{J} 2\alpha \\ &= -2C_4J\eta + 2J\alpha = -2C_4J\eta + \alpha \operatorname*{essVar}_{\Omega} u. \end{aligned}$$

Hence $L+\eta \geq -2C_4J\eta+\alpha\,\mathrm{essVar}_\Omega\, u$. As η was arbitrary, we conclude that $L \geq \alpha\,\mathrm{essVar}_\Omega\, u$, as needed. This proves the liminf inequality for $u \in BPV(\Omega;\{-1,1\})$.

Step 4. It remains to prove the limsup inequality for $u \in BPV(\Omega;\{-1,1\})$. This entails finding a sequence $\{u_n\}_n \subset W^{2,2}(\Omega)$ for which $u_n \to u$ in $L^2(\Omega)$ and $\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) \leq \alpha\,\mathrm{Var}_\Omega\, u$.

First, recall the definition of Φ as in (1.5) and the family $\mathscr{J}$ as in (1.6). Define the subfamily

$$\mathscr{J}^{(n)} := \left\{ h \in \mathscr{J} : \frac{\int_0^1 |h''|^2\,dx}{\int_0^1 (h^2-1)^2\,dx} \le \frac{1}{\varepsilon_n} \right\}.$$

Since $\varepsilon_n \to 0^+$, it is clear that $\bigcup_{n=1}^{\infty} \mathscr{J}^{(n)} = \mathscr{J}$. Setting

$$S_n := \left\{ \Phi(h) : h \in \mathscr{J}^{(n)} \right\},$$

$$S := \left\{ \Phi(h) : h \in \mathscr{J} \right\},$$

we have $\bigcup_{n=1}^{\infty} S_n = S$, and it can be shown that $\lim_{n\to\infty} \inf S_n = \inf S$. We note also that α, as defined in (1.8), may be written as $\alpha = \frac{2}{3^{3/4}} \inf S$.

For all n, find $h_n \in \mathscr{J}^{(n)}$ for which

$$\inf S_n \le \Phi(h_n) \le \frac{1}{n} + \inf S_n. \tag{3.12}$$

As in Step 2, suppose u "jumps" at the points $x_1, x_2, \cdots, x_J \in \Omega$, and find pairwise disjoint intervals $(a_i, b_i) \subseteq \Omega$ with $x_i \in (a_i, b_i)$. Fix x_i such that u jumps from -1 to 1 at x_i. Let ζ_n be a sequence of positive reals with $\zeta_n \to 0^+$ that we shall choose later. We define u_n in (a_i, b_i) as

$$u_n(x) := \begin{cases} -1, & a_i < x < x_i - \zeta_n/2 \\ h_n\left(\frac{x - x_i + \zeta_n/2}{\zeta_n}\right), & x_i - \zeta_n/2 \le x \le x_i + \zeta_n/2 \\ 1, & x_i + \zeta_n/2 < x < b_i \end{cases}.$$

We define u_n similarly over all other intervals (a_j, b_j), and in $\Omega \setminus \bigcup_{j=1}^{J}(a_j, b_j)$ we let u_n agree with u. Since $h_n(0_+) = -1$, $h_n(1_-) = 1$, and $h_n'(0_+) = h_n'(1_-) = 0$, we have that $u_n \in W^{2,2}(\Omega)$. Moreover, $u_n \to u$ almost everywhere as a consequence of $\zeta_n \to 0^+$, and $\{|u_n|\}_n$ is equi-integrable, so by Vitali's Convergence Theorem we have $u_n \to u$ in $L^2(\Omega)$. It remains to verify the limsup inequality. We have

$$\begin{aligned} F_{\varepsilon_n}(u_n) &= \sum_{i=1}^{J} \int_{a_i}^{b_i} \left[\varepsilon_n^{-1}(u_n^2-1)^2 + \varepsilon_n^3 |u_n''|^2\right] dx \\ &= \sum_{i=1}^{J} \int_{a_i}^{b_i} \left[\varepsilon_n^{-1} \left(h_n\left(\frac{x - x_i + \zeta_n/2}{\zeta_n}\right)^2 - 1 \right)^2 + \frac{\varepsilon_n^3}{\zeta_n^4} \left| h_n''\left(\frac{x - x_i + \zeta_n/2}{\zeta_n}\right) \right|^2 \right] dx, \end{aligned}$$

and after changing variables we get

$$\begin{aligned} F_{\varepsilon_n}(u_n) &= \sum_{i=1}^{J} \zeta_n \int_0^1 \left[\varepsilon_n^{-1} \left(h_n(y)^2 - 1\right)^2 + \frac{\varepsilon_n^3}{\zeta_n^4} |h_n''(y)|^2 \right] dy \\ &= \sum_{i=1}^{J} \frac{\zeta_n}{\varepsilon_n} \int_0^1 \left(h_n(y)^2 - 1\right)^2 dy + \frac{\varepsilon_n^3}{\zeta_n^3} \int_0^1 |h_n''(y)|^2\, dy. \end{aligned} \tag{3.13}$$

We now choose ζ_n so that we obtain the equality case in the AM-GM inequality. Specifically, we choose

$$\zeta_n := \varepsilon_n \left(\frac{3 \int_0^1 |h_n''(y)|^2 \, dy}{\int_0^1 \left(h_n(y)^2 - 1\right)^2 \, dy} \right)^{1/4} .$$

Showing that this choice is valid for all sufficiently large n amounts to proving that $\zeta_n \to 0^+$. Indeed, since $h_n \in \mathscr{J}^{(n)}$, we have that

$$\zeta_n \leq \varepsilon_n \left(\frac{3}{\varepsilon_n} \right)^{1/4} = 3^{1/4} \varepsilon_n^{3/4} \to 0^+.$$

With the choice of ζ_n justified, we now continue the argument in (3.13) by using the choice of ζ_n and (3.12) to write

$$F_{\varepsilon_n}(u_n) = \sum_{i=1}^{J} \frac{4}{3^{3/4}} \left(\int_0^1 (h_n^2 - 1)^2 \, dy \right)^{3/4} \left(\int_0^1 |h_n''|^2 \, dy \right)^{1/4} \leq \frac{4J}{3^{3/4}} \left(\frac{1}{n} + \inf S_n \right).$$

Taking the limsup, we obtain

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) \leq \limsup_{n\to\infty} \frac{4J}{3^{3/4}} \left(\frac{1}{n} + \inf S_n \right) = \frac{2 \operatorname{Var}_\Omega u}{3^{3/4}} (\inf S) = \alpha \operatorname*{Var}_\Omega u.$$

This completes the proof. □

Chapter 4

Boundary Conditions

The Gamma limit will change upon restricting to the boundary conditions $u(a_+) = a_\varepsilon$ and $u(b_-) = b_\varepsilon$ for $a_\varepsilon \to -1$ and $b_\varepsilon \to 1$ as $\varepsilon \to 0^+$. A portion of the work needed to account for the boundary conditions has already been done in the proof of Theorem 3.5. Intuitively, every jump in the interior of (a, b) induces a factor of α, whereas a jump at the boundary induces a factor of $\beta(t)$ depending on the height of the jump. Recall that $\beta(t)$ is defined as in the statement of Theorem 1.1, and we will again reference the family $\mathscr{J}_1(t)$ and the functional Φ, defined in (1.7) and (1.5) respectively.

We first introduce a new family of functions $\mathscr{J}_2(t)$, defined as

$$\mathscr{J}_2(t) := \left\{u \in W^{2,2}_{\text{loc}}(-\infty, 0) : u(0) = t \text{ and } u(x) = -1 \text{ for all } x \le -L \text{ for some } L > 0\right\} \tag{4.1}$$

for a parameter $t \in \mathbb{R}$. We associate with each $u \in \mathscr{J}_2(t)$ the constant L_u, where $-L_u$ is the first time that u reaches -1 and remains at this value indefinitely. That is,

$$L_u := \inf \{L > 0 : u(x) = -1 \text{ for all } x \le -L\}. \tag{4.2}$$

We also define a new functional $\Psi : W^{2,2}_{\text{loc}}(-\infty, 0) \to \overline{\mathbb{R}}$ via

$$\Psi(u) := \int_{-\infty}^{0} \left[(u^2 - 1)^2 + |u''|^2\right] dx.$$

Lemma 4.1

$$\beta(t) = \frac{4}{3^{3/4}} \inf_{u \in \mathscr{J}_1(t)} \Phi(u) = \inf_{v \in \mathscr{J}_2(t)} \Psi(v).$$

Proof. We first note that if $t = -1$ then both $\mathscr{J}_1(t)$ and $\mathscr{J}_2(t)$ contain the constant function -1, thus $\inf_{u \in \mathscr{J}_1(t)} \Phi(u) = 0$ and $\inf_{u \in \mathscr{J}_2(t)} \Psi(u) = 0$. Now assume that $t \neq -1$.

Consider $u \in \mathscr{J}_1(t)$. Take $L > 0$ depending only on u (that we shall choose later), and define $v \in \mathscr{J}_2(t)$ via

$$v(x) := \begin{cases} -1, & x \leq -L \\ u(x/L+1), & -L < x < 0 \end{cases}.$$

With the change of variables $y := x/L + 1$, we have

$$\begin{aligned} \Psi(v) &= \int_{-L}^{0} \left[(v(x)^2-1)^2 + |v''(x)|^2\right] \, dx \\ &= \int_{-L}^{0} \left[(u(x/L+1)^2-1)^2 + \frac{1}{L^4}|u''(x/L+1)|^2\right] \, dx \\ &= \int_{0}^{1} \left[L(u(y)^2-1)^2 + \frac{1}{L^3}|u''(y)|^2\right] \, dy. \end{aligned} \tag{4.3}$$

By examination of the equality case of the AM-GM inequality, this is precisely $\frac{4}{3^{3/4}}\Phi(u)$ for a proper choice of L. Specifically, one may take $L := \left(\frac{3\int_0^1 |u''|^2 \, dx}{\int_0^1 (u^2-1)^2 \, dx}\right)^{1/4}$, which is well-defined from the assumption that $t \neq -1$. This gives $\frac{4}{3^{3/4}}\Phi(u) = \Psi(v) \geq \inf_{w\in \mathscr{J}'_\infty(t)} \Psi(w)$, and taking the infimum gives $\inf_{u\in\mathscr{J}_1(t)} \Phi(u) \geq \inf_{w\in\mathscr{J}'_\infty(t)} \Psi(w)$.

On the other hand, if $v \in \mathscr{J}_2(t)$, then from taking $u(x) := v\left(L_v(x-1)\right)$, where L_v is defined in (4.2), we get from using the same computations done in (4.3) and applying the AM-GM inequality that

$$\begin{aligned} \Psi(v) &= \int_{0}^{1} \left[L_v(u(y)^2-1)^2 + \frac{1}{L_v^3}|u''(y)|^2\right] \, dy \\ &\geq \frac{4}{3^{3/4}} \left(\int_0^1 (u(y)^2-1)^2 \, dy\right)^{3/4} \left(\int_0^1 |u''(y)|^2 \, dy\right)^{1/4} \\ &= \frac{4}{3^{3/4}}\Phi(u), \end{aligned}$$

so that $\Psi(v) \geq \frac{4}{3^{3/4}} \inf_{u\in\mathscr{J}_1(t)} \Phi(u)$. Thus $\inf_{v\in\mathscr{J}_2(t)} \Psi(v) \geq \frac{4}{3^{3/4}} \inf_{u\in\mathscr{J}_1(t)} \Phi(u)$, finishing the proof. □

We will now prove several crucial results concerning the family $\mathscr{J}_2(t)$ and the functional Ψ. The first is a compactness result.

Lemma 4.2 *Let $\Omega \subseteq \mathbb{R}$ be a bounded open set and let $u_n \in W^{2,2}(\Omega)$ be such that*

$$M := \sup_{n\in\mathbb{N}} \int_\Omega \left[(u_n^2-1)^2 + |u_n''|^2\right] \, dx < \infty.$$

Then there exist a subsequence $\{u_{n_k}\}_k$ and $u \in W^{2,2}(\Omega)$ such that

1. $u_{n_k} \rightharpoonup u$ *in* $W^{2,2}(\Omega)$,

2. $u_{n_k} \to u$ *uniformly, and*

3. $u'_{n_k} \to u'$ *uniformly.*

Proof. Find $A > 0$ large enough so that $z^2 \le (z^2-1)^2$ for all $|z| \ge A$. Then

$$\int_\Omega u_n^2\,dx \le \int_{\{|u_n|<A\}} u_n^2\,dx + \int_{\{|u_n|\ge A\}} u_n^2\,dx \le A^2\mathcal{L}^1(\Omega) + A\int_\Omega (u_n^2-1)^2 \le A^2\mathcal{L}^1(\Omega) + AM,$$

so $\{u_n\}_n$ is uniformly bounded in $L^2(\Omega)$. Since $\{u''_n\}_n$ is uniformly bounded in $L^2(\Omega)$ as well, we may apply Theorem 2.13 with some $l < \mathcal{L}^1(\Omega)$ to deduce that $\{u'_n\}_n$ is uniformly bounded in $L^2(\Omega)$. Hence $\{u_n\}_n$ is uniformly bounded in $W^{2,2}(\Omega)$.

By Theorem 2.17, there exist a subsequence $\{u_{n_k}\}_k$ and $u \in W^{2,2}(\Omega)$ such that $u_{n_k} \rightharpoonup u$ in $W^{2,2}(\Omega)$. Moreover, since $\{u_{n_k}\}_k$ and $\{u'_{n_k}\}_k$ are both bounded in $W^{1,2}(\Omega)$, we have by Morrey's Embedding (Theorem 2.14) that both $\{u_{n_k}\}_k$ and $\{u'_{n_k}\}_k$ are bounded in $C^{0,1/2}(\Omega)$. In particular, both sequences are uniformly bounded and uniformly equicontinuous, so by the Ascoli-Arzela Theorem we may extract a further subsequence $\{u_{n_{k_j}}\}_j$ such that $u_{n_{k_j}} \to u$ uniformly and $u'_{n_{k_j}} \to u'$ uniformly.

□

Next, we show that we may "force" a bound on L_u for functions $u \in \mathscr{I}_2(t)$ (see (4.1) and (4.2)).

Lemma 4.3 *Let* $M, \eta > 0$. *Then there exists a constant* $L_{M,\eta} > 0$, *depending only on* M *and* η, *such that for every* $t \in \mathbb{R}$ *and* $u \in \mathscr{I}_2(t)$ *with* $\Psi(u) \le M < \infty$, *there exists* $v \in \mathscr{I}_2(t)$ *such that* $\Psi(v) \le \Psi(u) + O(\eta)$ *and* $v(x) = -1$ *for all* $x \le -L_{M,\eta}$, *that is,* $L_v \le L_{M,\eta}$.

Proof. **Step 1.** For ease, we may assume that $\eta < \frac{1}{2}$. We begin by constructing a function $v_1 \in \mathscr{I}_2(t)$ such that $\Psi(v_1) \le \Psi(u) + O(\eta)$ and $\mathcal{L}^1(v_1^{-1}((0,2))) \le C$ for a constant C depending only on M and η.

First, let us apply Theorem 3.2 to the interval $(-L_u - 1, 0)$ with $\varepsilon = 1$ to obtain

$$\int_{-L_u}^{-1} |u'|\,dx \le C' \int_{-L_u-1}^{0} \left[(u^2-1)^2 + |u''|^2\right] dx \le C'M \tag{4.4}$$

for a universal constant C'. We may assume that $C' > 4$.

Let $K := \frac{(C'+1)(M+1)}{\eta}$ and $C := 2K+3$. Let $E := u^{-1}((0,2))$. If $\mathcal{L}^1(E) \le 2K+3$, then set $v_1 := u$. Otherwise, we have $\mathcal{L}^1(E \cap (-\infty,-1)) > 2K+2$, and consider $E_1 := (-\infty, y) \cap E$,

$F := [y,z] \cap E$, and $E_2 := (z,-1) \cap E$, where y, z are chosen such that $\mathcal{L}^1(E_1) = \mathcal{L}^1(E_2) = K$ and $\mathcal{L}^1(F) = \mathcal{L}^1(E) - 2K > 2$.

We use the fact that $u > 0$ over E_1, the inequality (4.4), and the inclusion $E_1 \subseteq (-L_u, -1)$ to write

$$\begin{aligned} M &\geq \int_{-L_u-1}^{0} \left[(u^2-1)^2 + |u''|^2\right] dx \\ &\geq \frac{1}{2}\int_{E_1} (u^2-1)^2\,dx + \frac{1}{2}\int_{-L_u-1}^{0} \left[(u^2-1)^2 + |u''|^2\right] dx \\ &\geq \frac{1}{2}\int_{E_1} (u-1)^2\,dx + \frac{1}{2C'}\int_{-L_u}^{-1} |u'|^2\,dx \\ &\geq \frac{1}{2}\int_{E_1} (u-1)^2 + \frac{1}{2C'}|u'|^2\,dx, \end{aligned}$$

so there exists $x_1 \in E_1$ such that $\frac{1}{2}(u(x_1)-1)^2 + \frac{1}{2C'}|u'(x_1)|^2 \leq \frac{M}{\mathcal{L}^1(E_1)} = \frac{M}{K} \leq \frac{\eta}{C'}$. In particular we have $(u(x_1)-1)^2 \leq \frac{2\eta}{C'}$ and $|u'(x_1)|^2 \leq 2\eta$. By the assumptions on C' and η, we note that $|u(x_1)-1| < \frac{1}{2}$ and $|u'(x_1)| < 1$.

We now apply Lemma 3.6 to u over the interval $[x_1, x_1+1]$, with $A := u(x_1) - 1$ and $m := u'(x_1)$, to see that there exists $\tilde{u}_1 \in C^\infty([x_1, x_1+1])$ such that $\tilde{u}_1(x_1) = u(x_1)$, $\tilde{u}_1'(x_1) = u'(x_1)$, $\tilde{u}_1(x_1+1) = 1$, and $\tilde{u}_1'(x_1+1) = 0$, with $\sup_{[x_1,x_1+1]} |\tilde{u}_1 - 1| \leq |u(x_1)-1| + \frac{1}{2}|u'(x_1)| < 1$,

$$\int_{x_1}^{x_1+1} (\tilde{u}_1 - 1)^2 \leq C''((u(x_1)-1)^2 + |u'(x_1)|^2),$$

and

$$\int_{x_1}^{x_1+1} |\tilde{u}_1'|^2 \leq C''((u(x_1)-1)^2 + |u'(x_1)|^2),$$

for a universal constant $C'' > 0$. Since $|\tilde{u}_1 - 1| < 1$ on $[x_1, x_1+1]$, we may write

$$(\tilde{u}_1^2 - 1)^2 = (\tilde{u}_1 - 1)^2(\tilde{u}_1 + 1)^2 \leq (\tilde{u}_1 - 1)^2(2+1)^2 = 9(\tilde{u}_1 - 1)^2,$$

so we conclude that

$$\begin{aligned} \int_{x_1}^{x_1+1} \left[(\tilde{u}_1^2-1)^2 + |\tilde{u}''|^2\right] dx &\leq \int_{x_1}^{x_1+1} 9(\tilde{u}_1 - 1)^2\,dx + \int_{x_1}^{x_1+1} |\tilde{u}''|^2\,dx \\ &\leq 10C''((u(x_1)-1)^2 + |u'(x_1)|^2) \\ &\leq 10C''\left(\frac{2\eta}{C'} + 2\eta\right) = O(\eta). \end{aligned}$$

Here and henceforth, we assume that η is sufficiently small so that $O(\eta) < 1$. We construct $x_2 \in E_2$ and $\tilde{u}_2 \in C^\infty([x_2-1, x_2])$ in a similar manner so that $\sup_{[x_2-1,x_2]} |\tilde{u}_2 - 1| < 1$ and $\int_{x_2-1}^{x_2} (\tilde{u}_2 - 1)^2 + |\tilde{u}_2''|^2\,dx = O(\eta)$.

Now define $\tilde u$ as

$$\tilde u(x) := \begin{cases} u(x), & -\infty < x < x_1 \\ \tilde u_1(x), & x_1 \le x \le x_1+1 \\ 1, & x_1+1 < x < x_2-1 \\ \tilde u_2(x), & x_2-1 \le x \le x_2 \\ u(x), & x_2 < x < 0 \end{cases}. \tag{4.5}$$

Note that this is well-defined in the sense that $x_1+1 < x_2-1$ because $x_2 - x_1 > z - y \ge \mathcal{L}^1(F) \ge 2$. Moreover our choices for $\tilde u_1$ and $\tilde u_2$ ensure that $u \in W^{2,2}_{\mathrm{loc}}(-\infty, 0)$, and

$$\begin{aligned} \Psi(\tilde u) &= \int_{(-\infty,x_1)\cup(x_2,0)} \left[(\tilde u^2-1)^2 + |\tilde u''|^2\right] dx + \int_{[x_1,x_1+1]\cup[x_2-1,x_2]} \left[(\tilde u^2-1)^2 + |\tilde u''|^2\right] dx \\ &\le \int_{(-\infty,x_1)\cup(x_2,0)} \left[(u^2-1)^2 + |u''|^2\right] dx + \int_{[x_1,x_1+1]\cup[x_2-1,x_2]} \left[(u^2-1)^2 + |u''|^2\right] dx + O(\eta) \\ &\le \Psi(u) + O(\eta). \end{aligned}$$

We define

$$v_1(x) := \begin{cases} \tilde u(x - x_2 + x_1 + 2), & -\infty < x < x_2 - 1 \\ \tilde u(x), & x_2 - 1 < x < 0 \end{cases}.$$

In essence, we have "deleted" an interval in which $\tilde u = 1$. We still have $v_1 \in W^{2,2}_{\mathrm{loc}}(-\infty, 0)$ and $v_1(0_+) = u(0_+) = t$, with $\lim_{x\to-\infty} v_1(x) = -1$, so that $v_1 \in \mathscr{J}_2(t)$. Furthermore, $\Psi(v_1) = \Psi(\tilde u) \le \Psi(u) + O(\eta)$. Finally, we see that

$$\begin{aligned} \mathcal{L}^1(\{x<0 : 0 < v_1(x) < 2\}) &\le \mathcal{L}^1(\{x \in (-\infty, x_1+1) \cup (x_2-1, 0) : 0 < \tilde u(x) < 2\}) \\ &\le \mathcal{L}^1(\{x \in E_1 \cup E_2 : 0 < u(x) < 2\}) \\ &\qquad + \mathcal{L}^1([x_1, x_1+1]) + \mathcal{L}^1([x_2-1, x_2]) + \mathcal{L}^1((-1,0)) \\ &= 2K + 3 = C, \end{aligned}$$

which was our goal.

Step 2. Now we can construct v with $L_v \le L_{M,\eta}$, where we take $L_{M,\eta} := C + M + K + 3$. Define the set $G := \{-L_{M,\eta} + 1 < x < -1 : -2 < v_1(x) \le 0\}$, and observe that

$$\begin{aligned} C + M + K + 1 &\le \mathcal{L}^1((-L_{M,\eta}+1, -1)) \\ &\le \mathcal{L}^1(\{-L_{M,\eta}+1 < x < -1 : 0 < v_1(x) < 2\}) \\ &\qquad + \mathcal{L}^1(\{-L_{M,\eta}+1 < x < -1 : |v_1(x)| > 2\}) + \mathcal{L}^1(G). \end{aligned}$$

Since $\mathcal{L}^1(\{-L_{M,\eta}+1 < x < -1 : 0 < v_1(x) < 2\}) \le C$ by construction, and

$$M + 1 \ge \Psi(u) + 1 \ge \Psi(v_1) \ge \int_{\{|v_1|>2\}} \left[(v_1^2-1)^2 + |v_1''|^2\right] dx \ge \mathcal{L}^1(\{x<0 : |v_1| > 2\}),$$

we deduce that $\mathcal{L}^1(G) \ge K$.

We will now bound $\int_G (v_1+1)^2 + |v_1'|^2\,dx$. By Theorem 3.2, we have that

$$\int_G |v_1|^2\,dx \le \int_{-L_\eta+1}^{-1} |v_1|^2\,dx \le C' \int_{-L_\eta}^{0} \left[(v_1^2-1)^2 + |v_1''|^2\right]\,dx \le C'\Psi(v_1) \le C'(M+1),$$

and since $(v_1+1)^2 \le (v_1^2-1)^2$ over G, we evidently have that

$$\int_G (v_1+1)^2\,dx \le \int_G (v_1^2-1)^2\,dx \le \Psi(v_1) \le \Psi(u) + O(\eta) \le M+1.$$

In sum, we have that $\int_G \left[(v_1+1)^2 + |v_1'|^2\right]\,dx \le (C'+1)(M+1)$. Thus there exists $x_3 \in G$ such that

$$(v_1(x_3)+1)^2 + |v_1''(x_3)|^2 \le \frac{(C'+1)(M+1)}{\mathcal{L}^1(G)} \le \frac{(C'+1)(M+1)}{K} \le \eta.$$

Hence, as we did before to u, we may use Lemma 3.6 to modify v_1 in the interval (x_3-1, x_3) and obtain a function $v \in W^{2,2}_{\mathrm{loc}}(-\infty,0)$ for which $v(x) = -1$ for all $x < x_3 - 1$, $v(x) = v_1(x)$ for $x_3 < x < 0$, and $\int_{x_3-1}^{x_3} \left[(v_1^2-1)^2 + |v_1''|^2\right]\,dx \le O(\eta)$.

This function v satisfies $\Psi(v) \le \Psi(v_1) + O(\eta) \le \Psi(u) + O(\eta)$ and, since $-L_{M,\eta}+1 < x_3$, we have that $v(x) = -1$ for all $x < -L_{M,\eta}$, where $L_{M,\eta}$ depends only on M and η, as desired. □

Now we arrive at our first major result.

Lemma 4.4 *The function β, as defined in* (1.9), *is continuous.*

Proof. **Step 1.** We claim that $\beta(t)$ is decreasing over $t \le -1$ and increasing over $t \ge -1$.

First note that $\beta(t) \ge 0$ for all t and $\beta(-1) = 0$. Now suppose $s, t \in \mathbb{R}$ satisfy either $-1 < s \le t$ or $t \le s < -1$. We show that $\beta(s) \le \beta(t)$ by proving that for all $v \in \mathscr{I}_1(t)$ we may find $u \in \mathscr{I}_1(s)$ for which $\Phi(u) \le \Phi(v)$.

Indeed, since $v(1_-) = t$ and $v(0_+) = -1$, there must exist $T \in (0,1)$ for which $v(T) = s$. Take $u(x) := v(Tx)$. Clearly, $u \in \mathscr{I}_1(s)$, and we have

$$\int_0^1 (u^2-1)^2\,dx = \int_0^T T^{-1}(v^2-1)^2\,dx \le \int_0^1 T^{-1}(v^2-1)^2\,dx$$

and

$$\int_0^1 |u''|^2\,dx = \int_0^T T^3|v''|^2\,dx \le \int_0^1 T^3|v''|^2\,dx.$$

Thus

$$\begin{aligned}\Phi(u) &= \left(\int_0^1 (u^2-1)^2\,dx\right)^{3/4}\left(\int_0^1 |u''|^2\,dx\right)^{1/4}\\ &\le \left(\int_0^1 T^{-1}(v^2-1)^2\,dx\right)^{3/4}\left(\int_0^1 T^3|v''|^2\,dx\right)^{1/4}\\ &= \left(\int_0^1 (v^2-1)^2\,dx\right)^{3/4}\left(\int_0^1 |v''|^2\,dx\right)^{1/4}\\ &= \Phi(v),\end{aligned}$$

as needed.

Step 2. Fix $t_0 \in \mathbb{R}$. We show that $\limsup_{t\to t_0}\beta(t) \le \beta(t_0)$. It is sufficient to prove that for any $v \in \mathscr{J}_1(t_0)$, we may pick $u_t \in \mathscr{J}_1(t)$ for each $t \in \mathbb{R}$ such that $\lim_{t\to t_0}\Phi(u_t) = \Phi(v)$. This is because if we fix $\eta > 0$, then we may take $v \in \mathscr{J}_1(t_0)$ such that $\beta(t_0) \le \frac{4}{3^{3/4}}\Phi(v) \le \beta(t_0)+\eta$, take u_t for each $t \in \mathbb{R}$ as above, and then choose $\delta > 0$ so small that $|\Phi(v) - \Phi(u_t)| < \eta$ for all t with $|t - t_0| < \delta$, so that

$$\beta(t) \le \frac{4}{3^{3/4}}\Phi(u_t) \le \frac{4}{3^{3/4}}(\Phi(v)+\eta) \le \beta(t_0) + \eta + \frac{4}{3^{3/4}}\eta,$$

which is enough.

For $v \in \mathscr{J}_1(t_0)$, we take $u_t \in \mathscr{J}_1(t)$ to be $u_t := \frac{1+t}{1+t_0}(v+1) - 1$. Then

$$\int_0^1 (u_t^2-1)^2\,dx = \left(\frac{1+t}{1+t_0}\right)^4 \int_0^1 (v+1)^2\left(v+1-2\frac{1+t_0}{1+t}\right)^2 dx.$$

Taking the limit, we obtain

$$\lim_{t\to t_0}\int_0^1 (u_t^2-1)^2\,dx = \lim_{t\to t_0}\int_0^1 (v+1)^2\left(v+1-2\frac{1+t_0}{1+t}\right)^2 dx = \int_0^1 (v^2-1)^2\,dx$$

because $(v+1)^2\left(v+1-2\frac{1+t_0}{1+t}\right)^2$ converges to $(v^2-1)^2$ pointwise as $t \to t_0$ and, since v is bounded, we have that $(v+1)^2\left(v+1-2\frac{1+t_0}{1+t}\right)^2$ is uniformly bounded for all t sufficiently near t, which is enough to apply dominated convergence.

We also have

$$\lim_{t\to t_0}\int_0^1 |u_t''|^2\,dx = \lim_{t\to t_0}\left(\frac{1+t}{1+t_0}\right)^2\int_0^1 |v''|^2\,dx = \int_0^1 |v''|^2\,dx.$$

Altogether, we see that

$$\begin{aligned}\lim_{t\to t_0}\Phi(u_t) &= \left(\lim_{t\to t_0}\int_0^1 (u_t^2-1)^2\,dx\right)^{3/4}\left(\lim_{t\to t_0}\int_0^1 |u_t''|^2\,dx\right)^{1/4}\\ &= \left(\int_0^1 (v^2-1)^2\,dx\right)^{3/4}\left(\int_0^1 |v''|^2\,dx\right)^{1/4}\\ &= \Phi(v).\end{aligned}$$

This completes the proof that β is upper semi-continuous.

Step 3. We are now ready to show that $\liminf_{t\to t_0}\beta(t)\geq\beta(t_0)$.

Letting $t_n\to t_0$ be arbitrary, we just need to show that $\liminf_{n\to\infty}\beta(t_n)\geq\beta(t_0)$.

By the monotone properties that we have proven about β in Step 1, we see that $\{\beta(t_n) : n\in\mathbb{N}\}$ is bounded by a constant M. Specifically, we may take

$$M=\max\left\{\beta\left(\sup_{n\in\mathbb{N}} t_n\right),\beta\left(\inf_{n\in\mathbb{N}} t_n\right)\right\}.$$

Next, by extraction of a subsequence, we assume that there exists the limit $L := \lim_{n\to\infty}\beta(t_n)$. Fix $\eta>0$ and select $\tilde{u}_n\in\mathscr{J}_2(t_n)$ such that $\beta(t_n)\leq\Psi(\tilde{u}_n)\leq\beta(t_n)+\frac{1}{n}$. Since $\Psi(\tilde{u}_n)\leq M+1$ for all n, we may use Lemma 4.3 to find $u_n\in\mathscr{J}_2(t_n)$ such that $u_n(x)=-1$ for all $x\leq -L_{M,\eta}$, where $L_{M,\eta}$ depends only on M and η, and $\Psi(u_n)\leq\Psi(\tilde{u}_n)+O(\eta)$, so that

$$\beta(t_n)\leq\Psi(u_n)\leq\beta(t_n)+\frac{1}{n}+O(\eta). \tag{4.6}$$

Note that $\{\Psi(u_n)\}_n$ is uniformly bounded. In particular, $\sup_{n\in\mathbb{N}}\int_{-m}^0 [(u_n^2-1)^2+|u_n''|^2]\,dx<\infty$ for each $m\in\mathbb{N}$.

Consider $m=2$. By Lemma 4.2 we may extract a subsequence $\{u_{2,n}\}_n$ of $\{u_n\}_n$ for which $u_{2,n}\rightharpoonup v_2$ in $W^{2,2}(-2,0)$ for some $v_2\in W^{2,2}(-2,0)$ and $u_{2,n}\to v_2$ uniformly.

Inductively, for $m\geq 3$ we take $\{u_{m,n}\}_n$ to be a subsequence of $\{u_{m-1,n}\}_n$ such that $u_{m,n}\rightharpoonup v_m$ in $W^{2,2}(-m,0)$ for some $v_m\in W^{2,2}(-m,0)$ and such that $u_{m,n}\to v_m$ uniformly.

We claim that v_m extends v_{m-1} for all $m\geq 3$. Indeed, $u_{m-1,n}\to v_{m-1}$ almost everywhere, and since $\{u_{m,n}\}_n$ is a subsequence of $\{u_{m-1,n}\}_n$, we have that $u_{m,n}\to v_{m-1}$ almost everywhere over $(-m+1,0)$. Since $u_{m,n}\to v_m$ almost everywhere, it follows that v_{m-1} and v_m agree over $(-m+1,0)$ as needed.

It follows that there exists a unique $u:(-\infty,0)\to\mathbb{R}$ extending every v_m. Now consider $\{u_{n,n}\}_n$. For each m, $\{u_{n,n}\}_n$ may be viewed as a subsequence of $\{u_{m,n}\}_n$ for n large enough, so $u_{n,n} \to v_n = u$ almost everywhere in $(-m, 0)$. Thus $u_{n,n} \to u$ over $(-\infty, 0)$ almost everywhere. Similarly, we see that $u_{n,n} \rightharpoonup u$ in $W^{2,2}(-m, 0)$ for each $m \geq 2$. In particular, $u_{n,n} \rightharpoonup u$ in $W^{2,2}_{\text{loc}}(-\infty, 0)$.

Since $\{u_{n,n}\}_n$ is a subsequence of $\{u_n\}_n$, we let n_k be such that $\{u_{n_k}\}_k := \{u_{n,n}\}_n$.

To finish, we use the fact that $u_{n_k} \to u$ almost everywhere in $(-\infty, 0)$ to obtain the inequality

$$\int_{-\infty}^{0} (u^2 - 1)^2 \, dx \leq \liminf_{k\to\infty} \int_{-\infty}^{0} (u_{n_k}^2 - 1)^2 \, dx$$

by Fatou's Lemma. Next, we use the property that $u''_{n_k} \rightharpoonup u''$ in $L^2_{\text{loc}}(-\infty, 0)$ and the fact that $z \mapsto z^2$ is convex, together with Theorem 2.18, to conclude that

$$\int_{-\infty}^{0} |u''|^2 \, dx \leq \liminf_{k\to\infty} \int_{-\infty}^{0} |u''_{n_k}|^2 \, dx.$$

Combining these two inequalities gives $\Psi(u) \leq \liminf_{k\to\infty} \Psi(u_{n_k})$.

Now, on one hand, we have from (4.6) that $\Psi(u_{n_k}) \leq \beta(t_{n_k}) + \frac{1}{n_k} + O(\eta)$, and taking the liminf we have

$$L + O(\eta) = \liminf_{k\to\infty} \beta(t_{n_k}) + O(\eta) \geq \liminf_{k\to\infty} \Psi(u_{n_k}).$$

On the other hand, we claim that $\Psi(u) \geq \beta(t_0)$. It suffices to prove that $u \in \mathscr{J}_2(t_0)$. Indeed, since $u_{n_k} \to u$ almost everywhere, and $u_{n_k}(0_-) = t_{n_k}$ with $t_{n_k} \to t_0$, we must have $u(0_-) = t_0$. Moreover, $u_n(x) = -1$ for all $x \leq -L_{M,\eta}$ and for all n, so $u(x) = -1$ for all such x as well. Thus $u \in \mathscr{J}_2(t_0)$.

With $\Psi(u) \geq \beta(t_0)$, we have that

$$L + O(\eta) \geq \liminf_{k\to\infty} \Psi(u_{n_k}) \geq \Psi(u) \geq \beta(t_0).$$

But $\eta > 0$ was arbitrary, so $L \geq \beta(t_0)$, which is what we wanted to show. □

We now prove our main result, Theorem 1.1.

Proof. As in the proof of Theorem 3.5, we need only consider the case $u \in BPV(\Omega; \{-1, 1\})$.

For the liminf inequality, let $u_n \to u$ in $L^2(\Omega)$. We may assume that $u_n(a_+) = a_{\varepsilon_n}$ and $u_n(b_-) = b_{\varepsilon_n}$ for all n.

Fix $\eta > 0$, and suppose that $u(b_-) = -1$. Find an interval $(b-\delta, b)$ in which $u = -1$, and now for all large enough n we follow Step 2 of the proof of Theorem 3.5 where we "modify" u_n in $(b, b-\delta)$ so that it becomes "affixed" to -1 at $b-\delta$. That is, we find $v_n : (b-\delta, b) \to \mathbb{R}$ such that $v_n \in W^{2,2}(b-\delta, b)$, $v_n(b-\delta_-) = -1$, $v'_n(b-\delta_-) = 0$, $v_n(b_-) = u_n(b_-) = b_{\varepsilon_n}$, and

$$\int_{b-\delta}^{b} \left[\varepsilon_n^{-1}(v_n^2 - 1)^2 + \varepsilon_n^3 |v''_n|^2\right] dx \leq \int_{b-\delta}^{b} \left[\varepsilon_n^{-1}(u_n^2 - 1)^2 + \varepsilon_n^3 |u''_n|^2\right] dx + \eta.$$

Now, as in Step 3 of the proof of Theorem 3.5, we may change variables and apply the AM-GM inequality to eventually obtain the bound

$$\int_{b-\delta}^{b} \left[\varepsilon_n^{-1}(v_n^2-1)^2 + \varepsilon_n^3|v_n''|^2\right] dx \geq \frac{4}{3^{3/4}} \inf_{u\in\mathscr{J}_1(b_{\varepsilon_n})} \Phi(u) = 2\beta(b_{\varepsilon_n}).$$

If instead $u(b_-)=1$, then by a symmetrical argument, we instead obtain the term $2\beta(-b_{\varepsilon_n})$. Both of these terms may be written as $2\beta(-\operatorname{sgn}(u(b_-))b_{\varepsilon_n})$.

Using the same argument, we obtain the term $2\beta(-\operatorname{sgn}(u(a_+))a_{\varepsilon_n})$, and we recover the term $\alpha \operatorname{essVar}_\Omega u$ as in Steps 2 and 3 of the proof of Theorem 3.5.

We deduce that

$$\liminf_{n\to\infty} G_{\varepsilon_n}(u_n) \geq \alpha \operatorname*{essVar}_\Omega u + \liminf_{n\to\infty} 2\beta(-\operatorname{sgn}(u(a_+))a_{\varepsilon_n}) + 2\beta(-\operatorname{sgn}(u(b_-))b_{\varepsilon_n}),$$

and so we get

$$\liminf_{n\to\infty} G_{\varepsilon_n}(u_n) \geq \alpha \operatorname*{essVar}_\Omega u + 2\beta(-\operatorname{sgn}(u(a_+))a_0) + 2\beta(-\operatorname{sgn}(u(b_-))b_0)$$

by continuity of β.

For the limsup inequality, we use a construction similar to that done in Step 5 of the proof of Theorem 3.5. We begin by strengthening the continuity result on $\beta(t)$. Define the constant

$$\beta_\varepsilon(t) := \inf\left\{\Psi(v) : v \in \mathscr{J}_2(t), L_v \leq \frac{1}{\sqrt{\varepsilon}}\right\},$$

and let $t_0 \in \mathbb{R}$ with $t_n \to t_0$. We claim that $\lim_{n\to\infty} \beta_{\varepsilon_n}(t_n) = \beta(t_0)$. To see this, fix $\eta > 0$ and for each $n \in \mathbb{N}$ take $v_n \in \mathscr{J}_2(t_n)$ for which $\beta(t_n) \leq \Psi(v_n) \leq \beta(t_n) + \frac{1}{n}$. The continuity of β ensures that $\{\Psi(v_n)\}_n$ is bounded by a constant $M > 0$. By Lemma 4.3 we construct $\tilde{v}_n$ for which $\Psi(\tilde{v}_n) \leq \Psi(v_n) + O(\eta) \leq \beta(t_n) + \frac{1}{n} + O(\eta)$ and $L_{\tilde{v}_n} \leq L_{M,\eta}$ where $L_{M,\eta}$ depends only on M and η. Particularly $L_{M,\eta}$ has no dependence on n, thus $L_{\tilde{n}_n} \leq \frac{1}{\sqrt{\varepsilon_n}}$ for all n large enough. For all such n we write

$$\beta_{\varepsilon_n}(t_n) \leq \Psi(\tilde{v}_n) \leq \beta(t_n) + \frac{1}{n} + O(\eta),$$

and so by taking the limsup we obtain

$$\limsup_{n\to\infty} \beta_{\varepsilon_n}(t_n) \leq \beta(t_0) + O(\eta)$$

by continuity of β. As $\eta > 0$ was arbitrary, we get $\limsup_{n\to\infty} \beta_{\varepsilon_n}(t_n) \leq \beta(t_0)$. But we clearly also have $\beta_{\varepsilon_n}(t_n) \geq \beta(t_n)$, and taking the liminf finishes the proof of the claim.

We now turn to the proof of the limsup inequality. As we did for the liminf inequality, assume the case $u(b_-) = -1$ and find $\delta > 0$ small enough so that $u = -1$ in the interval $(b-\delta, b)$. We will define u_n over $(b-\delta, b)$. Take $v_n \in \mathscr{J}_2(b_{\varepsilon_n})$ satisfying $L_{v_n} \leq \frac{1}{\sqrt{\varepsilon_n}}$ such that

$$\beta_{\varepsilon_n}(b_{\varepsilon_n}) \leq \Psi(v_n) \leq \beta_{\varepsilon_n}(b_{\varepsilon_n}) + \frac{1}{n}. \tag{4.7}$$

Since $\varepsilon_n L_{v_n} \leq \sqrt{\varepsilon_n} \to 0$, we have $\varepsilon_n L_{v_n} < \delta$ for all n large enough. For all such n, we define

$$u_n(x) := \begin{cases} -1, & b-\delta < x \leq b - \varepsilon_n L_{v_n} \\ v_n\left(\frac{x-b}{\varepsilon_n}\right), & b - \varepsilon_n L_{v_n} < x < b \end{cases}.$$

This satisfies $u_n(b-\delta) = -1$, $u_n'(b-\delta) = 0$, and the boundary condition $u_n(b_-) = b_{\varepsilon_n}$. Moreover,

$$\begin{aligned}
\int_{b-\delta}^{b} &[\varepsilon_n^{-1}(u_n^2-1)^2 + \varepsilon_n^3 |u_n''|^2]\,dx \\
&= \int_{b-\varepsilon_n L_{v_n}}^{b} \left[\varepsilon_n^{-1}\left(v_n\left(\frac{x-b}{\varepsilon_n}\right)^2 - 1\right)^2 + \frac{\varepsilon_n^3}{\varepsilon_n^4} v_n''\left(\frac{x-b}{\varepsilon_n}\right)^2\right] dx \\
&= \int_{-L_{v_n}}^{0} \left[(v_n^2-1)^2 + |v_n''|^2\right]\,dx \\
&= \Psi(v_n) \leq \beta_{\varepsilon_n}(b_{\varepsilon_n}) + \frac{1}{n},
\end{aligned}$$

thus

$$\limsup_{n\to\infty} \int_{b-\delta}^{b} \left[\varepsilon_n^{-1}(u_n^2-1)^2 + \varepsilon_n^3 |u_n''|^2\right]\,dx \leq \limsup_{n\to\infty} \beta_{\varepsilon_n}(b_{\varepsilon_n}) + \frac{1}{n} = \beta(b_0)$$

by the claim. It remains to show that $u_n \to -1$ in $L^2(b-\delta, b)$.

Since $\varepsilon_n L_{v_n} \to 0$, we have that $u_n \to -1$ almost everywhere in $(b-\delta, b)$. By Vitali's convergence theorem, it suffices to prove that $\{|u_n|^2 \cdot 1_{(b-\delta,b)}\}_n$ is equi-integrable. Indeed, we have

$$\int_{b-\delta}^{b} (u_n^2-1)^2\,dx = \varepsilon_n \int_{-L_{v_n}}^{0} (v_n^2-1)^2\,dx = \varepsilon_n \Psi(v_n),$$

and since $\limsup_{n\to\infty} \Psi(v_n) \leq \lim_{n\to\infty} \beta_{\varepsilon_n}(b_{\varepsilon_n}) + \frac{1}{n} < \infty$, we have that $\lim_{n\to\infty} \int_{b-\delta}^{b} (u_n^2 - 1)^2\,dx = 0$, so equi-integrability follows from Lemma 3.4.

Our construction for u_n over $(b-\delta, b)$ provides us with the term $\beta(b_0)$ for the Gamma limit in the case that $u(b_-) = -1$. If instead $u(b_-) = 1$, we may negate our construction to instead acquire the term $\beta(-b_0)$. The term in both cases is equal to $\beta(b_0 \operatorname{sgn} u(b_-))$. A symmetrical construction near the endpoint a yields the term $\beta(a_0 \operatorname{sgn} u(a_+))$. Lastly we may define u_n away from the endpoints as in Step 5 of the proof of Theorem 3.5 in order to recover the term $\alpha \operatorname{essVar}_\Omega u$, completing the proof. □

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