Talk Outline

Monday, October 6, 2025 8:27 PM

(1) · To solve wave eq, Fourier in space

· Get $\hat{u} = \cos(2\pi|\xi|t)\hat{f} + \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\hat{g}$

Let's take $g = 0$ tbh (should not be too different from $f=0$)

(2) Is the fixed-time sol'n operator $f \mapsto u|_{t=t_0}$ L^p-bounded?

Thm (Peral): $\|u|_{t=t_0}\|_{H^{s,p}} \lesssim \|f\|_p \iff s = -(n-1)\left|\frac{1}{2} - \frac{1}{p}\right|$

English: Regularity is lost

(2.5) $\|u\|_{H^{s,p}} := \|(1+\sqrt{-\Delta})^s u\|_p := \|((1+|\xi|)^s \hat{u})^\vee\|_p =: \|(1-\Delta)^{s/2} u\|_p$

$\nparallel$

$\|u\|_{W^{s,p}} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{\|x-y\|^{n+sp}}\right)^{1/p}$

Both "Sobolev"...

This is b/c waves can join up and cause major interference at a single point in time.

(3) But if we average a little in time...

Conj (Sogge): $\forall \varepsilon > 0$, $\|(1+\sqrt{-\Delta})^{-\varepsilon} u\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n \times [1,2])} \lesssim \|f\|_{L^{\frac{2n}{n-1}}}$

$\varepsilon > 0$ as small as we want so this is a gain of regularity compared to before!

This is equiv. to:

$\forall \varepsilon > 0$, $\|u\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n \times [1,2])} \lesssim \|f\|_{H^{\varepsilon, \frac{2n}{n-1}}(\mathbb{R}^n)}$

(by replacing f with $(1+\sqrt{-\Delta})^s f$)

(maybe need $(1-\Delta)^{s/2}$ instead)

(3.5) More generally it is conj. that $\|u\|_{L^p(\mathbb{R}^n \times [1,2])} \lesssim \|f\|_{H^{\varepsilon,p}}$ for (some condition on p and ε) but $p = \frac{2n}{n-1}$ is the strongest.

Fun facts:
- Resolved for $n=2$ (by Hong?)
- First pf for some p by Wolff, $p > 74$

---

(4) Kakeya conj: Any Besi. $\subseteq \mathbb{R}^n$ has Hausdorff dim. $\geq n$.

Kakeya Max'l: $\left\|\sum_T 1_T\right\|_{\frac{n}{n-1}} \lessapprox 1$

There is a range of intermediate conjectures:

$\left\|\sum_T 1_T\right\|_p \lessapprox \frac{1}{\delta^{n-1-\frac{n}{p}}}$ $\forall p \geq \frac{n}{n-1}$

Kakeya Max'l
$\Downarrow$
Kakeya (H)
$\Downarrow$
Kakeya (Mink.)

This shows a Besi $\subseteq \mathbb{R}^n$ has dim $(=: d(n)) \geq \frac{p}{p-1}$

n=3:

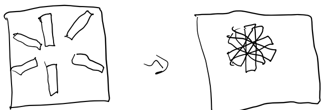
- $p = \infty$ trivial $\leadsto d(n) \geq 1$ — 1
- $p = 2$: Cordoba $\leadsto d(n) \geq 2$ ('77, Davies '71) — 2
- $p = \frac{n+1}{n-1}$: (Drury?) $\leadsto d(n) \geq \frac{n+1}{2}$ ("Bush") ('91) — 2
- $p = \frac{n+2}{n}$: Wolff $\leadsto d(n) \geq \frac{n+2}{2}$ ("Hairbrush") ('95) — 2.5
- Bourgain $\leadsto d(n) \geq \frac{13n+12}{25}$ ('98)
- (Katz, Łaba, Tao) $\leadsto d_M(3) > 2.5000000001$ ('99) — 2.5000...1
- (Wang, Zahl) $\leadsto d(3) = 3$ ('24) — 3

(5) Sosoe $\Longrightarrow$ Kakeya $\qquad \forall \varepsilon\, \exists C_\varepsilon\, \forall \delta>0\, \forall \{T_\omega\},\ \|\sum 1_{T_\omega}\|_{\frac{n}{n-1}} \le C_\varepsilon \frac{1}{\delta^\varepsilon}$

If Kakeya false then Kakeya max'l false.

so $\exists \varepsilon_0 \forall C\ \exists \delta, \{T_\omega\},\ \|\sum 1_{T_\omega}\|_{\frac{n}{n-1}} \ge C \frac{1}{\delta^{\varepsilon_0}}$.

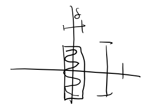
Rough Idea:



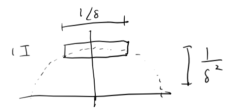
$t=0$ (f) $\qquad$ $1<t<2$ (u)

explosion! make $\|u\| \le C\|f\|$ false!

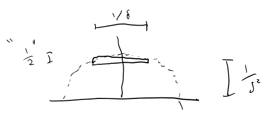
Step 1: Let f be a wave packet, $f(x) = \phi(x'/\delta, x_n)\, e^{\frac{2\pi i x_n}{\delta^2}} \approx$



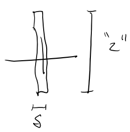
Then $\hat f(\xi) = \delta^{n-1} \phi\left(\delta \xi', \xi_n - \frac{1}{\delta^2}\right) \approx$



Now $\hat u = \cos(2\pi|\xi|t)\, \hat f(\xi) \approx$

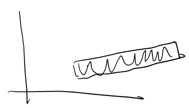


$\Rightarrow$ $u \approx$

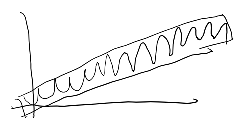


when $1<t<2$, $|\cos(2\pi|\xi|t)|$ has period like b/w $\frac{1}{2}$ and 1 (in $|\xi|$)
So can expect a ⌒ should always exist?

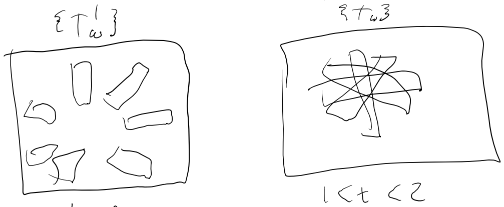
Step 2: If $f \approx$



then $u \approx$



Step 3:



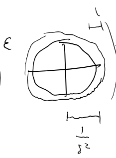
$\{T'_\omega\}$ $\qquad$ "$\{T_\omega\}$"

$t=0$ $\qquad$ $1<t<2$

If $f = \sum_{\omega\in\Omega} \phi_\omega$, $\phi_\omega \approx 1_{T'_\omega} e^{2\pi i \cdots}$, then

$$u \approx \sum_{\omega\in\Omega} 1_{T_\omega} e^{2\pi i \cdots} \quad \text{for like}$$

Now plug this into $\|u\| \le C_\varepsilon \|f\|$.

RHS: $\|f\|_{H^{\varepsilon, \frac{2n}{n-1}}} = \|((1+|\xi|)^\varepsilon \hat f)^\vee\|_{\frac{2n}{n-1}} \sim \|((1+|\xi|)^\varepsilon$  $)^\vee\|_{\frac{2n}{n-1}}$

$$\sim \left\|\left(\frac{1}{\delta^{2\varepsilon}} \hat f\right)^\vee\right\|_{\frac{2n}{n-1}}$$

$$\sim \frac{1}{\delta^{2\varepsilon}} \|f\|_{\frac{2n}{n-1}}$$

$$\sim \frac{1}{\delta^{2\varepsilon}} |\mathrm{supp}\, f|^{\frac{n-1}{2n}} \sim \frac{1}{\delta^{2\varepsilon}} \left(\frac{1}{\delta^{n-1}} \cdot \delta^{n-1}\right)^{\frac{n-1}{2n}} = \frac{1}{\delta^{2\varepsilon}}$$

LHS: Not so clear b/c there may be cancellation among the overlapping tubes...

Fix: Instead take $f = \sum_{\omega} \varepsilon_\omega \phi_\omega$.

Now $u \sim \sum \varepsilon_\omega 1_{T_\omega} e^{i\cdots}$ (for long enough)

Khinchine: $\| \sum \varepsilon_\omega g_\omega \|_{L^p(\Omega \times X)} \sim \| \sqrt{\sum |g_\omega|^2} \|_{L^p(X)}$

So (probabilistic method), $\exists$ signs so that

$$\|u\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^n \times [1,2])} \gtrsim \left(\int_1^2 \int_{\mathbb{R}^n} \sqrt{\sum_\omega 1_{T_\omega}}^{\frac{2n}{n-1}} \right)^{\frac{n-1}{2n}}$$

$$= \| \sum_\omega 1_{T_\omega} \|_{\frac{n}{n-1}}^{1/2} \geq \frac{C^{1/2}}{\delta^{\varepsilon/2}}$$

$\therefore \quad \frac{C^{1/2}}{\delta^{\varepsilon_0/2}} \leq \frac{C_\varepsilon}{\delta^{2\varepsilon}}$ Take $\varepsilon = \varepsilon_0/4$

this can't hold for all $C \geq 0$ lol $\square$

---

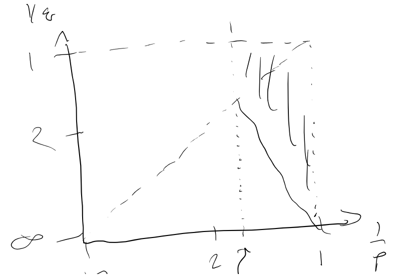
⑥ Bochner–Riesz: $S_1^\varepsilon f := \left((1-|\xi|)_+^\varepsilon \hat{f} \right)^\vee$ is L^p-bdd

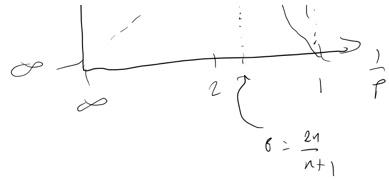
$\forall \quad \frac{2n}{n+1} < p < \frac{2n}{n-1}$

Restriction: $\|\hat{f}\|_{L^q(S)} \lesssim \|f\|_p \quad \forall \quad p < \frac{2n}{n+1}$,

$p' \geq \frac{n+1}{n-1} q$,

S non-vanishing curvature





Sogge $\Rightarrow$ B-R $\Rightarrow$ Restriction $\Rightarrow$ Kakeya max'l $\Rightarrow$ Kakeya (H) $\Rightarrow$ K.K (M)