NYU Advanced Calculus Workshop

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Contents

0 Notation and Conventions

0.1 Multvariable Calculus

- **Vector Fields and Scalar Fields:** Vector fields are always denoted by capital letters. Scalar fields are always denoted by lowercase letters.
- **Gradients and Jacobians:** For a function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f is the gradient of f. For a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$
- **Integration:**
  - I will not be writing $\iint$, $\iiint$, etc. and will instead represent all integrals as $\int$.
  - $\int_C F \cdot ds$ is the line integral of a vector field F over a path C.
  - dx indicates integration over $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$, depending on context. You may be familiar with seeing dA or dV instead. I may instead use $d(x,y)(\equiv dA)$ to clarify integration over $\mathbb{R}^2$, and similarly write $d(x,y,z)(\equiv dV)$ to clarify integration over $\mathbb{R}^3$.
  - dS indicates a surface integral.
  - In writing an integral we will often omit the independent variable for brevity/clarity, e.g. $\int_\Omega f\,dx := \int_\Omega f(x)\,dx$.
  - If, however, the variable is important (particularly when multiple integrals are involved), I may emphasize the name of the variable under the integral, e.g.

$$\int_{x\in E} f(x)\,dx := \int_E f(x)\,dx.$$

- ν is the unit outward normal to the implied surface. (I am allergic to using n or $\hat{n}$.)
- We write $\operatorname{div} F$ for divergence and $\operatorname{curl} F$ for curl (instead of $\nabla \cdot F$ and $\nabla \times F$, respectively).
- $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the basis vectors of $\mathbb{R}^3$. For example $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k} = (3,4,5)$.

0.2 Analysis

- $B_n(x,r)$ is the n-dimensional ball centered at x with radius r. That is, $B_n(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$.
- For a set U, ∂U denotes the boundary of U.
- log is the natural log.

1 Day 1: Integral Spam

1.1 Area and Volume

- Usually you want to find area/volume of a region by slicing the region into lower-dimensional cross sections and integrating over the length/area of these sections.
- Consider using polar coordinates or other coordinate systems if such methods seem relevant.

Example 1.1 (Stolen from the GRE): Find the volume of the region bounded by $y = x^2$, $y = 2 - x^2$, $z = 0$, and $z = y + 3$.

Solution. This is

$$\begin{aligned}\int_{x=-1}^{1}\int_{y=x^2}^{2-x^2}\int_{z=0}^{y+3} 1\,dz\,dy\,dx &= \int_{x=-1}^{1}\int_{y=x^2}^{2-x^2} y+3\,dy\,dx \\ &= \int_{x=-1}^{1}\frac{1}{2}(2-x^2)^2-\frac{1}{2}(x^2)^2+3(2-2x^2)\,dx \\ &= \int_{x=-1}^{1} 8-8x^2\,dx \\ &= 16-\frac{16}{3} = \boxed{\frac{32}{3}}\end{aligned}$$

■

Example 1.2: Find the volume of the set

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2<1, y^2+z^2<1, x^2+z^2<1\}.$$

Solution. By symmetry we can restrict to the octant $x, y, z > 0$ and then multiply the answer we get by 8. A dumb expression for the volume is given by

$$\frac{V}{8} = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 1_{x^2+y^2<1}1_{y^2+z^2<1}1_{x^2+z^2<1}\,dz\,dy\,dx,$$

where we use the "indicator function"

$$1_P := \begin{cases} 1, & P \text{ is true} \\ 0, & P \text{ is false} \end{cases}.$$

This sets up a seemingly bare-bones integral, but this ends up working pretty well if you're having trouble imagining what the set looks like.

$$\begin{aligned}
\frac{V}{8} &= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\int_{z=0}^{1} 1_{y^2+z^2<1}1_{x^2+z^2<1}\,dz\,dy\,dx \\
&= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\int_{z=0}^{1} 1_{z<\sqrt{1-x^2}}1_{z<\sqrt{1-y^2}}\,dz\,dy\,dx \\
&= \int_{x=0}^{1}\int_{y=0}^{1} 1_{x^2+y^2<1}\min(\sqrt{1-x^2},\sqrt{1-y^2})\,dy\,dx \\
&= \int_{x,y>0,x^2+y^2<1} \min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y).
\end{aligned}$$

The min is ugly so we now use symmetry again by restricting to the region where $x<y$. This gives

$$\begin{aligned}
\frac{V}{16} &= \int_{x,y>0,x^2+y^2<1} \min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y) \\
&= \int_{x,y>0,x<y,x^2+y^2<1} \min(\sqrt{1-x^2},\sqrt{1-y^2})\,d(x,y) \\
&= \int_{x,y>0,x<y,x^2+y^2<1} \sqrt{1-y^2}\,d(x,y) \\
&= \int_{y=0}^{\sqrt{2}/2}\int_{x=0}^{y} \sqrt{1-y^2}\,dx\,dy + \int_{y=\sqrt{2}/2}^{1}\int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\,dy \\
&= \int_{y=0}^{\sqrt{2}/2} y\sqrt{1-y^2}\,dy + \int_{y=\sqrt{2}/2}^{1} 1-y^2\,dy \\
&= \int_{u=1}^{1/2} -\tfrac{1}{2}\sqrt{u}\,du + \left(1-\frac{\sqrt{2}}{2}\right) - \frac{1}{3} + \frac{\sqrt{2}}{12} \\
&= \frac{1}{3} - \frac{(1/2)^{3/2}}{3} + \left(1-\frac{\sqrt{2}}{2}\right) - \frac{1}{3} + \frac{\sqrt{2}}{12} \\
&= \frac{1}{3} - \frac{\sqrt{2}}{12} + 1 - \frac{\sqrt{2}}{2} - \frac{1}{3} + \frac{\sqrt{2}}{12} \\
&= 1 - \frac{\sqrt{2}}{2}.
\end{aligned}$$

So $\boxed{V = 8(2-\sqrt{2})}$. ■

1.2 Change of Variables

Let g be a smooth bijection, and let f be sufficiently nice. Then:

$$\begin{aligned}\int_{x\in E} f(x)\,dx \text{ “} &= \int_{g(y)\in E} f(g(y))\,d(g(y))\,dy \text{ ”} \\ \text{“} &= \int_{g(y)\in E} f(g(y))\frac{d(g(y))}{dy}\,dy \text{ ”} \\ &= \int_{y\in g^{-1}(E)} f(g(y))|\det Dg(y)|\,dy\end{aligned}$$

1.3 Polar and Spherical Coordinates

1.3.1 Polar

The *polar coordinates* are given by the change of variables

$$g(r,\theta) = (r\cos\theta, r\sin\theta),$$

which we can think of as the substitutions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The determinant of the Jacobian is

$$\det Dg(r,\theta) = \begin{vmatrix} \frac{\partial g_1}{\partial r}(r,\theta) & \frac{\partial g_1}{\partial \theta}(r,\theta) \\ \frac{\partial g_2}{\partial r}(r,\theta) & \frac{\partial g_2}{\partial \theta}(r,\theta) \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r,$$

and since $r > 0$ we simply have $|\det J_g(r,\theta)| = r$. Hence r is the “price” to pay in order to change to polar coordinates. Thus, for example, if $B_2(0,R)$ is the ball with radius R centered at $(0,0)$, then

$$\int_{B_r(0,R)} f(x,y)\,d(x,y) = \int_{r=0}^{R}\int_{\theta=0}^{2\pi} f(r\cos\theta, r\sin\theta)\cdot r\,d\theta\,dr.$$

1.3.2 Cylindrical

The *cylindrical coordinates* are given by the change of variables

$$g(r,\theta,z) = (r\cos\theta, r\sin\theta, z),$$

which we can think of as the substitutions:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

So cylindrical is actually pretty lame because it just tacks on a third dimension to polar coordinates. You can find that $|\det Dg(r,\theta,z)| = r$.

1.3.3 Spherical

The *spherical coordinates* are given by the change of variables

$$g(r,\theta,\phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \qquad 0 < \theta < \pi, 0 < \phi < 2\pi$$

which we can think of as the substitutions:

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

You can show that $|\det Dg(r,\theta,\phi)| = r^2\sin\theta$. Note that $\sin\theta > 0$ for $0 < \theta < \pi$, so this indeed always non-negative.

I personally find this hard to remember. I highly remember **understanding where spherical coordinates come from** instead of trying to memorize them. That way, you can always rederive them when you need them.

1.4 Path Integrals

Summary:

- A (parametrization of a) path γ is given by $\varphi : [a,b] \to \mathbb{R}^n$.
- Line integral of a *function* $f : \mathbb{R}^n \to \mathbb{R}$ over γ is given by

$$\int_\gamma f\,ds := \int_a^b f(\varphi(t))\|\varphi'(t)\|\,dt.$$

  - Does not depend on the parametrization, φ.

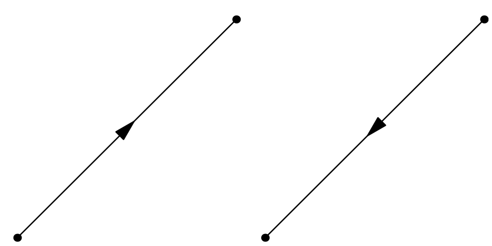
  - $\int_\gamma f\,ds$ does *not* depend on the orientation of γ. It's the same whether we go forward of backward.
  - Can think of $\int_\gamma f\,ds$ as "the area under f along the path γ", a picture best imagined for $f : \mathbb{R}^2 \to \mathbb{R}$.

- Line integral of a *vector field* $F : \mathbb{R}^n \to \mathbb{R}^n$ over γ is given by

$$\int_\gamma F \cdot ds := \int_a^b F(\varphi(t)) \cdot \varphi'(t)\,dt.$$

  - Does not depend on the parametrization either.
  - *Does* depend on the orientatiion of γ. In particular if $-\gamma$ is the path obtained by reversing the direction/orientation of γ, then $\int_{-\gamma} F \cdot ds = -\int_\gamma F \cdot ds$.
  - Can think of $\int_\gamma F \cdot ds$ as "how much F agrees with the velocity of γ as we travel along γ".
  - Other notations include:
    * $\int_\gamma F \cdot dr$
    * $\int_\gamma F$
    * In $\mathbb{R}^2$, $\int_\gamma M\,dx + N\,dy$, where $F = (M, N)$, i.e. M and N are the components of F.

- (For simplicity, we should assume that all functions and paths are smooth, but this assumption can be weakened to "Lipschitz".)



Two paths of the opposite orientation

Generally I'd say that line integrals of *vector fields* are more common and more useful, so I'd focus your energy on those.

Example 1.3: Let $f(x, y) = x^2 + y$. Let $F(x, y) = (xy, x - y)$. Let γ be the unit circle centered at $(0, 0)$ oriented counter-clockwise.

(a) Compute $\int_\gamma f\,ds$.

(b) Compute $\int_\gamma F \cdot ds$.

(c) Compute $\int_\gamma x\,dy$.

Solution. Let's use the parametrization $\varphi(t) = (\cos t, \sin t)$ over $t \in [0, 2\pi)$. Note that $\varphi'(t) = (-\sin t, \cos t)$, and that $\|\varphi'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$ for all t.

Part (a)

$$\int_\gamma f\, ds = \int_0^{2\pi} f(\cos t, \sin t)\|\varphi'(t)\|\, dt = \int_0^{2\pi} \cos^2 t + \sin t\, dt.$$

This is just $\int_0^{2\pi} \cos^2 t\, dt$ which is probably like π or something.

Part (b)

$$\int_\gamma F \cdot ds = \int_0^{2\pi} F(\cos t, \sin t) \cdot \varphi'(t)\, dt = \int_0^{2\pi} (\cos t \sin t, \cos t - \sin t) \cdot (-\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t \cos t + \cos^2 t - \sin t \cos t\, dt.$$

This evaluates to something.

Part (c)

I don't like this notation so I like to think of this as

$$\int_\gamma x\, dy = \int_\gamma (0, x) \cdot (dx, dy) = \int_\gamma (0, x) \cdot ds.$$

So this is

$$= \int_0^{2\pi} (0, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} \cos^2 t\, dt,$$

which is probably like π or something. ■

1.5 Surface Integrals

Now we're going to integrate over higher-dimensional things! For simplicity we'll stick to 2-dimensional manifolds in $\mathbb{R}^3$. Recall that a 2-dimensional manifold is, roughly speaking, a set $M \subseteq \mathbb{R}^3$ which is smoothly parametrized by a function $\varphi : U \to M$, with $U \subseteq \mathbb{R}^2$. φ is called a *chart*.

Definition 1.1 (Surface Integral)

Let $f : \mathbb{R}^3 \to \mathbb{R}$. The *surface integral* of f over M is given by

$$\int_M f\,dS := \int_U f(\varphi(u,v))\sqrt{\det[D\varphi(u,v)^T D\varphi(u,v)]}\,du\,dv,$$

where $\varphi : U \to M$ is a chart for M.
Other notations include $\int_M f\,dA$, $\int_M f\,d\sigma$, $\int_M f\,d\Sigma$, and $\int_M f\,d\mathcal{H}^2$.

The $\sqrt{\det[D\varphi(u,v)^T D\varphi(u,v)]}$ term (the "Jacobian") is the nasty part. Some people write it as $|||\varphi(u,v)|||$. A useful result for taming it is the *Cauchy-Binet formula*. In 2-dimensions, this formula says that

$$\begin{aligned}\det\left[\begin{pmatrix} a & b & c \\ d & e & f\end{pmatrix}\begin{pmatrix} a & d \\ b & e \\ c & f\end{pmatrix}\right] &= \begin{vmatrix} a & b \\ d & e\end{vmatrix}^2 + \begin{vmatrix} a & c \\ d & f\end{vmatrix}^2 + \begin{vmatrix} b & c \\ e & f\end{vmatrix}^2 \\ &= (ae-bd)^2 + (af-cd)^2 + (bf-ce)^2 \\ &= \|(a,b,c)\times(d,e,f)\|^2.\end{aligned}$$

So it's common to define/write

$$\int_M f\,dS := \int_U f(\varphi(u,v))\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\|\,du\,dv.$$

This makes some visual sense in that $\left\|\frac{\partial\varphi}{\partial u}(u,v)\times\frac{\partial\varphi}{\partial v}(u,v)\right\|$ is measuring the area of a parallelogram with "sides" $\frac{\partial\varphi}{\partial u}(u,v)$ and $\frac{\partial\varphi}{\partial v}(u,v)$, and is hence a "differential surface element" that in some sense "measures how curvy/distorted M is at (u,v)".

1.5.1 Integrating Over a Sphere

For a suitably decent function f, let us find a formula for the surface integral

$$\int_{\partial B(0,R)} f\,dS,$$

where $\partial B(0,R)$ is the surface of the ball of radius R centered at the origin. We can chart out this surface in coordinates via

$$\varphi(\phi,\theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi), \qquad 0<\phi<\pi, 0<\theta<2\pi.$$

We find that

$$D\varphi(\phi,\theta) = \begin{pmatrix} R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ -R\sin\phi & 0\end{pmatrix},$$

and so

$$|||D\varphi(\phi,\theta)||| = \sqrt{\begin{matrix}(R^2\sin\phi\cos\phi\cos^2\theta + R^2\sin\phi\cos\phi\sin^2\theta)^2 \\ +(R^2\sin\theta\sin^2\phi)^2 \\ +(-R^2\cos\theta\sin^2\phi)^2\end{matrix}}$$
$$= R^2\sqrt{\sin^2\phi\cos^2\phi + \sin^4\phi}$$
$$= R^2|\sin\phi| = R^2\sin\phi.$$

(Sanity check: Why does R^2 make sense, vis-a-vis, say, R or R^3?)

Thus

$$\int_{\partial B(0,R)} f\, dS = \int_0^{2\pi}\int_0^{\pi} f(R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)\cdot R^2\sin\phi\, d\phi\, d\theta.$$

(Normal people probably switch θ and ϕ around but I honestly don't care lmao i just rederive this every time)

1.5.2 Other Surfaces

Some manifolds, like the boundary of a cylinder, can't really be parametrized with a single chart. In that case, you divide the manifold into surfaces that can be (easily) parametrized.

Example 1.4: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as $f(x, y, z) = x^2 + yz$. Compute the surface integral $\int_C f\, dS$, where C is the curved surface of the upside-down circular cone with base $B_2(0, 1) \times \{1\}$ and vertex $(0, 0, 0)$.

Solution. Let's parameterize C with the chart $\varphi : (0, 1) \times (0, 2\pi)$ given by $\varphi(r, \theta) := (r\cos\theta, r\sin\theta, r)$. Then

$$\int_C f\, dS = \int_0^1\int_0^{2\pi} f(r\cos\theta, r\sin\theta, r)|||\varphi(r,\theta)|||\, d\theta\, dr.$$

We now compute the Jacobian $|||\varphi(r, \theta)|||$. First we note that

$$D\varphi(r,\theta) = \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 1 & 0\end{pmatrix},$$

so by the Cauchy-Binet formula we see that

$$\det(D\varphi(y)^T D\varphi(y)) = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}^2 + \begin{vmatrix}\cos\theta & -r\sin\theta \\ 1 & 0\end{vmatrix}^2 + \begin{vmatrix}\sin\theta & r\cos\theta \\ 1 & 0\end{vmatrix}^2$$
$$= r^2 + r^2\sin^2\theta + r^2\cos^2\theta = 2r^2,$$

thus

$$|||\varphi(r,\theta)||| = \sqrt{\det(D\varphi(y)^T D\varphi(y))} = \sqrt{2}r.$$

It follows that

$$\int_C f\, dS = \int_0^1 \int_0^{2\pi} (r^2\cos^2\theta + r^2\sin\theta)\cdot\sqrt{2}r\, d\theta\, dr = \sqrt{2}\int_0^1 \pi r^3\, dr = \boxed{\frac{\sqrt{2}\pi}{4}}.$$

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2 Day 2: Vector fields, Stokes, and Series

2.1 Vector Fields

For curious analysts: All instances of "smooth" in this section can be replaced with "Lipschitz".

Basic Facts:

- A *vector field* is a function that looks like $F : E^{(\subseteq \mathbb{R}^n)} \to \mathbb{R}^n$.
- F is *conservative* if there exists a *potential function*, $f : \mathbb{R}^n \to \mathbb{R}$, for which

$$\nabla f = F.$$

- f is a sort of "anti-derivative" for F, and hence satisfies the following version of the FTC: For $a, b \in \mathbb{R}^n$, we have

$$\int_C F \cdot dr = f(b) - f(a)$$

for any path C from a to b.

Example 2.1 (Winter 2017 #3): Consider the integral

$$I = \int_\Gamma \frac{x}{x^2+y^2}\,dx + y\frac{1-x^2-y^2}{x^2+y^2}\,dy$$

integrated over a path Γ.

(a) Show that I does not depend on the path Γ chosen to connect two fixed points.

(b) Compute I if Γ is a path joining $A = (0,1)$ to $B = (1,1)$.

2.1.1 Divergence

Facts:

- The *divergence* of a vector field F is a function $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}$ given by

$$\operatorname{div} F = \sum_{i=1}^n \frac{\partial F_1}{\partial x_i}.$$

- In 3D, this is
$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$
- Intuitively, $\operatorname{div} F(x, y, z)$ quantifies how much "F expands space at (x, y, z)".
- $\operatorname{div} F$ is often written as $\nabla \cdot F$.
- (*Bonus: This is not a contrived quantity, in fact it is actually quite "natural" because, surprisingly, divergence is independent of the (orthonormal) coordinate system chosen.*)

The following is the fundamental fact that pretty much explains why divergence is so important.

Theorem 2.1 (Divergence Theorem)

Let $U \subseteq \mathbb{R}^3$ be a bounded open set with smooth boundary. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then
$$\int_U \operatorname{div} F \, d(x, y, z) = \int_{\partial U} F \cdot \nu \, dS,$$
where for a point $p \in \partial U$, $\nu(p)$ denotes the *unit outward normal* to ∂U at p.

Notes:

- The motto for the Divergence Theorem is basically this: "If you have a liquid in a container, then the total pressure inside the liquid is equal to the total force the liquid exerts on the container." In layman's terms, "if thing go in then thing go out".
- **When to use:** This theorem gives you a great way to convert some nasty surface integrals to "normal" integrals. So if you hate surface integrals, the Divergence Theorem can save you.
- The surface integral $\int_{\partial U} F \cdot \nu \, dS$ may be phrased as "flux".
- (Bonus) By applying the Divergence Theorem, you can deduce the following variants of "integration by parts":
  - $\displaystyle\int_\Omega f \operatorname{div} G \, dx = \int_{\partial\Omega} f(G \cdot \nu) \, dS - \int_\Omega \nabla f \cdot G \, dx$
  - $\displaystyle\int_\Omega F \cdot \nabla g \, dx = \int_{\partial\Omega} (F \cdot \nu) g \, dS - \int_\Omega (\operatorname{div} F) g \, dx$ (literally the same as the previous one)

  It does not seem like these will appear on your exam, but being familiar with these is quite crucial for studying PDE.

2.1.2 Curl

The "amount" that a vector field "twists" space is a surprisingly revealing quantity.

- In $\mathbb{R}^2$, the *curl* of F is a function $\operatorname{curl} F : \mathbb{R}^2 \to \mathbb{R}$, with
$$\operatorname{curl} F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$
  How to remember the order:
  - *Method 1:* You *differentiate* a *function*, so the first column is *derivatives* and the second column is the *functions* in the "determinant"
  $$\operatorname{curl} F = \begin{vmatrix} \partial_x & F_1 \\ \partial_y & F_2 \end{vmatrix}.$$
  - *Method 2:* The prototypical "twisty" vector field to test against is $(-y, x)$. If you sketch this you'll find that this "twists counter-clockwise", so you should expect that $\operatorname{curl}(-y, x) > 0$.
- In $\mathbb{R}^3$, the *curl* of F is a vector field $\operatorname{curl} F : \mathbb{R}^3 \to \mathbb{R}^3$, with
$$\operatorname{curl} F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$
  How to remember the order:
  - *Method 1:* Same mnemonic as in $\mathbb{R}^2$, but you attach a third column with the basis vectors as in
  $$\operatorname{curl} F = \begin{vmatrix} \partial_x & F_1 & \hat{i} \\ \partial_y & F_2 & \hat{j} \\ \partial_z & F_3 & \hat{k} \end{vmatrix}.$$
  - *Method 2:* The above determinant is basically the "cross product" $\nabla \times F$, so if you know how to take a cross product then you know how to compute 3D curl.
  - *Method 3:* Test against the vector field $(-y, x, 0)$. You should expect that the components of $\operatorname{curl}(-y, x, 0)$ are non-negative.

A first important result regarding curl:

Theorem 2.2 (Green's Theorem)

Let $U \subseteq \mathbb{R}^2$ be a connected open set with (piecewise) smooth boundary. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth vector field. Then

$$\int_U \operatorname{curl} F \, d(x, y) = \int_{\partial U} F \cdot ds,$$

where ∂U is viewed as a path oriented counter-clockwise, and

$$\operatorname{curl} F := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Notes:

- This follows from the Divergence Theorem (why?).
- This is also an instance of Stoke's Theorem (why?).
- By applying Green's to certain vector fields (namely $(0, x)$, $(-y, 0)$, and $(\frac{-y}{2}, \frac{x}{2})$), you can find the area of any region by just examining its boundary.
- **When to use:**
  - Can be useful for turning line integrals into possibly "nicer" area integrals, especially when the curl of F is a nice quantity.
  - Is useful for finding the area of a region whose *boundary* is easy to parametrize, whereas the region itself is hard to parametrize.

Theorem 2.3 (Stoke's Theorem)

Let $M \subseteq \mathbb{R}^3$ be a smooth 2d manifold with smooth boundary ∂M. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\int_M \operatorname{curl} F \cdot \nu \, dS = \int_{\partial M} F \cdot ds,$$

where the orientation of ∂M and the choice of ν are selected so that if ν is pointing "up", then ∂M is travelling "counter-clockwise" when viewed from above.

Notes:

- "Intuition:" https://www.smbc-comics.com/comic/2014-02-24
- **When to use:** Often this is used to turn nasty line integrals into "nice" surface integrals, which are usually only "nice" when the curl ends up being a really simple quantity like 0.

2.2 Examples with Divergence, Green's, Stoke's

I am going to use the same notation as the exam. Use context to interpret the problems!

Example 2.2 (Fall 2007 #2): Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ defined by $\mathbf{F}(x, y, z) = (2x - y^2 - x^3, 3y - y^3, -x - z^3)$.
For a closed surface S in $\mathbb{R}^3$, consider $\int_S \mathbf{F}.\mathbf{n}dA$, the flux of $\mathbf{F}$ through S. Here $\mathbf{n}$ is chosen to be an outward normal. For what choice of S will $\int_S \mathbf{F}.\mathbf{n}dA$ be *maximal*? Explain your answer and compute $\int_S \mathbf{F}.\mathbf{n}dA$ in that case.

Example 2.3 (Fall 2008 #4): An object moves in the force field $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ starting at the origin and ending at some point $A(\xi, \eta, \zeta)$ that lies on the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. What is the maximum possible value of the work done $W = \oint \vec{F} \cdot d\vec{r}$?

Example 2.4 (Winter 2009 #2): Compute

$$\oint_L (y - z)dx + (z - x)dy + (x - y)dz$$

where L is the curve given by the intersection of the two surfaces

$$\begin{cases} x^2 + y^2 + z^2 = a^2, a > 0 \\ x + y + z = 0 \end{cases}$$

with counterclockwise orientation viewed from the positive x-axis.

Example 2.5 (Winter 2018 #1b): Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field such that, for every continuously differentiable function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ with compact support,

$$\int_{\mathbb{R}^3} \mathbf{f}(x) \cdot \nabla\varphi(x)\, dx = 0.$$

Show that the divergence of $\mathbf{f}$ is zero.

Example 2.6 (Winter 2019 #4): Let C be the closed curve formed by the intersection of the surface $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1+z^2\}$ with the plane $z = \frac{\sqrt{3}}{2}y$. Choose an orientation for C and compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for the vector field $\mathbf{F}(x,y,z) := (x^2+z, \sin y, \cos z)$.

Example 2.7 (Fall 2021 #3):

(a) [some nonsense that i don't want you worrying about]

(b) Let

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}, \qquad \mathbf{F} = \nabla f.$$

What is the flux of $\mathbf{F}$ through the surface of the unit sphere?

(c) Lastly, consiedr the vector fields:

$$\mathbf{G} = -r\sin\varphi\mathbf{i} + r\cos\varphi\mathbf{j} + \mathbf{k},$$

$$\mathbf{H} = \nabla \times \mathbf{G},$$

where (r,θ,φ) are the usual spherical coordinates for a point in $\mathbb{R}^3$:

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \arccos\frac{z}{r}, \qquad \varphi = \arctan\frac{y}{x}.$$

Compute the integral

$$\beta = \iint_S \mathbf{H} \cdot d\mathbf{A},$$

where S is the top half of the unit sphere, i.e.

$$S = \{x,y,z : x^2+y^2+z^2 = 1 \text{ and } z \geq 0\}.$$

Example 2.8 (I made this one up): Let $F(x,y) = (y^3, x - x^3)$. Find the maximum possible value of

$$\int_C F \cdot ds$$

over all simple closed curves C that are oriented counter-clockwise.

Solution. Let U be the region enclosed by C. Then by Green's Theorem,

$$\int_C F \cdot ds = \int_U \operatorname{curl} F \, d(x,y) = \int_U 1 - 3x^2 - 3y^2 \, d(x,y).$$

Evidently this is maximized by selecting the maximal region U for which the integrand $1 - 3x^2 - 3y^2$ is non-negative. After some musing, you can find that $U = B_2(0, 1/\sqrt{3})$ is a great choice. The maximum value in this case is

$$\int_{B_2(0,1/\sqrt{3}} 1 - 3x^2 - 3y^2 \, d(x,y) = 2\pi \int_0^{1/\sqrt{3}} (1 - 3r^2) r \, dr$$

by polar coordinates. This evaluates to something. ■

Example 2.9 (I also made this one up): Let

$$H := \{(x,y,z) : x^2 + y^2 + z^2 = 1, z > 1\}$$

be the upper half of the surface of the unit sphere. Compute

$$\int_H x^4 + y^4 + z^4 \, dS.$$

Solution.

$$\int_H x^4 + y^4 + z^4 \, dS = \int_H (x^3, y^3, z^3) \cdot \nu \, dS$$

$$= -\int_{\{x^2+y^2<1\}} (x^3, y^3, 0) \cdot (0,0,-1) \, dS + \int_{x^2+y^2+z^2<1, z>0} 3x^2 + 3y^2 + 3z^2 \, d(x,y,z)$$

$$= 3 \int_{\phi=0}^{\pi/2} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (r^2 \sin^2\phi \cos^2\theta + r^2 \sin^2\phi \sin^2\theta + r^2 \cos^2\phi) r^2 \sin\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{2\pi} \sin\phi \, d\theta \, d\phi = 2\pi$$

■

2.3 Series Convergence

2.3.1 Basics

Theorem 2.4 (Stupid Test)

If $\lim_{n\to\infty} |a_n| \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

Theorem 2.5 (p-Test et. al.)

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.
$\displaystyle\sum_{n=1}^{\infty} a^n$ converges for $|a| < 1$ and diverges for $|a| \geq 1$.

Theorem 2.6 (Direct Comparison Test)

Let $a_n \geq 0$ be a sequence. If you can find a b_n that *eventually dominates* a_n (i.e. there is N such that $b_n \geq a_n$ for all $n \geq N$), such that $\displaystyle\sum_{n=1}^{\infty} b_n < \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n < \infty$ i.e. converges.
Similarly, if instead you found a b_n for which eventually $0 \leq b_n \leq a_n$ forever, with $\displaystyle\sum_{n=1}^{\infty} b_n = +\infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n = +\infty$ i.e. diverges.

We're going to be implicitly invoking direct comparison quite a lot. I'll explain why this test is so useful/underrated in a bit.

Theorem 2.7 (Limit Comparison Test)

If two sequences are close together, they behave the same way. That is, if $a_n \geq 0$ and $b > 0$ are two sequences for which $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges iff $\displaystyle\sum_{n=1}^{\infty} b_n$ converges (so if either converges then the other converges, and if either diverges then the other diverges!).

Note: Most sources instead write "$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L \in (0, \infty)$", but I think this muddies the "best" way to think about using this test.

Limit comparison can be used to "clean up" junk and simplify a series. For example, if we are aiming to ascertain convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1},$$

then limit comparison says we can multiply by a term with limit 1 and the convergence behavior does not change. For instance we can multiply by $\frac{n^3+1}{n^3}$ to "convert" the series into

$$\sum_{n=1}^{\infty} \frac{n}{n^3},$$

which is far easier to reason with. Effectively, what we have done is erased the insignificant "1" term in $n^3 + 1$ and replaced it with n^3. If this is something you wish to do, consider limit comparison.

2.3.2 Positive and Negative Terms

Theorem 2.8 (Alternating Series Test)

If $a_n \geq 0$ is **monotone decreasing** and **tends to** 0, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Theorem 2.9 (Absolute Convergence)

If $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\sum_{n=1}^{\infty} a_n$ converges.

2.3.3 Geometric-y Tests

The classic example of a convergent series is the geometric series,

$$\sum_{n-1}^{\infty} r^n,$$

where $|r| < 1$. Intuitively, if we can ascertain whether a series "decays faster" than geometric, then it should converge. Capitalizing on this idea, we can notice that there are two ways to characterize the geometric sequence $a_n = r^n$:

- The *ratio* between terms, a_{n+1}/a_n, is r with $|r| < 1$.
- The nth *root* of each term, $\sqrt[n]{a_n}$, is r with $|r| < 1$.

So for an arbitrary sequence a_n, it is fairly intuitive that if we do *better* than either of the above properties, then we have convergence.

Please feel free to replace $\limsup$ *with* $\lim$ *if you are uncomfortable with* $\limsup$.

- If $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$, then this means that over the tail end of the sequence, the ratio between terms is better than geometric, so it converges.

- If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then this again means that over the tail end of the sequence, the nth root of terms is better than geometric, so it converges.

Of course, these observations are simply the ratio test and root test.

Theorem 2.10 (Ratio Test)

Consider the limit of the ratio between successive terms, $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$, if it exists.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- (If $L = 1$, you know nothing.)

Theorem 2.11 (Root Test)

Consider $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, if it exists.

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- (If $L = 1$, you know nothing.)

Remark 1: The hyper-analysis-savvy reader would be delighted to know that you can replace lim with lim sup.

Remark 2: You can use the root test to prove the ratio test.

2.3.4 Integral Test

If you only care about convergence, then numerous sums can be replaced by integrals.

Theorem 2.12 (Integral Test)

In the series $\sum_{n=1}^{\infty} f(n)$, You can replace the sum with an integral and nothing changes convergence-wise (as long as f is non-negative and monotone decreasing).

This is quite niche but if it just so happens that the summand "looks like something that you can integrate", consider this test.

2.3.5 (Semi-Optional) Summation by Parts and the Dirichlet Test

You should also know the Dirichlet test since that actually shows up in the writtens sometimes. But the Dirichlet test can be hard to remember, so I will instead frame it as a specific instance of a well-motivated general method.

First, recall the *integration by parts:* If $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x g(t)\,dt$, then

$$\begin{aligned}\int_0^\infty f(x)G(x)\,dx &= F(\infty)G(\infty) - F(0)G(0) - \int_0^\infty F(x)g(x)\,dx \\ &= F(\infty)G(\infty) - \int_0^\infty F(x)g(x)\,dx.\end{aligned}$$

(Here, $F(\infty) := \lim_{x\to\infty} F(x) = \int_0^\infty f(x)\,dx$ and similarly for $G(\infty)$.) In this way we may thus "move an (anti-)derivative" from one factor to the other, and hence converting fG to Fg.

Amazingly there is a discrete version of this formula, called *summation by parts*. My thesis here is that you can essentially derive it by simply mimicking integration by parts: If $A_n = \sum_{k=1}^n a_k$ and $B_n = \sum_{k=1}^n b_k$, then

$$\sum_{n=1}^{\infty} a_n B_n \stackrel{?}{=} A_\infty B_\infty - \sum_{n=1}^{\infty} A_n b_n.$$

Unfortunately this isn't *exactly* true, the indices are slightly wrong or something. But I don't give a fuck and neither should you. What is *definitely* true, and what is ultimately important here, is that one side converges iff the other side converges. Thus the motto: **for determining convergence, we may freely "move" a partial sum from one factor to the other.** To be more precise:

Fact: *With the above notation, assuming that "$A_\infty B_\infty$" $:= \lim_{n\to\infty} A_n B_n = 0$, we have that*

$$\sum_{n=1}^{\infty} a_n B_n \text{ converges} \iff \sum_{n=1}^{\infty} A_n b_n \text{ converges.}$$

A prototypical example that we may attack with this approach is the series $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$. We choose the "$a_n$" sequence to be a factor whose partial sums are somehow nice. It turns out that taking $a_n = \sin n$ is a great choice, because amazingly the sequence of partial sums $A_n = \sum_{k=1}^{n} \sin k$ is bounded (why?).

That leaves us with taking the "B_n" sequence to be $B_n = \frac{1}{\sqrt{n}}$. But what is b_n? You can find that the only way to have $B_n = \sum_{k=1}^{n} b_k$ is if we choose $b_1 = 1$ and $b_n = \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$ for $n > 1$. Convince yourself that this works: We "integrate" the $\sin n$ via a partial sum, and "differentiate" the $1/\sqrt{n}$ via differences.

Our bet is that the "integral" of $\sin n$ is tame enough, and in return we expect that $\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$ is somehow much nicer to work with. And this is true!

The summation by parts method tells us that provided that $A_nB_n \to 0$ (and it does, because A_n is bounded and $B_n \to 0$), we have that $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$ converges iff

$$\sum_{n=1}^{\infty} A_n b_n$$

converges. But A_n is bounded with some upper bound $M > 0$, and so

$$\left|\sum_{n=1}^{\infty} A_n b_n\right| \le \sum_{n=1}^{\infty} M \cdot |b_n| = M \sum_{n=1}^{\infty} b_n = M \cdot B_\infty < \infty,$$

so we have convergence!

If you review what properties were important in this argument, you'll find that $\sin n$ can be replaced with *any* sequence with *bounded partial sums*, and $\frac{1}{\sqrt{n}}$ can be replaced with *any* sequence that *is positive and decreases to 0*. This is Dirichlet's Test.

Theorem 2.13 (Dirichlet Test)

Suppose that:

- a_n is a sequence with bounded partial sums, i.e. $|\sum_{k=1}^{n} a_k| \le M$ for all n, for some large $M > 0$, and
- $b_n \ge 0$ be a decreasing monotone sequence that tends to 0.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

If you decide to practice this, seek problems from past exams which seem like they are begging for the Dirichlet test (there are at least two such problems!), and try to solve them *without looking at the above theorem*, because the point of this exposition is to get you to either derive it or bypass its necessity on the spot if it happens that you require this test. All you need to remember is the prototypical example, and that **it's ultimately just integration by parts**.

2.4 Convergence Test Tier List and Minor Examples

SS	Direct Comparison
S	Stupid Test, Limit Comparison
A	Alternating Series, Absolute Convergence
B	Ratio, Root, Integral
C	Dirichlet

- A-tier and above are by far the most useful. Direct comparison in particular is extremely underrated and people should abuse it much more often.
- Ratio and Root test are mid but can be useful. Integral test is good in specific instances.
- Dirichlet test is rare.
- I know I said that Limit Comparison was F tier during the actual workshop, but I think if presented "correctly" it can have S-tier usefulness for simplifying series.

Now I'll explain why I say that Direct Comparison is SS-tier. Often you can use it to erase certain terms completely: Consider

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}.$$

As noted earlier, this is a prototypical use-case for Limit Comparison. However, we can actually just use Direct Comparison by noting that $\frac{1}{n^3+1} \leq \frac{1}{n^3}$, and so

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which clearly converges.

This exact approach fails for a series such as

$$\sum_{n=2}^{\infty} \frac{n}{n^3-1},$$

in which case one can argue that Limit Comparison is now easier by "multiplying by $\frac{n^3-1}{n^3}$". But actually Direct Comparison can still be used in the following way: We can note that $\frac{1}{n^3-1} \leq \frac{1}{n^3/2}$ for *all n large enough*; say, $n \geq 10^{100}$. Then

$$\sum_{n=10^{100}}^{\infty} \frac{n}{n^3-1} \leq \sum_{n=10^{100}}^{\infty} \frac{n}{n^3}$$

which converges, thus the original sum converges because **the first one billion terms do not matter**! A finite number of terms, no matter how large, can never affect whether the sum converges!

Here's an example where Direct Comparison trumps Limit Comparison. For the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\sqrt{n}}},$$

it's tough to get a handle on the asymptotic nature of $2^{\sqrt{n}}$, so Limit Comparison isn't very usable. However, using the philosophy of throwing away the first 1 trillion terms, we may use Direct Comparison by arguing that

$$2^{\sqrt{n}} \geq n^2$$

for all $n \geq 10^{9999999}$ (why?). Hence

$$\sum_{n=10^{9999999}}^{\infty} \frac{1}{2^{\sqrt{n}}} \leq \sum_{n=10^{9999999}}^{\infty} \frac{1}{n^2} < \infty,$$

so the original series converges because the first 1 quadrillion terms do not matter.

2.5 Examples with Series: Convergence Tests

Example 2.10 (Winter 2008 #1): Find all the values $p \in \mathbb{R}$ such that the following series converges:

$$\sum_{k=2}^{\infty} (\log k)^{p \log k}$$

Example 2.11 (Fall 2008 #2): For each of the following, find the range of $x \in \mathbb{R}$ for which the series converges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{n^2+x^{2n}}$

Example 2.12 (Winter 2011 #3): Find the range of the parameter for which the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}\left(\frac{1-2x}{1+x}\right)^n$

(ii) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{(nx)^n}{n!}$

Example 2.13 (Winter 2017 #1): Consider the power series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} x^n.$$

Determine the radius of convergence R of the series. Determine whether the series conerges or diverges for $x = R$ and $x = -R$.

Example 2.14 (Fall 2019 #4): Let $A_n = \frac{a_1}{1+a_1} + \frac{a_2}{\sqrt{2}+a_2} + \cdots + \frac{a_n}{\sqrt{n}+a_n}$, $a_n > \frac{1}{\sqrt{n}}$, and consider the series,

$$\sum_{n=1}^{\infty} \frac{\cos(\sqrt{3}n + \frac{\pi}{3})}{A_n}.$$

Show that the series is convergent.

2.6 Example with Series: Bare Hands

Sometimes there simply are no big guns you can use to nuke problems. You'll have to tackle these series with nought but your bare hands.

Example 2.15 (Fall 2007 #5b): Does the series $\sum_{n\in S} \frac{1}{n}$ converge, where S consists of those positive integers whose decimal expansion does not contain the digit 1?

Example 2.16 (Winter 2018 #5): Show that the limit

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} \sin(\log n)$$

does not exist.

2.7 Convergence of Functions

There are numerous notions of function converge. For the writtens you should know two: pointwise convergence and uniform convergence.

Definition 2.1 (Pointwise Convergence)

A sequence of functions f_n converges *pointwise* to f if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all x.

Definition 2.2 (Uniform Convergence)

A sequence of functions f_n converges *uniformly* to f if

$$\lim_{n\to\infty} \sup_x |f_n(x) - f(x)| = 0.$$

The pointwise/uniform convergence of *series* are defined in terms of the pointwise/uniform convergence of their *partial sums*.

How to think about uniform convergence:

- To show that $f_n \to f$ uniformly, you want to find a really nice upper bound on $|f_n(x) - f(x)|$ that *doesn't have an x in it.* If the upper bound you find goes to 0 as $n \to \infty$, you win.
- To show that $\sum_{n=1}^{\infty} f_n$ converges uniformly, you want to find a really nice upper bound on $|\sum_{n=1}^{\infty} f_n(x)|$ that does not depend on x.

Prototypical Example 1: $\frac{\sin x}{n} \to 0$ uniformly (in x), because

$$\left|\frac{\sin x}{n}\right| \leq \frac{1}{n},$$

and $1/n$ is an upper bound that has no x in it, and moreover $1/n \to 0$.

M**-Test:** For uniform convergence of *series* in particular, there is a very nice test you can use: If you can find a nice upper bound on $|f_n(x)|$ that *does not involve* x, and

$$\sum_{n=1}^{\infty}(\text{Upper bound on } |f_n|) < \infty,$$

then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly!

Prototypical Example 2: $\sum_{n=1}^{\infty} \frac{\sin x}{n^2}$ converges uniformly, because we have the upper bound

$$\left|\frac{\sin x}{n^2}\right| \leq \frac{1}{n^2},$$

whose sum converges.

Now some examples from the actual writtens.

Example 2.17 (Winter 2021 #2): Define, for each $n \in \mathbb{N}$,

$$u_n(x) = \frac{x}{n^2 + x^2}, \qquad (x \geq 0).$$

(a) Show that $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[0, K]$ for every $K > 0$.

(b) Determine whether $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[0, \infty)$.

Example 2.18 (Fall 2019 #3): For $\alpha \geq 1$, the sequence of functions $\{f_n\}$ is defined by

$$f_n(x) = x^{\alpha} \ln\left(x + \frac{1}{n}\right), \qquad x \in (0, \infty).$$

Show that (a) $\{f_n\}$ is uniformly convergent when $\alpha = 1$; (b) $\{f_n\}$ is not uniformly convergent when $\alpha > 1$.

Example 2.19 (I made this one up): Consider the series

$$\sum_{n=1}^{\infty} nxe^{-nx^2}$$

for $x \in (0, \infty)$.

(a) Find all intervals $(a, b) \subseteq (0, \infty)$ over which the series converges pointwise.

(a) Find all intervals $(a, b) \subseteq (0, \infty)$ over which the series converges uniformly.

2.8 Swapping

Several questions on the writtens require you to do things like swap a limit and an integral, so we need to cover this. The theorems I list here are in decreasing order of importance, and this list is far from exhaustive.

2.8.1 Swap Limit and Integral

This is so important that the average measure theory class dedicates a ton of time to this. Unfortunately we don't have that luxury so here's a "cheap" result that should be good enough for the exam.

Theorem 2.14 (Swap Limit and Integral)

Suppose a sequence of functions f_n on a bounded interval $[a, b]$ converges *uniformly* to some f. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. We have

$$\left|\int_a^b f_n(x) - f(x)\,dx\right| \leq \int_a^b |f_n(x)-f(x)|\,dx \leq \int_a^b \sup_{[a,b]} |f_n-f|\,dx = (b-a)\sup_{[a,b]} |f_n-f| \xrightarrow{n\to\infty} 0.$$

$\square$

Comments:

- There are much better results like Dominated Convergence but I won't make you remember them.
- (Technically we need to assume that the functions involved are nice enough for their integral to "make sense", but whatever.)

2.8.2 Swap Derivative and Integral

A derivative is just a limit! So you can use the above result.

2.8.3 Swap Limit and Sum

Theorem 2.15

Let $f_n(t)$ be a function depending on a parameter t. Suppose that the sum $\sum_{n=1}^{\infty} f_n(t)$ converges *uniformly in* t. Then

$$\lim_{t\to t_0}\sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty}\lim_{t\to t_0} f_n(t).$$

Notes:

- A similar result can be used to swap two limits.
- This can be used to prove that an infinite sum is continuous. To be precise, if f_n is continuous for all n and the sum $\sum_{n=1}^{\infty} f_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ is continuous. (Of course, this is just an instance of "the uniform limit of continuous functions is continuous".)

2.8.4 Swap Derivative and Limit

Theorem 2.16

Let $f_n : I \to \mathbb{R}$ be a sequence of differentiable functions on an interval I. Suppose:

- f_n converges pointwise to a function f, and
- f_n' converges *uniformly* to a function g.

Then f is differentiable with $f' = g$.

Proof. We have

$$\int_{x_0}^{x} f_n'(t)\,dt = f_n(x) - f_n(x_0).$$

Now send $n \to \infty$. The left side converges to $\int_{x_0}^{x} g(t)\,dt$ because $f_n' \to g$ uniformly. The right side converges to $f(x) - f(x_0)$ because $f_n \to f$ pointwise. So

$$\int_{x_0}^{x} g(t)\,dt = f(x) - f(x_0).$$

Differentiating in x, we conclude that $f'(x) = g(x)$. □

2.8.5 Swap Derivative and Sum

A direct application of the previous theorem allows us to more closely study the regularity of a series!

Theorem 2.17

Let $f_n : I \to \mathbb{R}$ be a sequence of differentiable functions on an interval I. Suppose:

- $\displaystyle\sum_{n=1}^{\infty} f_n$ converges pointwise, and
- $\displaystyle\sum_{n=1}^{\infty} f_n'$ converges *uniformly.*

Then $\sum_{n=1}^{\infty} f_n$ is differentiable, and

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x).$$

By an inductive argument we have the following corollary: If

- $\displaystyle\sum_{n=1}^{\infty} f_n$ converges pointwise,
- $\displaystyle\sum_{n=1}^{\infty} f_n'$ converges uniformly,
- $\displaystyle\sum_{n=1}^{\infty} f_n''$ converges uniformly,
- $\displaystyle\sum_{n=1}^{\infty} f_n'''$ converges uniformly,
- ...
- and $\displaystyle\sum_{n=1}^{\infty} f_n^{(k)}$ converges uniformly,

then $\sum_{n=1}^{\infty} f_n$ is k-times differentiable!

2.8.6 Swap Integral and Integral

For finite intervals $[a, b]$ and $[c, d]$, when is it true that

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy?$$

I'll give you two possible "tests" you can use:

1. It is true when f is non-negative. (Tonelli's Theorem)
2. It is true when f is bounded. (Budget Fubini's Theorem)

For example, if f is continuous on $[a, b] \times [c, d]$, then f is bounded and so you can swap.

2.8.7 Swap Derivative and Derivative

Easy one: If $f \in C^2(\mathbb{R}^n)$ (i.e. f is twice-differentiable and all second derivatives are continuous), then you can swap partial derivatives, i.e.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

(This is the *Schwarz Theorem*.)

2.9 Examples on Swapping

Example 2.20 (Winter 2009 #5): Let $u_n(x) = x^n \log x, x \in (0, 1]$.

(a) Check the convergence and uniform convergence of $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ in $(0, 1]$.

(b) Compute $I = \displaystyle\int_0^1 \left(\sum_{n=1}^{\infty} x^n \log x \right) dx$.

Example 2.21 (Fall 2017 #4): Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 + x^2}$$

defines a continuously differentiable function of $x \in \mathbb{R}$.

Proof. First let's show that it is differentiable. By the "swap derivative and sum" theorem, we need only show that the sum of the derivatives converges uniformly. The series in question is

$$\sum_{n=1}^{\infty} \frac{d}{dx} \frac{\sin(nx)}{n^3 + x^2} = \sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2 + x^2} - \frac{2x\sin(nx)}{(n^3 + x^2)^2}.$$

It's not clear what the pointwise limit is, so it is wise to try the Weierstrass M-test. Let's get an upper bound on the nth term that's independent of x:

$$\left|\frac{n\cos(nx)}{n^3 + x^2} - \frac{2x\sin(nx)}{(n^3 + x^2)^2}\right| \le \frac{n}{n^3} + \left|\frac{2x}{n^3 + x^2}\right| \cdot \left|\frac{\sin(nx)}{n^3 + x^2}\right| \le \frac{1}{n^2} + \frac{1}{n^{3/2}} \cdot \frac{1}{n^3}$$

This upper bound converges when summed, so the M-test applies and we deduce uniform convergence of the derivatives. Hence the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3+x^2}$ is differentiable and its derivative is

$$\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3 + x^2} = \sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2 + x^2} - \frac{2x\sin(nx)}{(n^3 + x^2)^2}.$$

But we are not done yet because we actually want to show that the series is *continuously* differentiable. Thus it remains to show that $\sum_{n=1}^{\infty} \frac{n\cos(nx)}{n^2+x^2} - \frac{2x\sin(nx)}{(n^3+x^2)^2}$ is continuous. By "the uniform limit of continuous functions is continuous", we just need to show that this series converges uniformly. But we've already shown that, so we're done. $\square$

Example 2.22 (Fall 2020 #2): Consider the series $f(x) := \sum_{n=1}^{\infty} x^n \sin(x^n x)$ where x is a real variable.

(a) Is there a (positive length) interval over which f is well-defined and continuous?

(b) Is there a (positive length) interval over which f is differentiable? If so, is it infinitely differentiable anywhere in this interval?