21-236 Recitations

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Contents

1 Integrating Factors

Note: There are two competing letter choices for ODEs that I know of.

- The first one views the independent variable as x and the dependent variable/function as u for "unknown". So it would look something like $u'(x) = f(x, u(x))$.
- The second one views the independent variable as time, hence naming it t, and the dependent variable/function as x, so that $x(t)$ is viewed as a particle's position (its "x") at a time t. Hence ODEs look something like $x'(t) = f(t, x(t))$.

We will be using the first one because I think that's what Leoni will be using this semester. The second one certainly has its merits, but we're not quite in a context that needs these merits so it's ok.

1.1 Warm Up

Example 1.1: Solve the IVP

$$\begin{cases} u'(x) = x^2 \\ u(x_0) = u_0 \end{cases}.$$

Solution. It's as easy as you think it is. After integrating, we get $u(x) = u_0 + \frac{1}{3}(x - x_0)^3$. ■

Example 1.2: Solve the IVP

$$\begin{cases} u'(x) + \cos(x)u(x) = x^2 \\ u(x_0) = u_0 \end{cases}.$$

Solution. We use a clever idea: By multiplying each side by $e^{\sin(x)}$, we get

$$u'(x)e^{\sin(x)} + \cos(x)e^{\sin(x)}u(x) = x^2e^{\sin(x)},$$

and magically the left side can be simplified using the product rule!

$$\frac{d}{dx}\left[u(x)e^{\sin(x)}\right] = x^2e^{\sin(x)}$$

Integrating, we obtain

$$u(x)e^{\sin(x)} - u(x_0) = \int_{x_0}^{x} t^2e^{\sin(t)}\,dt,$$

and so the solution is given by $\boxed{u(x) = u_0 e^{-\sin(x)} + e^{-\sin(x)} \int_{x_0}^{x} t^2 e^{\sin(t)}\, dt}$. This idea is cute but also quite common! It is good to keep in mind, as it will pop up a lot in differential equations. ■

This ODE was easy to solve because it's *linear*. Namely, the term that contains the $u(x)$ term is linear in $u(x)$. It would *not* be linear if that term were, say, $\sin(x + u(x))$. That would be pretty nasty to work with.

However, there are some cases in which ODEs are pretty easy to solve. A class of such ODEs are called *exact*.

1.2 Exact Equations

Let's say that a vector field $F = (F_1, F_2)$ is conservative. Then it turns out that the differential equation

$$F_1(x, u(x)) + F_2(x, u(x))u'(x) = 0 \qquad (*)$$

is actually pretty easy to solve for... at least, implicitly.

To rigorously state the problem, we have a conservative $F : I \times J \to \mathbb{R}^2$, where I and J are intervals. Note that the assumption on what the domain looks like is quite necessary — the natural domain for solutions to differential equations are intervals, and such solutions ought to be continuous so their range will be an interval as well. We wish to solve the IVP

$$\begin{cases} F_1(x, u(x)) + F_2(x, u(x))u'(x) = 0 \\ u(x_0) = u_0 \end{cases}$$

for some given $x_0 \in I$ and $u_0 \in J$.

Since F is conservative, we may find a potential $g : I \times J \to \mathbb{R}$ for which $\nabla g = F$. Then our ODE becomes

$$\frac{\partial g}{\partial x}(x, u(x)) + \frac{\partial g}{\partial y}(x, u(x))u'(x) = 0,$$

which is just

$$(1, u'(x)) \cdot \nabla g(x, u(x)) = 0,$$

or

$$\frac{d}{dx} g(x, u(x)) = 0$$

by the chain rule. Thus the solution is given implicitly by $g(x, u(x)) = c$ for a constant c. That is, the solution follows a level set of the gradient potential! (Exercise: Why does this make A LOT of sense?)

Plugging in $x = x_0$ we see that c must be $g(x_0, u_0)$, so our implicit form for the solution is $\boxed{g(x, u(x)) = g(x_0, u_0)}$.

Note that since $I \times J$ is simply connected, we can simply verify that ϕ is irrotational and C^1 for this theory to work.

Let's see a concrete example.

Example 1.3: Assume that $u_0 \in \mathbb{R}$ and $x_0 > 0$. Find a solution to the IVP

$$\begin{cases} \sin(u(x)) + x\cos(u(x))u'(x) = 0 \\ u(x_0) = u_0 \end{cases}$$

for $u : [x_0, +\infty) \to \mathbb{R}$.

Solution. If we were to run the above logic, the vector field in question would be $F(x, y) = (\sin y, x\cos y)$, which is indeed conservative with potential $g(x, y) = x\sin y$. Thus the solution to the IVP is a function whose graph is a level set of this gradient potential. It follows that

$$x\sin(u(x)) = x_0\sin(u_0)$$

and so $\boxed{u(x) = \sin^{-1}\left(\frac{x_0\sin(u_0)}{x}\right)}$, which is well-defined because $x \geq x_0 > 0$. (By the way, why might this not be unique?) ■

We call ODEs of the form $(*)$ *exact* when F is conservative.

1.3 Integrating Factors

Consider now the ODE

$$2u(x) + xu'(x) = 0.$$

This is unfortunately not exact. The associated vector field is $F(x, y) = (2y, x)$, which is not irrotational. So solving this might be unfortunately hard. However, note that we are free to change the vector field by multiplying each side by whatever we want. For example, we can turn it into

$$2xu(x) + x^2u'(x) = 0$$

by multiplying by x. Now the associated vector field is $F(x, y) = (2xy, x^2)$, which *is* irrotational! We therefore call x an *integrating factor*.

Definition 1.1

An integrating factor is an expression you multiply a diffeq by to make things much nicer. (In this case, we're using integrating factors to make ODEs exact.)

In the context of vector fields, Fleming has a more precise definition.

Definition 1.2

Let $F : \Omega \to \mathbb{R}^N$ be a vector field. We call a function $\phi : \Omega \to \mathbb{R}$ an *integrating factor* if $\phi(x) \neq 0$ for all $x \in \Omega$, and ϕF is conservative.

Example 1.4: An integrating factor for the vector field $(yz, 2xz, 3xy)$ is given by yz^2. Indeed, $(y^2z^3, 2xyz^3, 3xy^2z^2)$ is irrotational, and unsurprisingly it is also the gradient of some function, namely xy^2z^3.

The natural question to ask is: When does an integrating factor exist? It turns out that if things are sane, there always exists an integrating factor that exists "locally", but that doesn't necessarily give us a good way to write it down.

A common dumb method for trying to find an integrating factor is to "pray" that the integrating factor, if it exists, depends only on one variable. To start, let's pray that for the vector field $F = (F_1, F_2)$, there exists a function ϕ such that $\phi(x)F(x, y) = (\phi(x)F_1(x, y), \phi(x)F_2(x, y))$ is irrotational. Doing the math, ϕ and F would have to satisfy

$$\frac{d\phi}{dx}(x)F_2(x, y) + \phi(x)\frac{\partial F_2}{\partial x}(x, y) = \phi(x)\frac{\partial F_1}{\partial y}(x, y).$$

Moving things around, this becomes

$$\frac{1}{\phi(x)}\frac{d\phi}{dx}(x) = \frac{\frac{\partial F_1}{\partial y}(x, y) - \frac{\partial F_2}{\partial x}(x, y)}{F_2(x, y)}.$$

Since the LHS depends only on x, we see that a necessary condition for this above to be sensible is for the RHS to depend only on x. But then this above would have the form

$$\frac{d\phi}{dx}(x) = \phi(x) \cdot [\text{some function of } x],$$

which we can solve!

Theorem 1.1

If $\frac{\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}}{F_2}$ depends only on x, then a solution ϕ to the differential equation

$$\phi'(x) = \left(\frac{\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}}{F_2}\right)(x)\phi(x)$$

will be an integrating factor, so that $(\phi F_1, \phi F_2)$ is irrotational.
A similar result occurs if $\frac{\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}}{F_1}$ depends only on y.

Example 1.5: "Solve" the differential equation

$$3x^2u(x) + 2xu(x) + u(x)^3 + (x^2 + u(x)^2)u'(x) = 0.$$

Solution. The corresponding vector field is $F(x, y) = (3x^2y + 2xy + y^3, x^2 + y^2)$. This is sadly not irrotational/conservative. However, we may note that

$$\frac{\frac{\partial F_1}{\partial y}(x, y) - \frac{\partial F_2}{\partial x}(x, y)}{F_2(x, y)} = 3,$$

which does not depend on y. So by the theorem, an integrating factor ϕ is given by solving the differential equation

$$\phi'(x) = 3\phi(x),$$

and easily we can pick $\phi(x) = e^{3x}$. We may now find that

$$\phi(x)F(x, y) = \nabla g(x, y)$$

where $g(x, y) = \int_0^y (x^2 + t^2)e^{3x}\, dt = (x^2y + y^3/3)e^{3x}$ is a gradient potential for $\phi(x)F(x, y)$. Hence a solution $u(x)$ is a level set of g, i.e.

$$(x^2u(x) + u(x)^3/3)e^{3x} = C$$

for a constant C. If you're really desperate, you may now proceed to use the cubic formula and obtain an explicit form for u. But I have better things to do. ■

2 The Winding Number and Simply Connected Sets

2.1 Winding Number

The winding number of a curve around some point x_0 is simply the number of times it goes "around" x_0. The way we make this rigorous is... uh...

Definition 2.1

Let γ be a closed curve and $(x_0, y_0) \in \mathbb{R}^2$. The *winding number* or *index* of γ around (x_0, y_0) is defined as

$$\operatorname{wind}(\gamma, x_0) := \frac{1}{2\pi} \int_\gamma \left(\frac{-(y-y_0)}{(x-x_0)^2 + (y-y_0)^2}, \frac{x-x_0}{(x-x_0)^2 + (y-y_0)^2} \right).$$

This is also just

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{-(\varphi_2(t) - y_0)\varphi_1'(t) + (\varphi_1(t) - x_0)\varphi_2'(t)}{(\varphi_1(t) - x_0)^2 + (\varphi_2(t) - y_0)^2} \, dt$$

for a parametrization φ of γ.

Here's a small digression on why this makes any sense whatsoever. If you've ever studied any complex analysis, you may be aware of a notion of integrating over curves in $\mathbb{C}$. The ugly integrand has a natural representation in complex analysis. If we take a parametrization $\varphi : [0, T] \to \mathbb{R}^2$ of the curve, then

$$\operatorname{wind}(\gamma, x_0) = \frac{1}{2\pi} \int_0^T \frac{-(\varphi_2(t) - y_0)\varphi_1'(t) + (\varphi_1(t) - x_0)\varphi_2'(t)}{(\varphi_1(t) - x_0)^2 + (\varphi_2(t) - y_0)^2} \, dt.$$

If we now view points (x, y) as complex numbers, this can be massaged into

$$= \frac{1}{2\pi} \int_0^T \frac{\operatorname{Re}(\frac{1}{i}\overline{(\varphi(t) - z_0)}\varphi'(t))}{|\varphi(t) - z_0|^2} \, dt$$

$$= \operatorname{Re}\left(\frac{1}{2\pi i} \int_0^T \frac{\varphi'(t)}{\varphi(t) - z_0} \, dt \right),$$

and now using the notion of contour integration *in the complex plane* (which you may or may not know) this is actually

$$= \operatorname{Re}\left(\frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz \right).$$

This is precisely the number of times γ revolves around z_0 by the Residue Theorem.

2.2 Examples

Keep in mind that this definition of winding number isn't just a formula for "stating the obvious". I view it as a bit more like a useful tool for rigorously studying this notion of "going around things".

Example 2.1: Let $\operatorname{wind}(\partial B(0,1),(0,0))$ where $\partial B(0,1)$ is oriented counter-clockwise.

Solution. With the obvious parametrization $\varphi(t) := (\cos t, \sin t)$, we have

$$\operatorname{wind}(\partial B(0,1),(0,0)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(-\sin t, \cos t)\cdot(-\sin t, \cos t)}{\cos(t)^2+\sin(t)^2} = 1,$$

to the surprise of absolutely nobody. ■

Example 2.2: Consider the limaçon γ with polar equation $r = 2\cos\theta - 1$, which may be parametrized by

$$\varphi(t) := ((2\cos t - 1)\cos t, (2\cos t - 1)\sin t), \qquad t \in [0, 2\pi].$$

Compute $\operatorname{wind}(\gamma, (1/2, 0))$.

Solution. Stuffing it into the definition, we have

$$\begin{aligned}\operatorname{wind}(\gamma,(1/2,0)) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\varphi_2(t)\varphi_1'(t) + (\varphi_1(t)-1/2)\varphi_2'(t)}{(\varphi_1(t)-1/2)^2+\varphi_2(t)^2}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{-\varphi_2(t)\varphi_1'(t) + (\varphi_1(t)-1/2)\varphi_2'(t)}{(\varphi_1(t)-1/2)^2+\varphi_2(t)^2}\,dt \\ &= \text{Eww} \\ &= \boxed{2}\end{aligned}$$

■

2.3 Simply Connected Sets

Recall that a set is simply connected if it is path-connected, and any closed curve inside it can be shrinked to a point. This shrinking is called a *homotopy*, and so the jargon would be "the curve is homotopic to a constant curve".

Recall:

Definition 2.2

- Curves γ_1 and γ_2 are *homotopic* if there exists a continuous $h : [0,1] \times [a,b]$ such that $h(0,t) = \varphi_1(t)$ and $h(1,t) = \varphi_2(t)$ where φ_1, φ_2 are parametrizations of γ_1, γ_2, respectively.

- γ_1 and γ_2 are *fixed-endpoint homotopic* if moreover $h(s,a) = \varphi_1(a) = \varphi_2(a)$ and $h(s,b) = \varphi_1(b) = \varphi_2(b)$ for all $s \in [0,1]$.

Observe that these notions of homotopic equivalence are both equivalence relations.

In lecture, we defined the following notion of being *null-homotopic*.

Definition 2.3

A closed curve γ is *null-homotopic* if it is fixed-endpoint homotopic to the constant curve which is equal to x_0 for all time, where $x_0 = \varphi(0) = \varphi(1)$ for every parametrization φ of γ.

This is a bit annoying to work with. Intuitively we would like to view a closed curve as being null-homotopic if it is homotopic to *some* constant curve (i.e. some point), not just a very specific one. Fortunately we can show that this easier notion of null-homotopic is equivalent.

Proof.

($\Longrightarrow$) Null-homotopic implies homotopic to *some* point

Trivial.

($\Longleftarrow$) Homotopic to *some* point implies null-homotopic

Let γ be a closed curve and let φ be a parametrization of γ. Suppose γ is homotopic to y_0, witnessed by a homotopy h so that $h(0,t) = \varphi(t)$ and $h(1,t) = y_0$ for all t.

The picture to have in mind is that "we'll do the homotopy but also draw the trail left behind by x_0 as it moves to y_0". To that end, consider the fixed-endpoint homotopy

$$h_1(s,t) := \begin{cases} h(2t, 0), & 0 \le t < s/2 \\ h\left(s, \frac{t-(s/2)}{1-s}\right), & s/2 \le t \le 1 - s/2 \\ h(2-2t, 0), & 1 - s/2 < t \le 1 \end{cases}.$$

This fixed-endpoint homotopy shows that γ is fixed-endpoint homotopic to the curve γ_1 parametrized by

$$\varphi_1(t) := \begin{cases} h(2t), & 0 \le t < 1/2 \\ h(2-2t), & 1/2 \le t \le 1 \end{cases},$$

which simply is a closed curve that goes from x_0 to y_0 and then back, along the same path. It remains to "pull back" γ_1 towards x_0. This may be done via the fixed-endpoint homotopy

$$h_2(s,t) := \begin{cases} h(2((1-s)t), & 0 \le t < 1/2 \\ h(2(1-s)(1-t)), & 1/2 \le t \le 1 \end{cases}.$$

So γ is fixed-endpoint homotopic to γ_1 which is fixed-endpoint homotopic to the constant curve at x_0, thus γ is null-homotopic. □

Great, this means that to verify that a curve is null-homotopic, we can just show that it is homotopic to *some* point x_0 that we are free to choose.

Theorem 2.1

Any convex $K \subseteq \mathbb{R}^N$ is simply connected.

Proof. That K is path-connected is obvious, since between any two points you can form a path via the line segment between them. Now take a closed curve γ in K parametrized by $\varphi : [0,T] \to K$, and for ease let us assume WLOG that $\vec{0} \in K$. We show that γ may be shrinked to $\vec{0}$. Indeed, consider the homotopy

$$h(s,t) := s\varphi(t).$$

Since $\vec{0} \in K$ and $\varphi(t) \in K$ we have $h(s,t) = s\varphi(t) \in K$ for all $s \in [0,1]$ by convexity. $t \mapsto h(0,t)$ parametrizes the constant curve with value $\vec{0}$ and $t \mapsto h(1,t)$ parametrizes γ, so γ is homotopic to a point. Hence K is simply connected. □

In particular, we see that $\mathbb{R}^N$ is simply connected for all N. (Of course, if you attended lecture, this is all just a special consequence of Example 25 which states that any star-shaped set is simply-connected. Indeed, convex sets are star-shaped!)

Let's take a look at some more specific examples.

Example 2.3: Is $\mathbb{R}^2 \setminus \{(0,0)\}$ simply connected?

Solution. No. If it were, then

$$\int_{\partial B(0,1)} \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

should be 0 (because the vector field would have to be conservative). But we know that this is actually 2π (when taking the counter-clockwise orientation). ■

Example 2.4: Is $\mathbb{R}^3 \setminus \{(0,0,0)\}$ simply connected?

Solution. Yes. It's obvious enough that it's pathwise-connected, but showing that every closed curve is null-homotopic is much more interesting. To help us prove this we shall look at some lemmas. This first lemma states intuitively that if two loops are null-homotopic, then so is their "sum".

Lemma 2.1

Let U be open, $x, y \in U$, let α be a path from x to y with range in U, and let β, γ be two paths from y to x with range in U. Then the compositions $\alpha\beta$ and $\beta^{-1}\gamma$ are two closed paths that "share" the path β, going in opposite directions. If both of these closed paths are null-homotopic, then so is $\alpha\gamma$.

(It may be helpful to view $\alpha\beta$ as the "initial curve" that we "lengthen" by "adding $\beta^{-1}\gamma$ to it" to get $\alpha\gamma$.)

Proof. Let $a, b, c : [0,1] \to U$ parametrize α, β, γ. Then $\alpha\gamma$ is parametrized by

$$ac(t) = \begin{cases} a(2t), & 0 \le t < 1/2 \\ b(2t-1), & 1/2 \le t \le 1 \end{cases}.$$

Step 1

Let's first show that $\alpha\gamma$ is homotopic to $(\alpha\beta)(\beta^{-1}\gamma)$, which may be parametrized by

$$(ab)(b^{-1}c)(t) = \begin{cases} a(4t), & 0 \le t < 1/4 \\ b(4t-1), & 1/4 \le t < 1/2 \\ b(3-4t), & 1/2 \le t < 3/4 \\ c(4t-3), & 3/4 \le t \le 1 \end{cases}.$$

Intuitively, we should imagine the curve $\alpha\gamma$ "growing an arm" at y, which extends to x along β. Indeed, a homotopy is given by

$$h(s,t) = \begin{cases} a(4t/(2-s)), & 0 \le t < (2-s)/4 \\ b(4t-s), & s/4 \le t < s/2 \\ b(3s-4t), & s/2 \le t < 3s/4 \\ c((4t-(2+s))/(2-s), & (2+s)/4 \le t \le 1 \end{cases}.$$

As s moves from 0 to 1, $h(s, \cdot)$ travels further down β and back.

Step 2

Since homotopic equivalence is an equivalence relation, it remains to show that $(\alpha\beta)(\beta^{-1}\gamma)$ is null-homotopic. Intuitively,

Let homotopies h_1 and h_2 witness the hypothesis that $\alpha\beta$ and $\beta^{-1}\gamma$ are null-homotopic, respectively. Specifically we may take these to be homotopies to the point x. Now consider the homotopy

$$H(s,t) = \begin{cases} h_1(s,2t), & 0 \leq t < 1/2 \\ h_2(s,2t-1), & 1/2 \leq t \leq 1 \end{cases}.$$

This works! Note that requiring these to be homotopies that shrink $\alpha\beta$ and $\beta^{-1}\gamma$ to the point x is what ensures that H is continuous at $t = 1/2$, because $H(s,\cdot)$ will always be two loops "glued at x". □

This intuitive lemma is invaluable for the next result.

Lemma 2.2

Let open sets U, V be simply connected. If $U \cap V$ is non-empty pathwise connected, then $U \cup V$ is simply connected.

Proof. Let γ be a closed curve with range in $U \cup V$. We will show that γ is null-homotopic.

Step 1

We will first divide γ into a bunch of paths, each of which stays entirely within either U or V, with both endpoints in $U \cap V$. This is not hard to do, but what we're really aiming for is to make the number of such paths *finite*, otherwise the implied induction in the next step falls apart.

To wit, parametrize γ via $\varphi : [0,1] \to U \cup V$. Note that $\varphi^{-1}(U)$ is a countable disjoint union of *relatively* open intervals $\bigcup_{i=1}^{\infty} I_i$, and similarly $\varphi^{-1}(V) = \bigcup_{j=1}^{\infty} J_j$. Now $(\bigcup_{i=1}^{\infty} I_i) \cup \left(\bigcup_{j=1}^{\infty} J_j\right)$ is an open cover of $[0,1]$ (in the relative topology), so we may pass to a finite subcover $[0, b_0) \cup \bigcup_{i=1}^{n-1} (a_i, b_i) \cup (a_n, 1]$.

We may pick and rearrange this subcover so that the right-endpoints are ordered (i.e. $0 < b_0 < b_1 < \ldots < b_{n-1} < 1$), and so that the cover is *minimal*. That is, it would not cover $[0,1]$ if any interval is deleted.

Now take any $0 \leq i \leq n-1$. Then b_i is covered by the interval (a_{i+1}, b_{i+1}), and *only* this interval (by minimality!). So $a_{i+1} < b_i$. Thus we may pick x_i with $a_{i+1} < x_i < b_i$. Since $x_i \in (a_i, b_i) \cap (a_{i+1}, b_{i+1})$, we have that $x_i \in U \cap V$ (I'm skipping some minor logical details here — convince yourself that this is a valid leap!).

Consequently, we have for each $1 \leq i \leq n-1$ that $\varphi([x_{i-1}, x_i]$ is the range of a curve contained completely inside either U or V, with endpoints $\varphi(x_{i-1}), \varphi(x_i)$ in $U \cap V$. If we assume WLOG via a "rotation homotopy" that $\varphi(0) = \varphi(1) \in U \cap V$, then we get the same nice property for $\varphi([0, x_0])$ and $\varphi([x_{n-1}, 1])$. This completes this step.

Step 2

Changing notation a bit, we have from the previous step that we may write γ as a composition (or "product"?) of paths $\gamma_1, \gamma_2, \cdots, \gamma_n$, each fully contained in either U or V, such that for each i the endpoints y_{i-1}, y_i of γ_i lie in $U \cap V$. (I'm playing fast and loose with the notion of "composing curves" here, hopefully you don't mind. If you're concerned, include some extra homotopies to get the time intervals to line up.)

Since $U \cap V$ is path-connected, we can find a path δ_i from y_i to y_0 with range in $U \cap V$. So $\gamma_1\delta_1$ is a closed curve with range in U, and hence is null-homotopic. We can say the same for $\delta_1^{-1}\gamma_2\delta_2$. Thus by the previous lemma, $\gamma_1\gamma_2\delta_2$ is null-homotopic. But so is $\delta_2^{-1}\gamma_3\delta_3$, so $\gamma_1\gamma_2\gamma_3\delta_3$ is null-homotopic. Inductively we deduce that $\gamma_1\gamma_2\ldots\gamma_n$ (maybe plus a constant curve) is null-homotopic. Done. □

Writing $\mathbb{R}^3 \setminus \{0,0,0\}$ as $U \cup V$ where

$$U = \{(x, y, z) : x = y = 0 \text{ and } z > 0 \text{ or } x, y \neq 0 \text{ and } z > -1\}$$

$$V = \{(x, y, z) : x = y = 0 \text{ and } z < 0 \text{ or } x, y \neq 0 \text{ and } z < 1\},$$

we see that U and V are both simply connected because they are star-shaped, and moreover $U \cap V = (\mathbb{R}^2 \setminus \{(0,0)\}) \times (-1, 1)$ which is path-connected. Thus $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is simply connected. ■

With that out of the way, I'd like to wrap up by discussing why certain conditions in Lemma 2.2 are quite necessary.

- It is critical that $U \cap V$ be pathwise-connected. Consider, for instance, taking two "bananas" in $\mathbb{R}^2$ and overlaying their ends. Their union is homeomorphic to a punctured disk and is hence not simply connected.
- It is of course incredibly important that U and V be simply connected. A counterexample is not hard to cook up.
- What about the condition that U and V be open? We needed openness in the proof to construct the open cover of $[0, 1]$. It turns out that if we don't require openness then things break. Here's a simple counterexample in $\mathbb{R}^2$:

$$U := \{(\cos t, \sin t) : 0 \leq t \leq \pi\}$$

$$V := \{(\cos t, \sin t) : \pi \leq t < 2\pi\}$$

3 Connectedness

3.1 Definition in Topological Spaces

We've been working with this notion of "path-connectedness" for a while now. It turns that the notion of *connectedness* is different from this.

Definition 3.1 (Disconnected, Connected)

Let (X, τ) be a topological space. A set $E \subseteq X$ is *disconnected* if there exist open sets $U_1, U_2 \subseteq X$ for which:

- $U_1 \cap E \neq \emptyset$ and $U_2 \cap E \neq \emptyset$,
- $U_1 \cap U_2 \cap E = \emptyset$ ("U_1 and U_2 are separate"), and
- $E \subseteq U_1 \cup U_2$ ("E is covered by U_1 and U_2").

E is *connected* if it is not disconnected.

This definition has a bit of an oddity. The second bullet only necessitates that $U_1 \cap U_2 \cap E = \emptyset$ instead of the possibly more intuitive condition $U_1 \cap U_2 = \emptyset$, meaning that in our definition, U_1 and U_2 are allowed to intersect outside of E.

It turns out this oddity is actually *more* intuitive. This is because we should view these conditions as saying that E is being separated by *relatively open* sets. The philosophy here is that we should only need to focus on what's happening in E, not anything outside of E. So if we frame the question as "is E separated, as a *topological space*?" then it is indeed more natural to view the game we're playing as happening entirely within E.

This philosophy will be reflected in Leoni's notes when he writes $E = (U_1 \cap E) \cup (U_2 \cap E)$ instead of $E \subseteq U_1 \cup U_2$. It's equivalent but it really emphasizes how we only care about what's inside E.

Let's begin with a simple example.

Example 3.1: Prove that if E is connected, then so is $\overline{E}$.

Proof. We'll prove the contrapositive. If $\overline{E}$ is disconnected, then we may find U_1, U_2 witnessing this. We claim that these open sets also witness that E is disconnected. Let's verify the conditions.

- Using 269-era arguments, you can show that $\overline{E} \cap U_1 \neq \emptyset$ implies $E \cap U_1 \neq \emptyset$. Similarly $E \cap U_2 \neq \emptyset$.

- $\emptyset \subseteq U_1 \cap U_2 \cap E \subseteq U_1 \cap U_2 \cap \overline{E} = \emptyset$ so $U_1 \cap U_2 \cap E = \emptyset$.
- $E \subseteq \overline{E} \subseteq U_1 \cup U_2$

□

3.2 ...it gets better in metric spaces!

If we upgrade the structure on X to a *metric space*, then we may actually remove this oddity! That is, in metric spaces, a set E can be separated by relatively open sets iff it can be separated by open sets!

Lemma 3.1

Suppose (X, d) is a metric space. Then in the definition of disconnectedness, we may replace the condition $U_1 \cap U_2 \cap E = \emptyset$ with $U_1 \cap U_2 = \emptyset$. That is, $E \subseteq X$ is disconnected iff there exist open sets $U_1, U_2 \subseteq X$ for which:

- $U_1 \cap E \neq \emptyset$ and $U_2 \cap E \neq \emptyset$,
- $U_1 \cap U_2 = \emptyset$. and
- $E \subseteq U_1 \cup U_2$.

Proof. One direction is trivial, so we do the other one. That is, suppose that E is disconnected, so that we find U_1, U_2 satisfying the properties, with the possibility that $U_1 \cap U_2 \neq \emptyset$. We will construct open sets V_1, V_2 satisfying the properties so that $V_1 \cap V_2 = \emptyset$.

Well, metric spaces let us draw balls, so let's draw a crapton of balls! For each $x \in U_1 \cap E$, we find $B(x, r_x) \subseteq U_1$, and for each $y \in U_2 \cap E$ we find $B(y, r_y) \subseteq U_2$. Now we simply take:

$$V_1 = \bigcup_{x \in U_1 \cap E} B(x, r_x/2)$$

$$V_2 = \bigcup_{y \in U_2 \cap E} B(y, r_y/2)$$

Let's prove that this works. It's clear that V_1, V_2 are open, $V_1 \cap E, V_2 \cap E \neq \emptyset$, and $E \subseteq V_1 \cup V_2$. It remains to show that $V_1 \cap V_2 = \emptyset$. To see this, we suppose otherwise and take $z \in V_1 \cap V_2$. Then there is $x \in V_1$ and $y \in V_2$ such that $d(x, z) < r_x/2$ and $d(y, z) < r_y/2$. So $d(x, y) < r_x/2 + r_y/2$. If WLOG $r_x < r_y$ then $d(x, y) < r_y$ so $x \in B(y, r_y)$. So $x \in U_2$, hence $x \in U_1 \cap U_2 \cap E$. But $U_1 \cap U_2 \cap E$ should be empty, contradiction! □

This lets us easily characterize all the connected sets in $\mathbb{R}$.

Example 3.2: Show that a set $I \subseteq \mathbb{R}$ is connected iff E is an interval.

Proof. ($\implies$) Suppose I is connected. Take $x, y \in I$, $x \leq y$. Let $x \leq z \leq y$. We want to show $z \in I$. Well, if not, then $(-\infty, z) \cup (z, \infty)$ is a separation of I by open sets, contradiction.

($\impliedby$) Suppose I is an interval. Suppose for contradiction that I is disconnected. Then we may find U_1, U_2 witnessing this, and moreover we may ensure that $U_1 \cap U_2 = \emptyset$ by the lemma. Take $x \in U_1 \cap I$, $y \in U_2 \cap I$, and assume WLOG that $x < y$. Let $z_0 = \sup\{z \in I : [x, z] \subseteq U_1 \cap I\}$. Note that $x < z_0 < y$, so $z_0 \in I$.

It follows that either $z_0 \in U_1$ or $z_0 \in U_2$. However neither is possible. On one hand, if $z_0 \in U_1$ then $[x, z_0 + \varepsilon] \subseteq U_1$ for a small $\varepsilon > 0$, contradicting the definition of z_0. On the other hand, if $z_0 \in U_2$ then $z_0 - \varepsilon \notin U_1$ for a small $\varepsilon > 0$, so $[x, z_0] \nsubseteq U_1$. □

3.3 ...it gets even better in normed spaces!

We defined a notion of *pathwise connected* in lecture, and now we have this weird other notion of *connected* which is seemingly different. It turns out that in normed spaces such as $\mathbb{R}^N$, they're equivalent for open sets!

Theorem 3.1

Let $(X, \|\cdot\|)$ be a normed space. Then an open set $U \subseteq X$ is connected iff it is pathwise connected.

Proof.

This proof is very fun! First, we leave the case $U = \emptyset$ to the philosophers and only consider $U \neq \emptyset$.

Forward Direction (Connected $\implies$ pathwise connected)

Assume U is connected. Take a point $x_0 \in U$. Let V be the set of points in U that can be reached from x_0 via a polygonal path.

CLAIM: V is open... (i.e. V is relatively open)

If $x \in V$, then we may find $B(x, r) \subseteq U$. For this r we will have $B(x, r) \subseteq V$. This is because for any $x' \in B(x, r)$ we may reach x' from x_0 by first drawing a polygonal path from x_0 to x (possible by definition of V) then drawing a segment from x to x' (possible because balls in normed spaces are convex).

CLAIM: ...uh, but also $U \setminus V$ is open... (i.e. V is relatively closed)

By similar reasoning, we see that if $x \in U \setminus V$, so that x_0 cannot reach x via a polygonal path, then x_0 cannot reach points in a small ball around x, since otherwise x_0 would be able to reach x.

CLAIM: ...so actually $V = U$, which is what we needed to show.

$U = V \sqcup (U \setminus V)$ is a separation of U via disjoint open sets, which would disconnect U unless either V or $U \setminus V$ is empty. Since $x_0 \in V$, the latter must hold, so $U \setminus V = \emptyset$ and $U = V$.

Backward Direction (Connected $\Longleftarrow$ pathwise connected)

Assume U is pathwise connected. Suppose for contradiction that U is disconnected, and find open sets U_1, U_2 that witness this. Take $x \in U_1 \cap E$, $y \in U_2 \cap E$, and find a path γ from x to y parametrized by $\varphi : [0,1] \to U$.

Now $V_1 := \varphi^{-1}(U_1)$ and $V_2 := \varphi^{-1}(U_2)$ are non-empty, disjoint, relatively open sets in $[0,1]$ whose union is $[0,1]$. So $[0,1]$ is disconnected, contradiction. $\square$

Remarks:

- We used openness to draw balls...
- ...and we used the fact that X is a normed space so that we can safely draw line segments in these balls by convexity!
- We implicitly, automagically proved some other properties in the above proof. For example, it is true in general topological spaces that pathwise connected implies connected, and that continuous functions preserve connectedness. If you're bored, piece together the proofs of these properties before Leoni does it in lecture.

3.4 Horrifying Counterexamples

3.4.1 We don't have Lemma 3.1 in topological spaces.

Take (X, τ) with $X = \{1,2,3\}$ and the open sets being $\emptyset, \{1,2\}, \{2,3\}, \{2\}$ and $\{1,2,3\}$. Then $E = \{1,3\}$ is disconnected, but the only way to disconnect it is via the open sets $U_1 = \{1,2\}$ and $U_2 = \{2,3\}$, which intersect outside of E.

3.4.2 There is an open set in a metric space which is connected but not pathwise connected. There is a set in a normed space which is connected but not pathwise connected.

Definition 3.2

The *Topologist's Sine Curve* is the set

$$\left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \leq 1\right\} \cup \{(0,0)\}.$$

(Note: Leoni's convention is to include the entire segment $\{0\} \times [-1,1]$ instead of just $\{(0,0)\}$. This doesn't change much, but some people call this the *closed Topologist's Sine Curve*.)

Call this set T. Viewing T as a subset of the normed space $\mathbb{R}^N$, we claim that T is connected. Indeed, suppose open sets U_1, U_2 separate T. WLOG $(0,0) \in U_1$. Then U_1 intersects the "squiggly part" $S := \{(x, \sin\frac{1}{x}) : 0 < x < 1\}$, but not all of it. U_2 contains the part of S not covered by U_1. From this, we see that U_1 and U_2 separate S, so S is disconnected. But S is connected because it is the image of the continuous map $x \in (0,1) \mapsto (x, \sin 1/x)$ and $(0,1)$ is connected, contradiction.

However, T is not pathwise connected. If it were, then there exists a path from $(1, \sin 1)$ to $(0,0)$ parametrized by $\varphi : [0,1] \to T$. Evidently $\varphi([0,1])$ must be connected, and for every $x \in (0,1)$ if $(x, \sin 1/x) \notin \varphi([0,1])$ then the open sets $(-\infty, x) \times \mathbb{R}$ and $(x, \infty) \times R$ disconnect $\varphi([0,1])$, which is bad. So it follows that $\varphi([0,1]) = T$. Since continuous functions send compact sets to compact sets, we deduce that T is compact. But it's not. (For instance, the point $(0,1) \notin T$ is an accumulation point.)

Thus, T is a set in a normed set which is connected but not pathwise connected. How about an *open set* in a *metric space* which is connected but not pathwise connected? It turns out that T is also an example of this! By passing to a relative topology, we may sacrifice the normed structure to force T to be open.

Specifically, we take our metric space to be T (endowed with the Euclidean metric), and the open set in question is the entire space T. Reasoning about relative topologies or otherwise, our arguments imply that T is an open connected set that is not pathwise connected.

4 Arc Length

4.1 Some Quick Examples of Connectedness

Example 4.1: Determine if the following sets are connected:

- $\mathbb{Q}^2$
- $(\mathbb{R}^2 \setminus \mathbb{Q}^2)$
- $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$
- $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ xor } y \in \mathbb{Q}\}$
- $\partial B_3(0, 1)$

Solution.

- $\mathbb{Q}^2$ is not connected, take $U_1 = \{x < \sqrt{2}\}$ and $U_2 = \{x > \sqrt{2}\}$.
- $(\mathbb{R}^2 \setminus \mathbb{Q}^2)$ is connected. Path connected always implies connected, so it is sufficient to show that it is path connected. Take $x, y \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and, using your favorite methodology, draw uncountably many paths from x to y that are disjoint (except at x and y). $\mathbb{Q}^2$ is countable, so surely one of these paths will not pass through an element of $\mathbb{Q}^2$.
- $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ is connected. It is possible to find a path between any two points of this set by following only horizontal and vertical line segments.
- $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ xor } y \in \mathbb{Q}\}$ is not connected. Consider $\{(x, y) : x > y\}$ and $\{(x, y) : x < y\}$.
- $\partial B_3(0, 1)$ is connected. We can in fact show that it is path connected. Intuitively this is obvious, but the main point of the exercise is to figure out the cleanest way to generate a path.

  For any two distinct points $x, y \in \partial B_3(0, 1)$, find a path between them in $\mathbb{R}^3$ that does not pass through $(0, 0, 0)$. (If x and y are not diametrically opposite, a straight line will do. Otherwise, you can do it in two segments.) Then project this path unto $\partial B_3(0, 1)$. The projection is continuous so the image is also a path.

■

4.2 Arc Length

Recall the following theorem.

Theorem 4.1

Let I be an interval and $f \in BPV(I;\mathbb{R}^d)$ (note that f is differentiable almost everywhere by Lebesgue Differentiation). Then

$$\int_I |f'|\,dx = \operatorname*{Var}_I f \iff f \text{ is AC.}$$

This will streamline our discussion on *arc length*.

Definition 4.1 (Arc Length)

Let γ be a curve in $\mathbb{R}^N$. The *length* of γ is given by

$$L(\gamma) := \operatorname*{Var}_I \varphi,$$

where $\varphi : I \to \mathbb{R}^N$ is a parametrization of γ.

Is this well-defined? We should verify that $\operatorname{Var}_I \varphi = \operatorname{Var}_J \psi$, where $\varphi : I \to \mathbb{R}^N$ and $\psi : I \to \mathbb{R}^N$ are two different parametrizations of γ. Well, by the definition of a curve, if both φ and ψ are parametrizations then they are equivalent, so that there exists a homeomorphism $h : I \to J$ such that $\varphi = \psi \circ h$. Fixing an arbitrary partition $t_0 < t_1 < \ldots < t_n$ in I, we have

$$\sum_{i=1}^{n} \|\varphi(t_i) - \varphi(t_{i-1})\| = \sum_{i=1}^{n} \|\psi(h(t_i)) - \psi(h(t_{i-1}))\| \le \operatorname*{Var}_J \psi,$$

and so by taking the sup on the LHS we get $\operatorname{Var}_I \varphi \le \operatorname{Var}_J \psi$. But there was nothing special about this order, so by the same logic we also get $\operatorname{Var}_J \psi \le \operatorname{Var}_I \varphi$ for free, hence the equality.

Example 4.2: The shortest path between two points is a straight line.

Proof. Let $x_0, y_0 \in \mathbb{R}^N$. Then for any curve γ with endpoints x_0 and y_0 and parametrization $\varphi \in C([a,b],\mathbb{R}^N)$, we have

$$L(\gamma) = \operatorname*{Var}_I \varphi \le \|\varphi(a) - \varphi(b)\| = \|x_0 - y_0\|,$$

and the length of $\|x_0 - y_0\|$ is obtained by a straight line. □

Note that if a curve γ is AC, so that it is parametrized by some AC $\varphi : I \to \mathbb{R}^N$, then the length is simply given by

$$L(\gamma) = \operatorname*{Var}_I \varphi = \int_I |\varphi'|\,dt.$$

For instance, it is unsurprising that the closed curve parametrized by $\varphi(t) := (\cos t, \sin t)$, $t \in [0, 2\pi]$ has length 2π. Since φ is AC, we can show this explicitly via the integral

$$\int_0^{2\pi} \|\varphi'(t)\| \, dt = \int_0^{2\pi} \|(-\sin t, \cos t)\| \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

But some curves are not so nice.

Example 4.3: Let $f : [0, 1] \to [0, 1]$ be the Devil's Staircase. Let γ be the curve parametrized by $\varphi(t) := (t, f(t))$, $t \in [0, 1]$. What is $L(\gamma)$?

Solution. Although $\varphi' = (1, 0)$ exists for almost every t, we certainly cannot say that the length is $\int_0^1 \|\varphi'\| \, dt = 1$. Though, it is true that this gives the bound $L(\gamma) \geq 1$.

I have no idea if there exists an AC parametrization. There probably isn't one. So this is one of those things where we need to use the definition via pointwise variation. If you think about it from the right angle, you can show that $L(\gamma) = 2$. ■

4.3 Rectifiable Curves

Definition 4.2

A curve γ is *rectifiable* if $L(\gamma) < \infty$. We say that γ is *locally rectifiable* if it admits a parametrization $\varphi \in BPV_{\text{loc}}(I; \mathbb{R}^N)$.

There certainly exist continuous curves with a parametrization on $[a, b]$ and infinite length. Take, for instance, a space-filling curve, or the graph of $x \sin(1/x)$ over $[-1, 1]$.

An example of a locally rectifiable curve which is not rectifiable is the graph of x^2.

Definition 4.3

A parametrization $\varphi : I \to \mathbb{R}^N$ of a curve γ is a *parametrization by arc length* if

$$\operatorname*{Var}_{[s,t]} \varphi = t - s$$

for all $s, t \in I$, $s < t$.

Some observations:

- Note that such a φ is Lipschitz and thus AC, so we can write

$$L(\gamma) = \int_0^{L(\gamma)} \|\varphi'\| \, dt \qquad (*)$$

for such curves γ.

- Since φ is 1-Lipschitz, we have $\|\varphi'\| \leq 1$ wherever it is differentiable, i.e. almost everywhere. But from $(*)$ we can in fact observe that

$$0 = \int_0^{L(\gamma)} 1 - \|\varphi'\| \, dt \geq 0,$$

which implies that $\varphi' = 1$ almost everywhere. This makes a lot of intuitive sense!

We evidently have the inclusions:

$$\text{Param. By Arc Length} \subseteq \text{Locally Rectifiable}$$

$$\text{Rectifiable} \subseteq \text{Locally Rectifiable}$$

An example of a curve which is rectifiable but not parametrizable by arc length is a constant curve $\varphi(t) := x_0$, $t \in [a,b]$, $a < b$. If there exists a parametrization $\psi = \varphi \circ h : [c,d] \to \mathbb{R}^N$ by arc length, then $0 = \operatorname{Var}_{[c,d]} \psi = d - c$. So $c = d$. That's kinda hard to pull off because then $[a,b] = h([c,d])$ is a singleton, despite the assertion $a < b$.

The issue here is that the parametrization φ "stops" for a long time. It turns out that if you prevent this from happening, then the curve will be parametrizable by arc length.

Theorem 4.2

Let γ be a continuous, locally rectifiable curve. Suppose γ may be parametrized by a $\varphi : I \to \mathbb{R}^N$ which is not constant on any proper subinterval of I. Then γ may be parametrized by arc length.

Proof. The idea is to revert the "lengthening" that φ is doing. To wit, pick a "start time" $t_0 \in I$ and use it to define the indefinite variation

$$v(t) := \begin{cases} \operatorname{Var}_{[t_0,t]} \varphi, & t \geq t_0 \\ -\operatorname{Var}_{[t,t_0]} \varphi, & t < t_0 \end{cases}. \qquad (**)$$

The assumption that φ is not constant on any proper subinterval implies that $v(t)$ is strictly increasing (why?)! So it has a continuous inverse $v^{-1} : v(I) \to I$. We claim that $\psi := \varphi \circ v^{-1}$ is a parametrization by arc length. Indeed, consider $s_1, s_2 \in v(I)$. Then it's easy to argue that

$$\operatorname*{Var}_{[s_1,s_2]} \psi = \operatorname*{Var}_{[v^{-1}(s_1), v^{-1}(s_2)]} \varphi,$$

and by definition of v this is

$$= v(v^{-1}(s_2)) - v(v^{-1}(s_1)) = s_2 - s_1$$

as needed. $\square$

Example 4.4: The graph of the Cantor function, as a curve, is AC. To be more precise, the curve parametrized by the obvious $\varphi(t) := (t, f(t))$, where f is the Cantor function, can be parametrized by arc length because φ isn't constant on any interval by virtue of the strictly increasing first component. So there exists such a parametrization ψ by arc length, and ψ is 1-Lipschitz and hence AC.

Here's another criterion which follows as a corollary from the previous one: AC curves that admit parametrizations that never "pause" for a moment are parametrizable by arc length.

Theorem 4.3

Let γ be a continuous curve with an AC parametrization $\varphi : I \to \mathbb{R}^N$ such that $\varphi' \neq 0$ almost everywhere. Then γ can be parametrized by arc length.

Proof. Easy. □

5 Completeness

5.1 Sequences

Sequences are really important! Lots of things in analysis become a lot more flexible when viewed in the context of sequences. You might remember constantly passing to sequences when justifying things for LDCT, for instance.

An example of sequences being very versatile is their ability to give an alternative characterization for closed and compact sets!

Definition 5.1 (Sequential Closure)

Let (X, τ) be a topological space. A subset $E \subseteq X$ is *sequentially closed* if for every sequence $x_n \in E$ with a limit $x \in X$, we have $x \in E$.

That is, E is literally closed under convergence.

Definition 5.2 (Sequential Compactness)

Let (X, τ) be a topological space. A subset $E \subseteq X$ is *sequentially compact* if every sequence $x_n \in E$ admits a subsequence that converges in E.

Intuitively, being sequentially closed should be the same thing as being closed, that kinda feels right. This is true in metric spaces!

Theorem 5.1

Let (X, d) be a metric space. Then $E \subseteq X$ is closed iff it is sequentially closed.

And, similarly for sequential compactness.

Theorem 5.2

Let (X, d) be a metric space. Then $E \subseteq X$ is compact iff it is sequentially compact.

The theorem for compactness will be done in lecture. Here we shall prove the theorem for closedness.

The proof is basically "remember from a year ago that sets are closed iff they contain all their accumulation points", but I'll write it out.

Proof. ($\implies$) Suppose E is closed. Take a sequence $x_n \in E$ that has a limit $x \in X$. Then x is an accumulation point of $\{x_n : n \in \mathbb{N}\}$, and is thus an accumulation point of E. Since closed sets contain their accumulation points, $x \in E$.

($\Longleftarrow$) Suppose E is sequentially closed. To show that E is closed, it suffices to prove that it contains all its accumulation points. Well, if somehow we found a point $x \in \operatorname{acc} E \setminus E$, then we can take $x_n \in \operatorname{acc} E$ with $x_n \to E$, so $x \in E$, contradiction. $\square$

We remark that it is not true that being sequential closed is not equivalent to being closed in topological spaces. A counterexample will be a part of your homework in the future. The silver lining is that you can get one direction: closed sets are always sequentially closed, even in topological spaces. The proof is not hard.

5.2 Completeness in Metric Spaces

There is an important relation between being complete and being closed.

Theorem 5.3

(X, d) be a complete metric space. Then for a subset $E \subseteq X$, we have

$$E \text{ is closed} \iff (E, d) \text{ is complete.}$$

Hence, in the context of Banach spaces, when people say "closed subspaces", they're basically talking about subspaces that are Banach.

Proof. ($\Longrightarrow$) Suppose E is closed. Let $x_n \in E$ be Cauchy in d. Then since $x_n \in X$, and X is complete, we have $x_n \to x$ for some $x \in X$. But E is sequentially closed, so $x \in E$.

($\Longleftarrow$) Suppose (E, d) is complete. Take a sequence $x_n \in E$ that converges in X. Then x_n is Cauchy in d, so $x_n \to x$ for some $x \in E$. So E is sequentially closed, and hence closed. $\square$

5.3 Relation to Uniform Continuity

The setting here is still metric spaces, but this result is so important that it deserves its own section!

There is something really neat about uniform continuity that I never got to talk about last semester because we just didn't quite have the technology to discuss it. When a function taking values in a complete metric space (like real-valued functions) is uniformly continuous, then you can extend it!

Theorem 5.4

Suppose X, Y are metric spaces, and that Y **is complete**. Let $E \subseteq X$ and let $f : E \to Y$ be uniformly continuous.
Then f can be extended continuously to all accumulations points of E (!!!).
In other words, we can find $\tilde{f} : \overline{E} \to Y$ continuous such that $\tilde{f}$ agrees with f over E.
Moreover, this extension is unique.

Before proving this, let us show a lemma.

Lemma 5.1

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be uniformly continuous. Then f sends Cauchy sequences to Cauchy sequences.

Proof. Take a Cauchy sequence $x_n \in X$. We claim that $f(x_n)$ is a Cauchy sequence in Y.

Indeed, fix $\varepsilon > 0$. Then

- since f is uniformly continuous, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, and now
- since x_n is Cauchy, there exists N so large that $d_X(x_m, x_n) < \delta$ for all $m, n \geq N$.

We claim that this works. Indeed, for all $m, n \geq N$, we have $d_X(x_m, x_n) < \delta$, and it follows that $d_Y(f(x_m), f(x_n)) < \varepsilon$ for all such m, n. Tada! □

Now we may prove the theorem.

Proof.

- Take $x_0 \in (\operatorname{acc} E) \setminus E$ (don't care about E, already know how to define over E).
- Since x_0 is an accumulation point, we can find $x_n \to x_0$.
- Since x_n converges, it is Cauchy.
- Since f is uniformly continuous, it sends Cauchy sequences to Cauchy sequences.
- Therefore, $\{f(x_n)\}_n \in Y$ is Cauchy.
- Y is complete, so we may conclude that $f(x_n)$ converges to some value in Y. **We define this value to be $\tilde{f}(x_0)$!** (Note that this is the *only* way to extend f to x_0 in a continuous way, so this proves uniqueness of the extension, provided that it exists.)

Over all other points of $x \in E$, we naturally define $\tilde{f}(x)$ to be $f(x)$. Hence we have defined a function $\tilde{f} : \overline{E} \to Y$. It remains to check that $\tilde{f}$ is continuous. We will, in fact, prove that it is uniformly continuous, to the surprise of probably nobody.

Take $\varepsilon > 0$ and find the corresponding $\delta > 0$ witnessing the uniform continuity of f over E. Take $x, y \in \overline{E}$ for which $d_X(x, y) < \delta/2$. Take $x_n, y_n \in E$ with $x_n \to x$ and $y_n \to y$ such that $f(x_n) \to f(x)$ and $f(y_n) \to f(y)$, as in the definition of f over $\overline{E}$ (if $x \in E$ then we can just take $x_n = x$ for all n, and same for y). Then

$$d_Y(\tilde{f}(x), \tilde{f}(y)) \le d_Y(\tilde{f}(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), \tilde{f}(y)).$$

By yet another triangle inequality, we have that

$$d_X(x_n, y_n) \le d_X(x_n, x) + d_X(x, y) + d_X(y, y_n) < \delta$$

for all large enough n, so that $d_Y(f(x_n), f(y_n)) < \varepsilon$ for all such n. Moreover we can get $d_Y(\tilde{f}(x), f(x_n)) < \varepsilon$ and $d_Y(f(y_n), \tilde{f}(y)) < \varepsilon$ for all n large enough. So $d_Y(\tilde{f}(x), \tilde{f}(y)) \le 3\varepsilon$, which is enough. □

5.4 Examples of Banach Spaces

5.4.1 Euclidean Space

$\mathbb{R}^N$ is possibly the most important Banach space of all! This is what lets a good number of other spaces be Banach.

5.4.2 Continuous bounded functions

Definition 5.3

Let (X, d) be a metric space. Then $C_b(X)$ is the space of all continuous and bounded functions $f : X \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$.

Theorem 5.5

$C_b(X)$ is Banach.

Proof. Apparently done in lecture. □

Corollary 5.1

$C(K)$ equipped with $\|\cdot\|_\infty$ is a Banach space for any compact K.

5.4.3 k-Continuously Differentiable Functions with Bounded Derivatives

Definition 5.4

Let $\Omega \subseteq \mathbb{R}^N$ be open. Then $C_b^k(\Omega)$ is the space of all $f : \Omega \to \mathbb{R}$ for which the derivative $\frac{\partial^\alpha f}{\partial x^\alpha}$ exists and is continuous and bounded for all multi-indices α with $|\alpha| \leq k$. The norm is given by

$$\|f\|_{C_b^k(\Omega)} := \|f\|_\infty + \sum_{1 \leq |\alpha| \leq k} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_\infty .$$

Theorem 5.6

$C_b^k(\Omega)$ is Banach.

Proof. Let's assume $N = 1$ so that this is easier to read. It's not much harder in higher dimensions.

Let $\{f_n\}_n$ be Cauchy in $\|\cdot\|_{C_b^k(\Omega)}$. Then $\{f_n\}_n$ is Cauchy in $\|\cdot\|_\infty$, and hence converges uniformly to a continuous function $f \in C_b(\Omega)$. But $\{f_n'\}_n$ is also Cauchy in $\|\cdot\|_\infty$, so $f_n' \to g \in C_b(X)$ uniformly. We're done if we can show that $f' = g$, since the argument for higher-order derivatives will follow inductively.

Here's one way. Take an $x_0 \in \Omega$. We'll show that $f'(x_0)$ exists and is $g(x_0)$. Take some $y_0 < x_0$ such that $(y_0, x_0) \subseteq \Omega$. Note that

$$\int_{y_0}^{x} g_n(t)\, dt = f_n(x)$$

for all $x \in (y_0, x_0]$. Now send $n \to +\infty$. Then $f_n(x) \to f(x)$, and since $g_n \to g$ uniformly, a domination argument shows that $\int_{y_0}^{x} g_n(t)\, dt \to \int_{y_0}^{x} g(t)\, dt$. Hence

$$\int_{y_0}^{x} g(t)\, dt = f(x).$$

This holds for all $x \in (y_0, x_0]$, so we may differentiate to obtain $g(x) = f'(x)$ for all such x, and particularly this holds at $x = x_0$ as needed. $\square$

The idea of the proof immediately implies the following corollary.

Corollary 5.2

$C^k([a, b])$ equipped with the $\|\cdot\|_{C_b^k([a,b])}$ norm is Banach for any compact interval $[a, b]$.

5.4.4 Hölder Continuous Functions

Theorem 5.7

Let $\Omega \subseteq \mathbb{R}^N$ be open. Then $C^{0,\alpha}(\Omega)$, equipped with the norm

$$\|f\|_{C^{0,\alpha}(\Omega)} := \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{\|x-y\|^\alpha} + \sup_\Omega |f|,$$

is a Banach space.

Proof. Let $\{f_n\}_n$ be Cauchy in this norm. Then it is Cauchy in $\|\cdot\|_\infty$, and since f_n is bounded (why?), it converges in $\|\cdot\|_\infty$ to some function $f \in C_b(\Omega)$. It's easy to show that $f \in C^{0,\alpha}(\Omega)$: For any $x, y \in \Omega$ we may write

$$\frac{|f(x)-f(y)|}{\|x-y\|^\alpha} = \lim_{n\to\infty} \frac{|f(x)-f(y)|}{\|x-y\|^\alpha} \leq \limsup_{n\to\infty} \|f_n\|_{C^{0,\alpha}(\Omega)} < \infty.$$

It remains to show that $f_n \to f$ in $\|\cdot\|_{C^{0,\alpha}(\Omega)}$. Fix $\varepsilon > 0$. Then we may find N_ε such that for $m, n \geq N_\varepsilon$ we have

$$\frac{|(f_m - f_n)(x) - (f_m - f_n)(y)|}{\|x-y\|^\alpha} \leq \|f_m - f_n\|_{C^{0,\alpha}(\Omega)} < \varepsilon.$$

Send $m \to +\infty$ and get

$$\frac{|(f - f_n)(x) - (f - f_n)(y)|}{\|x-y\|^\alpha} \leq \varepsilon$$

for all $n \geq N_\varepsilon$. Thus "$f_n \to f$ in the $C^{0,\alpha}(\Omega)$ seminorm" and also $f_n \to f$ in $\|\cdot\|_\infty$, so that $f_n \to f$ in $C^{0,\alpha}(\Omega)$ as desired.

$\square$

6 ODE Existence and Uniqueness

Recall that an IVP looks something like

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases},$$

for

- a function $f : I \times U \to \mathbb{R}^N$, $U \subseteq \mathbb{R}^N$,
- a "start time" $t_0 \in I$, and
- an initial data value $x_0 \in U$.

We solve for a differentiable function $u : I \to U$.

When we study ODEs, two questions arise:

1. Does a solution exist? (If not, does it exist *locally*?)
2. Is the solution unique?

Other questions we tend to ask include how the solution changes when we perturb the initial data, or how "stable" certain solutions are. But we won't study these questions here.

6.1 Existence

When do solutions exist? As it turns out, *pretty much always*, at least at a local level. All you need is a very mild condition: *continuity*.

Theorem 6.1 (Local Existence)

Let $I = [t_0, t_0 + T_0]$ be a compact interval, and let $f : I \times \overline{B_N(x_0, r)} \to \mathbb{R}^N$ be continuous. Then the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

admits a solution $x : [t_0, t_0 + \delta] \to \overline{B_N(x_0, r)}$ for some small $\delta > 0$.

Corollary 6.1 (Peano Existence Theorem)

Let $W \subseteq \mathbb{R} \times \mathbb{R}^N$ be open, let $f : W \to \mathbb{R}^N$ be continuous, and let $(t_0, x_0) \in W$. Then the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

admits a solution $x : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^N$ for some small $\delta > 0$.

So, for instance, there definitely exists a solution to the IVP

$$\begin{cases} x'(t) = \log\left(42 + \frac{\cos(x(t))^{\sin t}}{t^t + \sqrt{x(t)}}\right) \\ x(1) = 1 \end{cases}.$$

But we don't know how long this solution exists for around 1, and I definitely don't know what the solution actually is. Moreover, we thus far do not know if the solution in question is the *only* solution, i.e. we are unsure if we have uniqueness.

If you actually want to apply the local existence theorem carefully, we can take like $I = [1, 2]$ and $r = 0.5$. The function

$$f(t, z) := \log\left(42 + \frac{\cos(z)^{\sin t}}{t^t + \sqrt{z}}\right)$$

is definitely defined (and continuous) over all $(t, z) \in [1, 2] \times [0.5, 1.5]$.

How far can the solution go? Who knows, but at least we know that we can extend it "as far as it can possibly go".

Theorem 6.2 (Maximal Solutions)

Suppose $f : I \times U \to \mathbb{R}^N$ is a function, and $x'(t) = f(t, x(t))$ over some subinterval $J \subseteq I$. Then we may always extend x to a *maximal solution*. That is, we may extend x to exist over an interval J' with $J \subseteq J' \subseteq I$ such that no larger interval can support a solution satisfying $x'(t) = f(t, x(t))$.

6.2 Uniqueness

We will prove in lecture the following theorem.

Theorem 6.3 (Cauchy-Lipschitz-Picard-Lindelöf Uniqueness Theorem)

Let $f : [t_0, t_0 + T_0] \times \overline{B_N(x_0, r)} \to \mathbb{R}^N$ be continuous and *Lipschitz in space*. That is,

$$\|f(t, z_1) - f(t, z_2)\| \leq \|z_1 - z_2\|$$

for a constant $L > 0$. Then for all $T > 0$ small enough, there is a unique solution $x : [t_0, t_0 + T] \to \mathbb{R}^N$ to the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} .$$

To be specific, it is sufficient for T to be small enough such that $MT \leq r$ and $LT < 1$, where $M = \max\limits_{[t_0,t_0+T_0]\times\overline{B_N(x_0,r)}} |f|$.

Remark 1: The condition that $MT \leq r$ is highly necessary. It exists to make sure that $x(t)$ stays in $\overline{B_N(x_0, r)}$ for T amount of time. If we think of M as the "maximum possible speed of $x(t)$", then it logically follows that MT is the "maximum distance $x(t)$ can travel away from x_0", so it makes quite a bit of sense to have the constraint that $MT \leq r$.

Remark 2: We can remove the "$LT < 1$". It turns out that $MT \leq r$ is sufficient! Once we prove the theorem in lecture, consider this strengthening of the theorem as a fun exercise.

Remark 3: This is not the most general uniqueness theorem. Google the "Osgood Uniqueness Theorem" if you are interested in one generalization that I know of.

This theorem implies via an easy-ish argument that "Lipschitz-ness" will always prevent two different solutions from existing.

Corollary 6.2 (Locally Lipschitz is a Uniqueness Property)

Let $f : I \times U \to \mathbb{R}^N$ be *locally* Lipschitz in space (i.e. f is continuous and for every $(t, x) \in I \times U$ we can find $\delta > 0$ and $\varepsilon > 0$ small enough such that f is Lipschitz when restricted to $(t - \delta, t + \delta) \times (x - \varepsilon, x + \varepsilon)$). Let $(t_0, x_0) \in I \times U$. Suppose that x_1, x_2 are solutions to the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

over some time interval $J \subseteq I$ containing t_0. Then x_1 and x_2 are actually the same solution.

6.3 Global Existence and Uniqueness

The results thus far have tended to be more local in nature. It is good to have results that entail something about existence and uniqueness on the *whole* of a time interval, rather than just some of it. Here's the classic one.

Theorem 6.4 (Linear Growth Implies Global Existence)

Let $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ be continuous and locally Lipschitz in space. Let $(t_0, x_0) \in I \times \mathbb{R}^N$. Suppose that there are continuous functions $\alpha, \beta : I \to [0, \infty)$ such that

$$\|f(t, z)\| \leq \alpha(t) + \beta(t)\|z\|.$$

Then there exists a solution $u : I \to \mathbb{R}^N$ to the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}.$$

Remark: The solution must also be unique by the previous corollary.

6.4 Worked Examples

Example 6.1: Consider the IVP

$$\begin{cases} x'(t) = \arctan(x(t)) - \frac{1}{t} \\ x\left(\frac{4}{\pi}\right) = 1 \end{cases}.$$

1. Prove that the IVP admits a unique global solution on $(0, \infty)$.
2. Does the limit $\lim_{t \to 0^+} x(t)$ exist in $\overline{\mathbb{R}}$? What is it?

Solution. Let $f(t, z) = \arctan(z) - \frac{1}{t}$. Observe that f is C^1, and hence locally Lipschitz in space. Moreover, for $(t, z) \in (0, \infty) \times \mathbb{R}$, we may write

$$|f(t, z)| \leq \frac{1}{t} + |\arctan z| \leq \frac{1}{t} + |z|.$$

It follows by the global existence theorem that there exists a global and unique solution $x(t)$ over $(0, \infty)$.

Since we may bound arctan by $\pi/2$, we may note that

$$x'(t) \leq \frac{\pi}{2} - \frac{1}{t}, \qquad (*)$$

and so $x'(t) < 0$ for $0 < t < 2/\pi$. In particular $x(t)$ is monotone over $(0, 2/\pi)$ and so the limit $\lim_{x\to 0^+} x(t)$ must exist. To compute the limit, we may integrate $(*)$ over the interval $(t, 4/\pi)$ to find that

$$1 - x(t) \leq \frac{\pi}{2}\left(\frac{4}{\pi} - t\right) + \log 4 - \log t.$$

A bit of rearrangement lets us deduce that $\lim_{x\to 0^+} x(t) = +\infty$. ■

Example 6.2: Consider the IVP

$$\begin{cases} x'(t) = \frac{1}{x(t)-t^2} \\ x(0) = x_0 \end{cases},$$

where $x_0 > 0$.

1. Prove that the IVP admits a unique solution in a neighborhood of $t = 0$.
2. Prove that the solution exists over all of $[0, \infty)$.
3. Compute
$$\lim_{t\to+\infty} x(t),$$
where $x(t)$ is the unique solution.

Solution. Let $f(t, z) = \frac{1}{z-t^2}$. Note that f blows up when $z = t^2$, so there is little hope for an application of the linear growth criterion.

We'll have to use bare hands instead. Let us first establish local existence and uniqueness. Draw a small rectangle $[-a, a] \times [x_0 - \delta, x_0 + \delta]$ around $(0, x_0)$ that does not intersect the graph of t^2 (i.e. contains no points of the form (t, t^2)). This is possible because $x_0 > 0$. Note that f is well-defined over $[-\tau, \tau] \times [x_0 - a, x_0 + a]$, and moreover

$$\left|\frac{\partial f}{\partial z}(t, z)\right| = \left|\frac{-1}{(z - t^2)^2}\right| \leq \frac{1}{\min\limits_{[-\tau,\tau]\times[x_0-a,x_0+a]} |z - t^2|} < \infty,$$

so f is Lipschitz in space. By Cauchy-Lipschitz it follows that there exists a unique solution in a small interval $[-T, T]$.

In particular the solution x is unique over $[0, T]$. Now use the maximal existence theorem to extend x to $[0, \tau)$ where τ is as large as possible. Since f is locally Lipschitz in space, this extension is unique.

Assume for contradiction that $\tau < \infty$.

We claim that we may extend x to $[0, \tau]$. We need only show that the limit $\lim_{t\to\tau^-} x(t)$ exists in $\mathbb{R}$. If it does not, then since $x(t) > 0$ over $[0, \tau]$ (why?), we have that $x'(t) =$

$\frac{1}{x(t)-t^2} > 0$ over $[0, \tau]$. So x is increasing, and if the limit does not exist in $\mathbb{R}$ then the only possible reason is because $\lim_{x \to \tau^-} x(t) = +\infty$. It follows that $\lim_{x \to \tau^-} x'(t) = 0$. In particular $|x'(t)| \leq 1$ for all $t \in [\tau - \delta, \tau]$ for $\delta > 0$ small enough, and so

$$|x(t) - x(\tau - \delta)| \leq \int_{\tau-\delta}^{t} |x'(s)|\, ds \leq \int_{\tau-\delta}^{\tau} 1\, ds = \delta < \infty.$$

Sending $t \to \tau^-$, the LHS blows up, contradiction. (*Why can we apply the FTC?*)

So x may be extended to $[0, \tau]$. Now the motto is to "restart the IVP at time $t = \tau$". That is, consider the IVP

$$\begin{cases} y'(t) = \frac{1}{y(t)-t^2} \\ y(\tau) = x(\tau) \end{cases}.$$

By mimicking our arguments in the beginning (i.e. draw a small rectangle around $(\tau, x(\tau))$), there must exist a unique solution y over a small interval $[\tau, \tau + \varepsilon)$. But now

$$\tilde{x}(t) := \begin{cases} x(t), & 0 \leq t \leq \tau \\ y(t), & \tau < t < \tau + \varepsilon \end{cases}$$

is a solution to the *original* IVP that extends x, contradicting maximiality of τ.

We conclude that $\tau = +\infty$. That is, there exists a global unique solution over $[0, \infty)$.

We have $x(t) > t^2$ for all time (why?), so $\lim_{t \to +\infty} x(t) = +\infty$. ■

Example 6.3: Consider the IVP

$$\begin{cases} x'(t) = e^{x(t)^2} - e^{t^2} \\ x(0) = 0 \end{cases}.$$

1. Find the maximal interval (α, β) of existence.
2. Study the asymptotics of $x(t)$ as $t \to \beta^-$.

Solution. For any interval $0 \in I \subseteq \mathbb{R}$ we have that $f(t, z) := e^{z^2} - e^{t^2}$ is C^1 and bounded over I. Particularly we may write

$$\|f(t, z)\| \leq \|f\|_{C_b(I \times \mathbb{R})} + 0 \cdot \|z\|.$$

So by the global existence theorem we have a solution on I. But I was arbitrary, so we can argue that there exists a unique global solution on $\mathbb{R}$ (why?).

For asymptotics, we claim that $\lim_{x \to +\infty} x(t) + t = 0$. I have no proof of this that's nice so uh have fun. The Italian book Leoni gave me suggests showing that $x(t) < -\sqrt{t^2 - 1}$ or something.

■

7 More ODEs

7.1 Linear ODEs

Recall that an ODE that looks like

$$u'(t) + a(t)u(t) = b(t)$$

is called *linear*. I must emphasize that these are very solvable, and that you should keep the trick in mind! That is, the trick of using the *product rule* to make things simpler. If we multiply by $e^{\int_{t_0}^t a(s)\,ds}$ to get

$$u'(t)e^{\int_{t_0}^t a(s)\,ds} + a(t)u(t)e^{\int_{t_0}^t a(s)\,ds} = b(t)e^{\int_{t_0}^t a(s)\,ds},$$

then this is just

$$\frac{d}{dt}\left(u(t)e^{\int_{t_0}^t a(s)\,ds}\right) = b(t)e^{\int_{t_0}^t a(s)\,ds},$$

which we may solve by integration.

This is a sort of sly trick that you should always keep an eye out for. Here is an example that I had to solve for my blog a while back:

$$1 + u'(x)^2 = -2u(x)u''(x)$$

This is not a linear ODE whatsoever, but that does not mean that we can use a similar idea. If we multiply by $u(x)$, we get

$$u'(x) + u'(x)^3 = -2u(x)u'(x)u''(x).$$

If you stare really hard, this reduces to

$$\frac{d}{dx}\left[u(x) + u(x)u'(x)^2\right] = 0,$$

so we need only solve $u(x) + u(x)u'(x)^2 = c$ for some constant $c \in \mathbb{R}$, and this is separable. Speaking of which...

7.2 Separable ODEs

Consider the IVP

$$\begin{cases} \frac{dy}{dx} = xy \\ y(1) = 2. \end{cases}$$

One way people "solve" this is to "separate and integrate". People write

$$\frac{1}{y}\,dy = x\,dx$$

and then integrate to get

$$\log|y| = \frac{1}{2}x^2 + C.$$

People then conclude that

$$y = \pm e^{\frac{1}{2}x^2 + C}$$

for a constant C. This makes me a bit nervous for obvious reasons, so let's do it right.

Definition 7.1

An ODE

$$u'(t) = f(u(t))g(t)$$

is called *separable.*

Observations:

- If $f(z)$ has a root at $z = a$, then $u(t) = a$ is a solution to the ODE.
- Let $t_0, u_0 \in \mathbb{R}$, I an interval containing t_0, J an interval containing u_0. If $f : J \to \mathbb{R}$ is Lipschitz and $g : I \to \mathbb{R}$, then there exists a unique solution to the IVP

  $$\begin{cases} u'(t) = f(u(t))g(t) \\ u(t_0) = u_0 \end{cases}$$

  locally near t_0.
- If $f(z)$ has no roots, we may find an explicit solution to the ODE by dividing by $f(u(t))$ and applying the chain rule.

Example 7.1: Solve the IVP

$$\begin{cases} u'(t) = tu(t) \\ u(t_0) = u_0. \end{cases}$$

using the "separate and integrate" method, but without being bad.

Solution. Of course, one way to solve this is to argue that the solution exists globally and uniquely, and then use whatever methods you want to "guess" the correct solution, which can be verified to work by plugging it in. This approach is completely rigorous. But just for kicks, how would we show the unique existence of a solution (and calculate it) without such technology?

Let us first presume that $u_0 \neq 0$. ■

Example 7.2: Solve the IVP

$$\begin{cases} u'(t) = \frac{1+2t}{\cos(u(t))} \\ u(0) = \pi \end{cases}.$$

Solution. Start by studying $f(t,z) := \frac{1+2t}{\cos z}$. Since $u(0) = \pi$, we take the domain of f to be $\mathbb{R} \times (\pi/2, 3\pi/2)$ (note that we should be careful to dodge the zeroes of cosine). Over this interval, f is certainly C^1, so any solution we find must be unique over the time interval that is exists over. In particular, the maximal solution has to be unique. Let's find it.

If $u(t)$ is the maximal solution, then we may multiply of $\cos(u(t))$ to get

$$\cos(u(t))u'(t) = 1 + 2t.$$

We can write this as

$$\frac{d}{dt}[\sin(u(t))] = 1 + 2t.$$

Thus we may apply the FTC and $u(0) = \pi$ to deduce that

$$\sin(u(t)) - \sin(u(0)) = \sin(u(t)) = t + t^2.$$

Unfortunately we are now in dangerous territory. We cannot always take the arcsin of both sides. We now need to do some rigorous reasoning.

First, we have reasoned thus far that if $u(t)$ is the maximal solution, and its domain is I, then it must satisfy $\sin(u(t)) = t + t^2$ for all $t \in I$. But since $-1 \leq \sin(u(t)) \leq 1$, we must have $t + t^2 \in [-1, 1]$, so $t \in [-\varphi, \varphi^{-1}]$ where φ is the golden ratio. In particular, $I \subseteq [-\varphi, \varphi^{-1}]$. The point of this reasoning is to make a deduction that lets us take inverses without the dread of things possibly being undefined.

Now let's actually solve. Remember that the domain of f is $(\pi/2, 3\pi/2)$. That is, we know that $u(t)$ must live inside $(\pi/2, 3\pi/2)$ at all times. Inside this interval, the unique z for which $\sin z = a$ (where $a \in [-1.1]$) is given by $\pi - \arcsin(a)$ (draw a picture!). So $u(t)$ must satisfy $\boxed{u(t) = \pi - \arcsin(t + t^2)}$, which holds for all $t \in (-\varphi, \varphi^{-1})$, and we cannot extend this to either endpoint, so this must be the maximum interval of existence. ■

7.3 Homogenous ODEs

Definition 7.2

An ODE
$$x'(t) = f(t, x(t))$$
is *homogenous* if f is a function of $x(t)/t$. That is, we may write $f(t, x(t)) = g(x(t)/t)$ for a function f.

Observations:

- Why is it called homogenous? It is because if we have a solution to a homogenous ODE, then it is still a solution if we "zoom in". To be more precise, if $x'(t) = g(x(t)/t)$, then by taking $x_1(t) := kx(t/k)$, we have $x_1'(t) = x'(t/k) = g(x(t/k)/(t/k)) = g(x_1(t))$.
- To solve homogenous ODEs, you use the substitution $y(t) = x(t)/t$. Then the ODE becomes
$$y'(t) = \frac{x'(t)t - x(t)}{t^2} = \frac{1}{t}x'(t) - \frac{1}{t} \cdot \frac{x(t)}{t} = \frac{1}{t}\left[g(y(t)) - y(t)\right].$$
This is separable.

Example 7.3: The ODE
$$x'(t) = \frac{x(t)^2 + t^2}{x(t)t}$$
is homogenous, as it can be massaged into the form
$$x'(t) = \frac{(x(t)/t)^2 + 1}{x(t)/t}.$$
(insert motivation for "homogenous" and what it means)
To solve it, we substitute $y(t) = x(t)/t$. Then
$$y'(t) = \frac{\frac{y(t)^2+1}{y(t)} - y(t)}{t} = \frac{1}{t \cdot y(t)}.$$
Hence
$$\begin{aligned} 2y(t)y'(t) &= \frac{2}{t} \\ y(t)^2 - y(t_0)^2 &= 2\log|t/t_0| \\ x(t)^2 &= x(t_0)^2t^2/t_0^2 + 2\log|t/t_0|t^2, \end{aligned}$$
and which branch we choose will depend on t_0.

7.4 Solution by Series

Oftentimes, when we cannot find a nice closed-form solution to an ODE, we settle for a solution in the form of a series.

Theorem 7.1

Let $f_n : I \to \mathbb{R}$ be a sequence of differentiable functions. If $\sum_{n=1}^{\infty} f_n$ converges over I, and the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly, then

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Proof. Next recitation. □

Example 7.4: Solve the problem

$$\begin{cases} u''(t) + t^2 u'(t) - 4tu(t) = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$$

by series.

Solution. First we find a "formal" solution. Consider the formal power series $\sum_{n=0}^{\infty} a_n t^n$. We want such a series that "satisfies" the problem. We are given that $a_0 = 1$ and $a_1 = 0$. Hence we must solve

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} na_n t^{n+1} - \sum_{n=0}^{\infty} 4a_n t^{n+1} = 0.$$

Reindexing,

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=4}^{\infty}(n-3)a_{n-3} t^{n-2} - \sum_{n=3}^{\infty} 4a_{n-3} t^{n-2} = 0.$$

So $n(n-1)a_n + (n-7)a_{n-3} = 0$ for $n \geq 4$, $6a_3 - 4a_0 = 0$, and $2a_2 = 0$. This gives $a_2 = 0$ and $a_3 = 2/3$, which is enough to solve inductively for all coefficients.

To show that the resulting series works, we use the above theorem. ■

8 Even More ODEs

8.1 Swapping Limit and Derivative

Theorem 8.1

Let $U \subseteq \mathbb{R}^N$ be open, bounded, and convex. Let $f_n : U \to \mathbb{R}$ be a sequence of differentiable functions. If

1. $\nabla f_n \to g$ uniformly for some $g = (g_1, g_2, \cdots g_N)$, and
2. $f_n(x_0)$ converges for *some* $x_0 \in U$,

then $f_n \to f$ uniformly with $\frac{\partial f}{\partial x_i} = g_i (= \lim_{n\to\infty} \frac{\partial f_n}{\partial x_i})$.

Proof. **Step 1:** We first show that f_n converges uniformly. To do this, we need only show that it is Cauchy in $\|\cdot\|_\infty$.

Take $x \in U$. We'll draw the line between x and x_0, take a derivative along it so that the gradient pops out, and then play with it.

By the MVT (applied to the function $t \mapsto (f_m - f_n)((1-t)x_0 + tx)$), we have that $(f_m - f_n)(x) - (f_m - f_n)(x_0) = \nabla(f_m - f_n)(y) \cdot (x - x_0)$ for some y on the line segment between x_0 and x. Note that y depends on m, n, and x. We rearrange this to get

$$|(f_m - f_n)(x)| \le \|\nabla(f_m - f_n)(y)\| \cdot \|x - x_0\| + |(f_m - f_n)(x_0)|,$$

where we have applied Cauchy-Schwarz. Now, loosely speaking, note that:

- Since ∇f_n converges uniformly, we know that $\|\nabla(f_m - f_n)(y)\| \to 0$ uniformly.
- $\|x - x_0\|$ is uniformly bounded because U is bounded.
- $|(f_m - f_n)(x_0)| \to 0$ "uniformly" since there is no dependence on x.

So $|(f_m - f_n)(x)| \to 0$ uniformly as $m, n \to +\infty$. This is morally the end of this step, but let's make this more precise. Fix $\varepsilon > 0$.

- Since $\nabla f_n \to g$ converges uniformly, there exists N_1 such that $\|\nabla f_n - g\| < \varepsilon$ over U for all $n \ge N_1$. In particular, if $m, n \ge N_1$ and we take the point y from before, then

$$\|\nabla(f_m - f_n)(y)\| \le \|\nabla f_m(y) - g(y)\| + \|g(y) - \nabla f_n(y)\| < 2\varepsilon.$$

- $\|x - x_0\| \le \operatorname{diam}(U) < \infty$

- Since $\{f_n(x_0)\}_n$ converges, it is Cauchy in $\mathbb{R}$, so there is some N_2 for which $|f_m(x_0) - f_n(x_0)| < \varepsilon$ for all $m, n \geq N_2$.

Thus, if $m, n \geq \max(N_1, N_2)$, then

$$|(f_m - f_n)(x)| \leq (2\varepsilon)(\operatorname{diam}(U)) + \varepsilon,$$

which is enough.

We have hence shown that $f_n \to f$ uniformly for some f.

Step 2: Now let's show that f is differentiable, and that its gradient is $\nabla f = g$. To do this, it is sufficient to show that

$$\lim_{y \to x} \frac{|f(y) - f(x) - g(x) \cdot (y - x)|}{\|y - x\|} = 0.$$

For fixed x, y, n, let us write:

$$\begin{aligned}\frac{|f(y) - f(x) - g(x) \cdot (y - x)|}{\|y - x\|} &\leq \frac{|(f(y) - f(x)) - (f_n(y) - f_n(x))|}{\|y - x\|} \\ &+ \frac{|f_n(y) - f_n(x) - \nabla f_n(x) \cdot (y - x)|}{\|y - x\|} \\ &+ \frac{|\nabla f_n(x) \cdot (y - x) - g(x) \cdot (y - x)|}{\|y - x\|}.\end{aligned}$$

The second term will be killed when we send $y \to x$, so we just need to pick a good value of n that forces the first and third terms to be small for all $y \in U$.

For the first term, we again write

$$|(f_m - f_n)(y) - (f_m - f_n)(x)| \leq \|\nabla(f_m - f_n)(z)\| \cdot \|y - x\|$$

for some z on the line segment between x and y. Since ∇f_n converges uniformly, we know that $\|\nabla(f_m - f_n)(z)\| < \varepsilon$ for all z, provided that m, n are large enough. So

$$|(f_m - f_n)(y) - (f_m - f_n)(x)| \leq \varepsilon \|y - x\|$$

for all m, n large enough. Sending $m \to +\infty$, we end up with $|(f - f_n)(y) - (f - f_n)(x)| \leq \varepsilon \|y - x\|$ for all large enough n, which is what we needed.

For the third term, we just use Cauchy-Schwarz to bound it by $\|\nabla f_n(x) - g(x)\|$. Of course, this is bounded by ε for all large enough n.

Thus, we can pick n such that both $|(f - f_n)(y) - (f - f_n)(x)| \leq \varepsilon \|y - x\|$ and $\|\nabla f_n(x) - g(x)\| < \varepsilon$ hold. The triple triangle-inequality then reduces to

$$\frac{|f(y) - f(x) - g(x) \cdot (y - x)|}{\|y - x\|} \leq \varepsilon + \frac{|f_n(y) - f_n(x) - \nabla f_n(x) \cdot (y - x)|}{\|y - x\|} + \varepsilon,$$

which holds for all $y \in U$. Sending $y \to x$ gives

$$\limsup_{y\to x} \frac{|f(y) - f(x) - g(x)\cdot(y-x)|}{\|y-x\|} \le \limsup_{y\to x} \frac{|f_n(y) - f_n(x) - \nabla f_n(x)\cdot(y-x)|}{\|y-x\|} + 2\varepsilon = 2\varepsilon.$$

But $\varepsilon > 0$ was arbitrary, so actually the limit exists and is 0. □

Corollary 8.1 (Swapping Derivative and Limit)

Suppose $U \subseteq \mathbb{R}^N$ is open and let $f_n : U \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that:

- f_n converges pointwise.
- ∇f_n converges *locally uniformly*. (That is, for every $x \in U$ we have that ∇f_n converges uniformly over $B(x,r)$ for some $r > 0$.)

Then

$$\nabla\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} \nabla f_n(x).$$

Proof. Fix $x \in U$. Find $r > 0$ for which ∇f_n converges uniformly over $B(x,r)$, and call the limit g. $B(x,r)$ is open, bounded, and convex, and moreover $f_n(x)$ converges. Thus by the previous theorem we know that f_n converges to some f uniformly with $\nabla f = g$. That is,

$$\nabla\left(\lim_{n\to\infty} f_n(y)\right) = \nabla f(y) = g(y) = \lim_{n\to\infty} \nabla f_n(y)$$

for all $y \in B(x,r)$. In particular this holds for $y = x$. □

Corollary 8.2 (Swapping Derivative and Sum)

Suppose $U \subseteq \mathbb{R}^N$ is open and let $f_n : U \to \mathbb{R}$ be a sequence of differentiable functions. Suppose that:

- $\sum_{n=1}^\infty f_n$ converges.
- $\sum_{n=1}^\infty \nabla f_n$ converges locally uniformly.

Then

$$\nabla \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \nabla f_n(x).$$

Proof. Immediate. □

Now we're ready to have some fun. Let us consider the problem

$$\begin{cases} u''(t) = u(t) \\ u(0) = 1 \\ u'(0) = 0 \end{cases}.$$

We will solve this in several different ways.

8.2 Solution By Series

I didn't quite get to this methodology last time, so let's just get it over with here. We first find a *formal solution*. That is, we attempt to find

$$u(t) = \sum_{n=0}^{\infty} a_n t^n$$

which solves the problem, where we interpret this as a formal power series rather than a function. For this to make sense, we have to define its formal derivative

$$u'(t) := \sum_{n=1}^{\infty} n a_n t^{n-1}.$$

Then its formal second-order derivative would be

$$u''(t) := \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

The condition that $u'' = u$ implies that $a_{n-2} = n(n-1)a_n$ for all $n \geq 2$. The initial conditions $u(0) = 1$ and $u'(0) = 0$ give us the base cases $a_0 = 1$ and $a_1 = 0$. From this we conclude that $a_n = 0$ for all odd n, and $a_n = \frac{1}{n!}$ for all even n. Thus our formal solution is

$$u(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}.$$

Now let's interpret this as a *function*. Factorials kill powers, so this certainly converges, and hence this is well-defined.

We claim that in fact, the sum of the derivatives, $\sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!}$, converges locally uniformly over $\mathbb{R}$. Fixing $t_0 \in \mathbb{R}$, we observe that for all $t \in (t_0 - 42, t_0 + 42)$, we have

$$\left| \sum_{n=m}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \right| \leq \sum_{n=m}^{\infty} \frac{\max(t_0 - 42, t_0 + 42)^{2n-1}}{(2n-1)!} \xrightarrow{m \to +\infty} 0,$$

proving our claim. Thus we may swap the sum with a derivative to write

$$u'(t) = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!}.$$

This again converges pointwise, and $\sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!}$ converges locally uniformly by similar reasoning. So we may swap again to get

$$u''(t) = \sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!} = u(t).$$

Thus (after briefly checking the initial conditions *juuuusst to be sureee*) we see that this indeed satisfies the problem.

8.3 Reduction to first order

Great, we found *a* solution. But we should be at least a little sad because we haven't been able to apply those cool existence and uniqueness theorems to feel safe about what we're doing. Could there be another weird solution to the problem that isn't analytic?

It turns out that our existence and uniqueness theory can definitely be applied to higher-order ODEs! Let $v := (u, u') : I \to \mathbb{R}^2$. Then v satisfies

$$v' = \begin{bmatrix} u' \\ u'' \end{bmatrix} = \begin{bmatrix} u' \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix} = Av,$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is a first order ODE! We have reduced to solving the IVP

$$\begin{cases} v' = Av \\ v(0) = (1, 0) \end{cases}.$$

We can apply all those lovely theorems! The function $f : x \in \mathbb{R}^2 \mapsto Ax \in \mathbb{R}^2$ is certainly continuous, so a local solution clearly exists. It is also clearly C^1, so it is locally Lipschitz in space (...note that there is no dependence on time!), so solutions are unique... heck, for this particular ODE, we can even observe that

$$\|f(x)\| \leq \lambda \|x\|,$$

where λ is the **operator norm** of A! (i.e. it is $\sup_{\|x\|=1} \|Ax\|$, which is finite by compactness.) So the unique solution exists globally on $\mathbb{R}$. Cool! Well, we kinda already knew that from the last section, but it's nice that we can figure this out without doing any computations.

In general, we can always reduce higher-order ODEs to first-order ODEs in this way. The "sacrifice" is that the dimension of the range increases. A bit more explicitly, if we are faced with a problem of the form

$$\begin{cases} u^{(k)} = f(t, u, u', \cdots, u^{(k-1)}) \\ (u(t_0), u'(t_0), \cdots, u^{(k-1)}(t_0)) = (u_0, u_1, \cdots, u_{k-1}) \end{cases},$$

then by letting $v = (u, u', \cdots, u^{(k-1)})$, we have that

$$v' = (u', u'', \cdots, u^{(k)}) = g(t, v),$$

where

$$g(t, z_0, z_1, \cdots, z_{k-1}) := (z_1, z_2, \cdots, z_{k-1}, f(t, z_0, z_1, \cdots, z_{k-1})).$$

So we may apply our first-order ODE theory to the problem

$$\begin{cases} v' = g(t, v) \\ v(t_0) = v_0 \end{cases}$$

where $v_0 := (u_0, u_1, \cdots, u_{k-1})$.

8.4 Wacky Matrix Exponentials

You might be wondering if we can actually solve the "reduction" $v' = Av$ as-is. Well, since the solution to $\begin{cases} f' = af \\ f(0) = f_0 \end{cases}$ is $f(t) = e^{ta} f_0$, surely the solution to $\begin{cases} v' = Av \\ v(0) = (1, 0) =: v_0 \end{cases}$ is $v(t) = e^{tA} v_0$... right?

It turns out that this is completely legit. If you haven't seen matrix exponentials before, we define

$$e^M := \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

for square matrices M, which converges (and hence is well-defined) because factorials trump exponentials. In this case, I'm making the bold claim that

$$v(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n v_0.$$

To see that this is legit, we can show as before that $\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n$ converges locally uniformly (in each of its "four components"... view the entries of A^n as dumb constants!), and so we can pass a derivative through to get

$$v'(t) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n v_0 = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{n+1} v_0 = Av(t).$$

Nice. Now we recover the solution u to the original ODE by taking the first component of v. Whatever that is. (In this case the powers of A^n are easy to compute, so recovering the series using this method is quite doable.)

8.5 $u'' = f(u)$

Let $f : I \to \mathbb{R}$ be continuous. Then ODEs of the form

$$u'' = f(u)$$

may be solved as follows. First multiply each side by u' to obtain

$$u''u' = f(u)u'.$$

This magically simplifies as

$$\frac{d}{dt}\frac{1}{2}|u'|^2 = \frac{d}{dt}F(u),$$

where $F : I \to \mathbb{R}$ is a primitive for f. We may then deduce that

$$\frac{1}{2}|u'(t)|^2 = F(u) + C$$

for a constant C. This is separable!

Let's use this methodology to try and tackle our problem. From $u'' = u$, we multiply by u' to get $u''u' = u'u$, and so

$$\frac{d}{dt}|u'|^2 = \frac{d}{dt}|u|^2.$$

Integrating over $(0, t)$ gives

$$u'(t)^2 - 0 = u(t)^2 - 1.$$

So

$$\frac{u'(t)}{\sqrt{u(t)^2 - 1}} = 1,$$

and hence by integrating again we obtain

$$t = \int_0^t \frac{u'(s)}{\sqrt{u(s)^2 - 1}}\,ds = \int_{u(0)}^{u(t)} \frac{1}{\sqrt{x^2 - 1}}\,dx.$$

This integral indeed has a closed-form, but taking its inverse might be painful.

8.6 How normal people solve $u'' = u$

Generally speaking, for ODEs of the form $u^{(k)} + \sum_{n=0}^{k-1} a_n u^{(n-1)}$ for constants $\{a_n\}$, you form the characteristic polynomial $p(x) = x^k + \sum_{n=0}^{k-1} a_n u^{(n)}$ and look at its roots $\lambda_0, \cdots, \lambda_{k-1}$. If they are all distinct, then solutions are given by

$$u(t) = \sum_{j=0}^{k-1} c_j e^{\lambda_j t}$$

for constants $\{c_j\}$. Here I use the index j because the exponentials may be complex. If roots are repeated then the situation is dicier. But fortunately you need not memorize this theory to survive on an island, since it can be readily derived.

One particularly nice way is to view differentiation as a linear operator D. For instance, for $u'' = u$, we may write this $(D^2 - I)u = 0$, where I is the identity, and this may be "factored" as $(D + I)(D - I)u = 0$. Letting $v = (D - I)u = u' - u$, we have that $0 = (D + I)v = v' + v$. So we may solve for v and then solve for u. This is not handwavy - it is perfectly rigorous! You can convince yourself that the *decoupled* problem

$$\begin{cases} v = u' - u \\ v' + v = 0 \\ u(0) = 1 \\ u'(0) = 0 \end{cases}$$

is equivalent to the original problem, and that the lens of using D and "factoring" is merely an algebraic trick that lets us see this more easily.

Anyway, the solution to $v' + v = 0$ is clearly $v(t) = c_1 e^{-t}$, where we can compute $c_1 = -1$, and so now we need only solve $u' - u = -e^{-t}$. This is also easy, and the solution is given by $u(t) = c_2 e^t - \frac{e^{-t}}{2}$, where we find that $c_2 = \frac{1}{2}$. Thus $u(t) = \cosh t$.

If you haven't seen it before, try playing with an ODE such as $u'' - 2u' + u = 0$ or $u'' + u = 0$ using this methodology and seeing what happens.

8.7 Bernoulli Differential Equations

A *Bernoulli* ODE is of the form

$$u'(t) + a(t)u(t) = b(t)u(t)^\alpha,$$

where $\alpha \notin \{0, 1\}$.

To solve ODEs of this form, we make the substitution $v(t) = u(t)^{1-\alpha}$. Then

$$v'(t) = (1 - \alpha)u(t)^{-\alpha}u'(t) = (1 - \alpha)\left[b(t) - a(t)u(t)^{1-\alpha}\right] = (1 - \alpha)\left[b(t) - a(t)v(t)\right].$$

So $v'(t) + (1 - \alpha)a(t)v(t) = (1 - \alpha)b(t)$

Example 8.1: If $u(t)$ is the percentage of people with covid, and if we make the reasonable assumption that $u'(t)$ is proportional to $u(t)(1-u(t))$, then the percentage of people with covid is given by

$$u'(t) = ku(t)(1 - u(t))$$

where k determines the rate of propagation. Of course, this is separable, but we can also view this as a Bernoulli ODE. By making the substitution $v(t) = 1/u(t)$, we see that

$$v'(t) = \frac{-u'(t)}{u(t)^2} = k\left(1 - \frac{1}{u(t)}\right) = k - kv(t).$$

Now this is easy to solve.

$$\begin{aligned}
v'(t) + kv(t) &= k \\
v'(t)e^{kt} + kv(t)e^{kt} &= ke^{kt} \\
\frac{d}{dt}\left[v(t)e^{kt}\right] &= ke^{kt} \\
v(t)e^{kt} - u_0^{-1} &= e^{kt} - 1 \\
v(t) &= 1 - e^{-kt} + \frac{e^{-kt}}{u_0} \\
u(t) &= \frac{e^{kt}u_0}{u_0(e^{kt} - 1) + 1}
\end{aligned}$$

9 Compact Embeddings and Separability

9.1 Ascoli-Arzela and Precompactness

Recall Ascoli-Arzela.

Theorem 9.1 (Ascoli-Arzela)

Let (X, d) be separable and let $\mathcal{F} \subseteq C_b(X)$ be (1) bounded when evaluated at every $x \in X$, and (2) equicontinuous. Then for every $f_n \in \mathcal{F}$ there exists a subsequence $f_{n_k} \in \mathcal{F}$ such that $f_{n_k} \to f \in C_b(X)$ uniformly.

A bunch of people often state Ascoli-Arzela in the lens of *precompactness*.

Definition 9.1 (Precompact)

Let $(X, \|\cdot\|_X)$ be a normed space. We say that a subset $E \subseteq X$ is *precompact* (...or *relatively compact*) if $\overline{E}$ is compact.

Note that $E \subseteq X$ is precompact iff for every $x_n \in E$ there is a subsequence x_{n_k} which converges in X.

Proof. ($\implies$) $x_n \in \overline{E}$ and $\overline{E}$ is sequentially compact, so there is a subsequence x_{n_k} which has a limit in $\overline{E} \subseteq X$.

($\impliedby$) If $x_n \in \overline{E}$, then for every x_n there is some $y_{n,m} \in B(x_n, 1/m) \cap E$. By hypothesis, there is a subsequence y_{n_k,n_k} of $\{y_{n,n}\}_n$ for which $y_{n_k,n_k} \to x \in X$. Evidently $x \in \overline{E}$. It remains to prove that $x_{n_k} \to x$. But

$$\|x_{n_k} - x\| \leq \|x_{n_k} - y_{n_k,n_k}\| + \|y_{n_k,n_k} - x\| \leq \frac{1}{n_k} + \|y_{n_k,n_k} - x\| \xrightarrow{k \to +\infty} 0.$$

$\square$

Indeed, this equivalence shows that Ascoli-Arzela is a statement about precompactness: For separable X, bounded subsets of $C_b(X)$ that are equicontinuous are precompact.

9.2 Embeddings

Definition 9.2 (Continuous Embedding)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces with $X \subseteq Y$. Then:

- The *inclusion map* from X into Y is simply the map $i : X \to Y$ that sends elements of X to themselves, i.e. $i : x \mapsto x$.
- We say that X is *continuously embedded* into Y if the inclusion map is continuous, and we write $X \hookrightarrow Y$.

Notes:

- $X \hookrightarrow Y$ iff there exists a constant C such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$.
- What this means is that the inclusion *preserves convergence*. If a sequence in X converges with respect to $\|\cdot\|_X$, then it must converge with respect to $\|\cdot\|_Y$.
- "By going from X to Y, it becomes *easier* to converge."

Definition 9.3 (Compact Embedding)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces with $X \subseteq Y$. We say that X is *compactly embedded* into Y if $X \hookrightarrow Y$ and every $\|\cdot\|_X$-bounded sequence in X has a subsequence $\|\cdot\|_Y$-converging in Y. We say "$X \hookrightarrow Y$ is compact" or write $X \subset\subset Y$.

Note that if $X \hookrightarrow Y$, then $X \hookrightarrow Y$ is compact iff any bounded subset of $(X, \|\cdot\|_X)$ is precompact in $(Y, \|\cdot\|_Y)$.

A bunch of the spaces that we have studied / will study are related by embeddings.

For this first example, recall that $C^{0,\alpha}(\Omega)$ is endowed with the norm

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_\infty + |u|_{C^{0,\alpha}(\Omega)},$$

where

$$|u|_{C^{0,\alpha}(\Omega)} := \sup_{x \ne y} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha}.$$

Theorem 9.2

Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, and let $0 < \alpha < \beta \le 1$. Then the inclusion $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is compact.

Proof. We first show that $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$, i.e. the inclusion map from $C^{0,\beta}(\Omega)$ to $C^{0,\alpha}(\Omega)$ is continuous.

Take $u \in C^{0,\beta}(\Omega)$. Then for any $x, y \in \Omega$ we may write

$$\|x-y\|^\beta = \|x-y\|^\alpha \cdot \|x-y\|^{\beta-\alpha} \le C\|x-y\|^\alpha,$$

where we may take $C = (\operatorname{diam}\Omega)^{\beta-\alpha}$. Now

$$\frac{|u(x)-u(y)|}{\|x-y\|^\alpha} \le C \cdot \frac{|u(x)-u(y)|}{\|x-y\|^\beta} \le C|u|_{C^{0,\beta}(\Omega)},$$

and taking the sup gives $|u|_{C^{0,\alpha}(\Omega)} \le C|u|_{C^{0,\beta}(\Omega)}$. Thus

$$\|u\|_{C^{0,\alpha}(\Omega)} = \|u\|_\infty + |u|_{C^{0,\alpha}(\Omega)} \le \max(1,C)\|u\|_\infty + \max(1,C)|u|_{C^{0,\beta}(\Omega)} \le \max(1,C)\|u\|_{C^{0,\beta}(\Omega)},$$

proving that $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$.

Now we may show that $C^{0,\beta}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ is compact. Take any sequence $u_n \in C^{0,\beta}(\Omega)$ bounded in $\|\cdot\|_{C^{0,\beta}(\Omega)}$. Then $\{u_n\}_n$ is equicontinuous and uniformly bounded, so by Ascoli-Arzela there exists a subsequence u_{n_k} such that $u_{n_k} \to u \in C(\Omega)$ uniformly. In fact, we may note for later that $u \in C^{0,\beta}(\Omega)$. This is because $|u_{n_k}(x) - u_{n_k}(y)| \le M\|x-y\|^\beta$ for all k, where M is an upper bound on $\|u_{n_k}\|_{C^{0,\beta}(\Omega)}$, and we may send $k \to \infty$ and use simple pointwise convergence of $u_{n_k} \to u$.

We know that $\|u_{n_k} - u\|_\infty \to 0$, so it remains to show that $|u_{n_k} - u|_{C^{0,\alpha}} \to 0$. For $x, y \in \Omega$, we may write

$$\begin{aligned}
&\frac{|(u_{n_k}-u)(x) - (u_{n_k}-u)(y)|}{\|x-y\|^\alpha} \\
&= \left(\frac{|(u_{n_k}-u)(x) - (u_{n_k}-u)(y)|}{\|x-y\|^\beta}\right)^{\alpha/\beta} |(u_{n_k}-u)(x) - (u_{n_k}-u)(y)|^{1-\alpha/\beta} \\
&\le |u_{n_k} - u|_{C^{0,\beta}(\Omega)}^{\alpha/\beta} \left(\|u_{n_k}-u\|_\infty + \|u_{n_k}-u\|_\infty\right)^{1-\alpha/\beta} \\
&\le \left(|u_{n_k}|_{C^{0,\beta}(\Omega)} + |u|_{C^{0,\beta}(\Omega)}\right)^{\alpha/\beta} \left(2\|u_{n_k}-u\|_\infty\right)^{1-\alpha/\beta}.
\end{aligned}$$

Taking the sup gives

$$|u_{n_k} - u|_{C^{0,\alpha}} \le \left(|u_{n_k}|_{C^{0,\beta}(\Omega)} + |u|_{C^{0,\beta}(\Omega)}\right)^{\alpha/\beta} \left(2\|u_{n_k}-u\|_\infty\right)^{1-\alpha/\beta}.$$

Since $\|u_{n_k} - u\|_\infty \to 0$, $\{|u_{n_k}|_{C^{0,\beta}(\Omega)}\}_k$ is bounded, and $|u|_{C^{0,\beta}(\Omega)} < \infty$ (since we proved $u \in C^{0,\beta}(\Omega)$), we may conclude. □

As a corollary, we have the continuous inclusions

$$C^{0,1}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \hookrightarrow C(\Omega),$$

all of which are compact.

Why do we care about such compactness results? One reason is that the ability to extract convergent subsequence is crucial for the existence of minimizers.

Example 9.1: Let us consider the problem

$$\min \left\{ |u|_{C^{0,1}([0,2023])} : u \in C^{0,1}([0,2023]), u(0) = 0, u(2023) = 1 \right\}.$$

Does this minimum exist? Note that the set

$$E := \{u \in C^{0,1}([0,2023]), u(0) = 0, u(2023) = 1\}$$

is nonempty (take $u(x) = x/2023$) and bounded from below by 0, so E has an infimum $I := \inf E \geq 0$. Take a sequence $u_n \in E$ for which $|u_n|_{C^{0,1}([0,2023])} \to I$. Since convergent sequences are bounded, we have that $|u_n|_{C^{0,1}([0,2023])}$ is bounded from above by a constant $M > 0$.
But now it follows that $\{u_n\}_n$ is bounded in $\|\cdot\|_{C^{0,1}([0,2023])}$ (...why is it bounded in $\|\cdot\|_\infty$?), and since $C^{0,1}([0,2023]) \hookrightarrow C([0,2023])$ is compact, it follows that there exists a subsequence u_{n_k} converging uniformly to some u. Now since

$$|u_{n_k}(x) - u_{n_k}(y)| \leq |u_{n_k}|_{C^{0,1}[0,2023]} \cdot \|x - y\|,$$

we may send $k \to +\infty$ to get $|u(x) - u(y)| \leq I\|x - y\|$. Hence $u \in C^{0,1}([0,2023])$ with $|u|_{C^{0,1}([0,2023])} \leq I$. But $u(0) = 0$ and $u(2023) = 1$, so $u \in E$ and hence $|u|_{C^{0,1}([0,2023])} \geq I$. We conclude that $|u|_{C^{0,1}([0,2023])} = I$. That is, the minimum exists!

9.3 Separability

Theorem 9.3

Let (X, d) be separable. Then (E, d) is separable for any $E \subseteq X$.

Proof. Let $\{x_n\}_n \in X$ be a countable dense set. Then for each m, $\bigcup_{n=1}^\infty B(x_n, 1/m)$ covers E. For each n, m for which $B(x_n, 1/m)$ intersects E, pick a point $y_{n,m} \in B(x_n, 1/m)$, and let I be the collection of indices (n, m) for which we have chosen a $y_{n,m}$. Then $\{y_{n,m}\}_{(n,m)\in I}$ is countable. Moreover, for any $y \in E$ and any $\varepsilon > 0$, we may pick $1/m < \varepsilon/2$ and find an $x_n \in B(y, 1/m)$. But then $y \in B(x_n, 1/m)$, hence $B(x_n, 1/m)$ intersects E, so there is $y_{n,m} \in B(x_n, 1/m) \cap E$. Now $d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) \leq 2/m < \varepsilon$. $\square$

What are examples of separable spaces?

- $\mathbb{R}^N$ and any finite-dimensional normed space
- Any compact metric space
- $L^p(E)$ for $1 \leq p < \infty$
- $C(K)$ for $K \subseteq \mathbb{R}^N$ compact

What are examples of NOT separable spaces?

- $L^\infty(E)$
- $C_b(\mathbb{R}^N)$
- $BPV(I)$

To see that $C_b(\mathbb{R}^N)$ is not separable, suppose otherwise, and let $\{g_n\}_n \in C_b(\mathbb{R}^N)$ be dense. Take a bump $\varphi \in C_c([-1/2, 1/2]^N)$ for which $\varphi(0) = 1$ and $\|\varphi\|_\infty = 1$. Then consider

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi(x - ne_1).$$

For each n, we may choose $a_n \in \{-1, 1\}$ for which $|g_n(n) - a_n| \geq 1$. Then we must have $\|f - g_n\|_\infty \geq 1$ for all n.

10 Lebesgue Spaces

10.1 A Generalization of Dominated Convergence

There is a more powerful version of the LDCT, called *Vitali's Theorem*. To set it up, we need two useful definitions. We begin with equi-integrability.

Definition 10.1 (Equi-integrability)

A family $\mathcal{F} \subseteq L^1(X, \mathfrak{M}, \mu)$ is *equi-integrable* if the following two conditions hold:

- ("Equi-tightness") For all $\varepsilon > 0$ we can find a "big" set $E \in \mathfrak{M}$ with $\mu(E) < \infty$ such that
$$\int_{X \setminus E} |f| \, d\mu < \varepsilon$$
for all $f \in \mathcal{F}$.
- ("Equi-AC") For all $\varepsilon > 0$ we can find $\delta > 0$ such that
$$\int_F |f| \, d\mu < \varepsilon$$
for all $F \in \mathfrak{M}$ with $\mu(F) < \delta$, and for all $f \in \mathcal{F}$.

Notes:

- When $(X, \mathfrak{M}, \mu)$ is a finite measure space, we may throw out the equi-tightness condition.
- Equi-tightness may be thought of as the condition "the family $\mathcal{F}$ doesn't explode horizontally".
- Equi-AC may be thought of as the condition "the family $\mathcal{F}$ doesn't explode vertically".

To get our feet wet, let's prove something lame.

Lemma 10.1

Let $f \in L^1(X, \mathfrak{M}, \mu)$ (i.e. f is integrable). Then $\{f\}$ is equi-integrable.

Proof. You know already that the AC condition holds for a single f. So we need only show that we have tightness. Let us assume first that $f \geq 0$. We may take a sequence of simple functions $\varphi_n \nearrow f$, and pick n large enough so that $\int_X f - \varphi_n \, d\mu < \varepsilon$. φ_n, by virtue of being a simple function, may be expressed as

$$\varphi_n = \sum_{k=1}^{m} c_k \cdot 1_{E_k}$$

for $c_k > 0$ and $E_k \in \mathfrak{M}$. Since f is integrable, so is φ_n, and thus $\mu(E_k) < \infty$ for each k. Take $F = \bigcup_{k=1}^{m} E_k$. We claim this works. Indeed,

$$\int_{X \setminus F} f \, d\mu = \int_{X \setminus F} f - \varphi_n \, d\mu + \int_{X \setminus F} \varphi_n \, d\mu < \varepsilon + 0.$$

$\square$

This result, although a bit boring, has some corollaries.

Corollary 10.1

Any finite subset $\{f_1, f_2, \cdots, f_n\} \in L^1(X, \mathfrak{M}, \mu)$ is equi-integrable.

Corollary 10.2

Let $\mathcal{F} \subseteq L^1(X, \mathfrak{M}, \mu)$. Suppose there is some $g \in L^1(X, \mathfrak{M}, \mu)$ for which that $|f| \leq |g|$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is equi-integrable.

The next definition we need is a notion of function convergence in measure spaces, called *convergence in measure*.

Definition 10.2 (Convergence in Measure)

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Let $\{f_n\}_n, f : X \to \mathbb{R}$ be $\mathfrak{M}$-measurable. We say f_n *converges to* f *in measure*, and we write $f_n \xrightarrow{\mu} f$, if for all $\varepsilon > 0$ we have that

$$\lim_{n \to \infty} \mu(\{|f_n - f| \geq \varepsilon\}) = 0.$$

Notes:

- This is the analogue of probability's *convergence in probability*.
- In a finite measure space, $f_n \to f$ a.e. implies $f_n \xrightarrow{\mu} f$. But **this is not true in general!**
- $f_n \to f$ in $L^p(X)$ implies $f_n \xrightarrow{\mu} f$.
- For more relationships, see https://www.johndcook.com/blog/modes_of_convergence/ which does a great job of summarizing everything that's true. (Don't worry about the "AU".)

We are now ready to state Vitali's Theorem.

Theorem 10.1 (Vitali Convergence Theorem)

Let $\{f_n\}_n, f \in L^p(X, \mathfrak{M}, \mu)$. Then

- $\{|f_n|^p\}_n$ is equi-integrable and
- $f_n \xrightarrow{\mu} f$

if and only if $f_n \to f$ in $L^p(X, \mathfrak{M}, \mu)$.

Proof. Homework. $\square$

Let's show that Vitali is stronger than the LDCT by proving that if $f_n \to f$ a.e. and the f_n are dominated by some integrable g, then $\{f_n\}_n$ is equi-integrable and $f_n \xrightarrow{\mu} f$.

Since $|f_n| \leq g$ with g integrable, Corollary 10.2 shows that the family $\{|f_n|\}_n$ is equi-integrable. Moreover, since $f_n \to f$ a.e., we have by LDCT (lol) that $f_n \to f$ in L^1. This implies convergence in measure.

It follows that Vitali is applicable in more situations than the LDCT. If you are trying to apply an LDCT-like argument but it's not quite working, consider using Vitali.

10.2 Other Useful Tools

From Remark 144 in the lecture notes:

Theorem 10.2 (Lp implies a.e. up to a subsequence)

If $f_n \to f$ in $L^p(X)$, then there is a subsequence f_{n_k} for which $f_{n_k} \to f$ a.e.

From Theorem 151 in the lecture notes:

Theorem 10.3 ("Continuity of the Integral")

Let $f \in L^p(\mathbb{R}^N)$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx < \varepsilon$$

for all $\|h\| < \delta$.

10.3 Some Problems

Example 10.1 (CMU Measure and Integration Basic Exam): Suppose $I = [0,1]$, $f_n : I \to \mathbb{R}$ is Lebesgue measurable for all $n \in \mathbb{N}$, and

$$\int_I |f_n|^2 \, dx \le 5 \qquad \forall n \in \mathbb{N}.$$

Suppose moreover that $f_n(x) \to 0$ as $n \to \infty$ for every $x \in I$.

(a) Does it necessarily follow that $\lim_{n\to\infty} \int_I |f_n|^2 \, dx = 0$?

(b) Does it necessarily follow that $\lim_{n\to\infty} \int_I |f_n| \, dx = 0$?

Solution. (a) is false. For example, we can take $f_n = \sqrt{n} \cdot 1_{(0,1/n)}$.

(b) is true. Since $[0,1]$ has finite measure, the convergence $f_n \to 0$ a.e. implies that $f_n \to 0$ in measure. We claim that $\{|f_n|\}_n$ is equi-integrable. Again, since $[0,1]$ has finite measure, it is sufficient to prove that it is "equi-AC".

Fix $\varepsilon > 0$. Then if $\delta = \varepsilon^2/5$, then for any F with $\mathcal{L}^1(F) < \delta$ we have that

$$\int_F |f_n| \, dx \le \left(\int_F |f_n|^2 \, dx\right)^{1/2} \left(\int_F 1^2 \, dx\right)^{1/2} \le \sqrt{5} \cdot \sqrt{\mathcal{L}^1(F)} < \sqrt{5\delta} = \varepsilon.$$

Now by Vitali we have that $f_n \to 0$ in $L^1([0,1])$, as needed. ∎

Example 10.2 (U of Washington Analysis Qualification Exam): Let $1 < p < \infty$. Let $f_n \in L^p([0,1])$ be a sequence bounded in $L^p([0,1])$. Suppose that $f \in L^1([0,1])$ and $f_n \to f$ in $L^1([0,1])$.

(a) Show that $f \in L^p([0,1])$.

(b) True or false: We must have $f_n \to f$ in $L^p([0,1])$.

Solution. Let $M = \sup_n \int_0^1 |f_n|^p \, dx$. Since $f_n \to f$ in $L^1([0,1])$, we may extract a subsequence f_{n_k} which converges to f a.e., so that

$$\int_0^1 |f|^p \, dx \le \int_0^1 \liminf_{k\to\infty} |f_{n_k}|^p \, dx \le \liminf_{k\to\infty} \int_0^1 |f_{n_k}|^p \, dx \le M.$$

It is not true in general that $f_n \to f$ in $L^p([0,1])$. Take $f = 0$ and $f_n = n^{1/p} \cdot 1_{(0,1/n)}$. ∎

Example 10.3 (CMU Measure and Integration Basic Exam):

(a) Show that there are measure spaces for which $\bigcap_{1\le p<\infty} L^p \neq L^\infty$.

(b) Fix $f \in \bigcap_{1\le p<\infty} L^p$. Prove that the map $p \mapsto p\log\|f\|_{L^p}$ is convex on $[1,\infty)$.

Solution.
(a) Consider $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with $\mu(\{n\}) := \frac{1}{n!}$. Consider $f(n) = n$. Then for every $1 \le p < \infty$ we have

$$\int_{\mathbb{N}} |f|^p \, d\mu = \sum_{n=1}^\infty n^p \mu(\{n\}) = \sum_{n=1}^\infty \frac{n^p}{n!} < \infty$$

so $f \in L^p(\mathbb{N}, 2^{\mathbb{N}}, \mu)$. But we do not have $f \in L^\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu)$. This is because for every $M > 0$, we have that $\{|f| > M\}$ has positive μ-measure, so $\|f\|_{L^\infty(\mathbb{N},2^{\mathbb{N}},\mu)} = +\infty$.

(b) Let $p, q \in [1,\infty)$ and let $\theta \in (0,1)$. Then we would like to show that

$$\log \int_X |f|^{(1-\theta)p+\theta q} \, d\mu \overset{?}{\le} (1-\theta)\log \int_X |f|^p \, d\mu + \theta \log \int_X |f|^q \, d\mu,$$

or

$$\int_X |f|^{(1-\theta)p+\theta q} \, d\mu \overset{?}{\le} \left(\int_X |f|^p \, d\mu\right)^{1-\theta} \left(\int_X |f|^q \, d\mu\right)^{\theta}.$$

Fortunately this is immediate by Hölder with exponents $\frac{1}{1-\theta}$ and $\frac{1}{\theta}$. ■

Example 10.4 (UCLA Analysis Qualification Exam): Let $1 < p < \infty$. Let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$ where q is the Hölder conjugate of p. Prove that the convolution

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y) \, dy$$

is continuous, and that moreover $\lim_{\|x\|\to\infty}(f*g)(x) = 0$.

Solution. Fix $x_1, x_2 \in \mathbb{R}^N$. Then

$$|(f*g)(x_1) - (f*g)(x_2)| \le \int_{\mathbb{R}^N} |f(x_1 - y) - f(x_2 - y)| \cdot |g(y)| \, dy$$

$$\le \left(\int_{\mathbb{R}^N} |f(x_1-y) - f(x_2-y)|^p \, dy\right)^{1/p} \|g\|_{L^q(\mathbb{R}^N)}.$$

Fix $\varepsilon > 0$. Get a corresponding $\delta > 0$ from continuity of the integral. Then if $\|x_1 - x_2\| < \delta$, we have that

$$|(f*g)(x_1) - (f*g)(x_2)| \le \varepsilon^{1/p}\|g\|_{L^q(\mathbb{R}^N)},$$

which is enough to deduce continuity of $f * g$. (The argument shows that it is, in fact, uniformly continuous!)

Now we show that $(f * g)$ vanishes at infinity. Fix $\varepsilon > 0$. Find R_f and R_g such that $\int_{\mathbb{R}^N \setminus B(0,R_f)} |f|^p \, dx < \varepsilon$ and $\int_{\mathbb{R}^N \setminus B(0,R_g)} |g|^p \, dx < \varepsilon$. Take $R = \max(R_1, R_2)$. Now, for all $x \in \mathbb{R}^N$ with $\|x\| \geq 2R$, we have that

$$\begin{aligned}
|(f * g)(x)| &\leq \int_{\mathbb{R}^N} |f(x-y)g(y)| \, dy \\
&\leq \int_{\mathbb{R}^N \setminus B(0,R)} |f(x-y)g(y)| \, dy + \int_{\mathbb{R}^N \setminus B(x,R)} |f(x-y)g(y)| \, dy \\
&\leq \left(\int_{\mathbb{R}^N \setminus B(0,R)} |g|^q \, dy \right)^{1/q} \|f\|_{L^p(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N \setminus B(0,R)} |f|^p \, dy \right)^{1/p} \|g\|_{L^q(\mathbb{R}^N)} \\
&\leq \varepsilon^{1/q} \|f\|_{L^p(\mathbb{R}^N)} + \varepsilon^{1/p} \|g\|_{L^q(\mathbb{R}^N)},
\end{aligned}$$

which is enough to say that $\lim_{\|x\| \to \infty} (f * g)(x) = 0$. ■

Example 10.5: Show that if $(X, \mathfrak{M}, \mu)$ is a finite measure space and $f : X \to \mathbb{R}$ is measurable, then

$$\lim_{p \to \infty} \|f\|_{L^p(X)} = \|f\|_{L^\infty(X)}.$$

Solution.

Let $M = \|f\|_{L^\infty(X)}$. On one hand, since $f \leq M$ a.e., we have that

$$\left(\int_X |f|^p \, d\mu \right)^{1/p} \leq \left(\int_X M^p \mu \right)^{1/p} = M \mu(X)^{1/p},$$

so sending $p \to +\infty$ gives

$$\limsup_{p \to \infty} \|f\|_{L^p(X)} \leq M.$$

On the other hand, if we take a small $\varepsilon > 0$, then $E := \{|f| > M - \varepsilon\}$ has positive measure, so we may write

$$\left(\int_X |f|^p \, d\mu \right)^{1/p} \geq \left(\int_E |f|^p \, d\mu \right)^{1/p} \geq \left(\int_E (M - \varepsilon)^p \, d\mu \right)^{1/p} = (M - \varepsilon) \mu(E)^{1/p},$$

so sending $p \to +\infty$ gives

$$\liminf_{p \to \infty} \|f\|_{L^p(X)} \geq M - \varepsilon.$$

Now send $\varepsilon \to 0^+$. ■

Remark: Supposedly this is still true if instead of X being a finite measure space, you knew only that $f \in L^p(X)$ for some $1 \leq p < \infty$.

11 Helly's Selection Theorem and Sobolev Spaces

11.1 Helly's Selection Theorem

A major theme in this cousre has been *compactness*: The art of taking a sequence that is bounded in one sense and extracting a subsequence that converges in another sense. *Helly's Selection Theorem* is a prime example of a *compactness result*.

Theorem 11.1 (Helly's Selection Theorem)

Let I be an interval and $f_n : I \to \mathbb{R}$. Suppose that

- $\{\operatorname{Var}_I f_n\}_n$ is bounded, and
- there is some $x_0 \in I$ for which $\{f_n(x_0)\}_n$ is bounded.

Then there exists a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise, for some $f : I \to \mathbb{R}$.

Proof. A bunch of proofs rely on first making a nice assumption on what we have to work with. This is one of them. Remember that any $g \in BPV(I)$ can be written as the difference between two monotone functions? What this means is that it essentially suffices to assume that each f_n is monotone.

Step 1. We first reduce to the case in which each f_n is increasing. Indeed, suppose that we have proven the theorem under the assumption that the sequence of functions is increasing. If f_n is not necessarily increasing, we may write it as a difference $f_n := v_n - (v_n - f_n)$, where v_n is the indefinite pointwise variation starting at some fixed point (say, z_0) in I. That is,

$$v_n(x) := \begin{cases} \operatorname{Var}_{[z_0,x]} f_n, & x \geq z_0 \\ -\operatorname{Var}_{[x,z_0]} f_n, & x < z_0 \end{cases}$$

We know classically that v_n and $v_n - f_n$ are both increasing. So now we just need to justify applying the theorem for increasing sequences of functions to each of them. Indeed:

- $\operatorname{Var}_I v_n = \sup_I v_n - \inf_I v_n = \operatorname{Var}_I f_n$, which is bounded, and you can argue that $|v_n(x_0)| \leq |f_n(x_0)| + \operatorname{Var}_I f_n$, which is again bounded.
- $\operatorname{Var}_I(v_n - f_n) \leq \operatorname{Var}_I v_n + \operatorname{Var}_I f_n$ which is bounded by the previous bullet, and $v_n(x_0) + f_n(x_0)$ is also bounded by the previous bullet.

Great. From the first bullet, we may extract a subsequence v_{n_k} of v_n which converges pointwise in I. From the second bullet, we may extract a subsequence $v_{n_{k_j}} - f_{n_{k_j}}$ of $v_{n_k} - f_{n_k}$

which converges pointwise in I. Since $v_{n_{k_j}}$ also converges pointwise in I, we conclude that $f_{n_{k_j}} = v_{n_{k_j}} + (f_{n_{k_j}} - v_{n_{k_j}})$ converges pointwise in I.

Thus it indeed is sufficient the prove the statement for when f_n is increasing. Assume that this is the case.

Step 2: Our goal in this step is to get convergence in a dense set. Let $E := \{q_i\}_{i=1}^{\infty}$ enumerate all rationals in I.

Observe that $\{f_n(x)\}_n$ is a bounded sequence for all x. (To be specific, if $\mathrm{Var}_I f_n \leq M$ and $|f_n(x_0)| \leq M$ for all n, then $|f_n(x)| \leq |f_n(x_0)| + |f_n(x) - f_n(x_0)| \leq M + \mathrm{Var}_I f_n \leq 2M$.) Thus we may do some silly diagonalization where we repeatedly apply Bolzano-Weierstrass to extract subsequences forever.

- Extract a subsequence $\{f_{1,n}\}_n$ of $\{f_n\}_n$ for which $f_{1,n}(q_1)$ converges to a value that we shall call $g(q_1)$.
- Extract a subsequence $\{f_{2,n}\}_n$ of $\{f_{1,n}\}_n$ for which $f_{2,n}(q_2)$ converges to a value that we shall call $g(q_2)$.
- Extract a subsequence $\{f_{3,n}\}_n$ of $\{f_{2,n}\}_n$ for which $f_{3,n}(q_3)$ converges to a value that we shall call $g(q_3)$.
- Extract a subsequence $\{f_{4,n}\}_n$ of $\{f_{3,n}\}_n$ for which $f_{4,n}(q_4)$ converges to a value that we shall call $g(q_4)$.
- ...

In this way, we define a function $g : E \to \mathbb{R}$, and moreover the diagonalization $\{f_{n,n}\}_n$ converges to g pointwise in $\{q_i : i \in \mathbb{N}\}$.

Step 2.5: In this step, we categorize the remaining points in $I \setminus E$ into two categories to be dealt with separately.

Note that g is increasing. Indeed, take $q_i, q_j \in E$ with $q_i < q_j$. Then $f_{n,n}(q_i) \leq f_{n,n}(q_j)$ because $f_{n,n}$ is increasing. Send $n \to +\infty$ to get $g(q_i) \leq g(q_j)$.

It follows that g is "continuous on I" except at countably many (possibly irrational) points $x_1, x_2, \cdots, \in I$. Rigorously speaking, pick some increasing extension of g to the whole of I. Call the extension $\tilde{g}$. (It doesn't matter which one we choose. Take e.g. $\tilde{g}(x) := \sup_{q_n < x} g(q_n)$ for $x \notin E$, $\tilde{g}(x) := g(x)$ for $x \in E$.) We may classify the remaining points we need to deal with (i.e. those points in $I \setminus E$) into two categories: The points $F := \{x_i\}_{i=1}^{\infty}$ at which $\tilde{g}$ is discontinuous, and all other points $I \setminus E \setminus F$, where $\tilde{g}$ is continuous.

Step 3: In this step, we find a further subsequence which converges in the discontinuity set F (as well as E).

This is "easy". Note that F is countable, so we may repeat the wacky diagonalization trick where we extract a million subsequences of $f_{n,n}$ to end up with a subsequence $f_{n,n,n}$ which converges pointwise on both E *and* F. As before, for $x \in F$ we let $g(x)$ be the pointwise limit of $f_{n,n,n}(x)$. (This could disagree with $\tilde{g}$!) This indexing notation is cursed so let f_{n_k} be the subsequence we have extracted so far, converging on $E \cup F$.

Step 4: Finally we show that the subsequence we obtained actually converges everywhere else in I.

All the other points are points of continuity of $\tilde{g}$. We claim that $f_{n_k}(x_0)$ converges for every x_0 where $\tilde{g}$ is continuous, and that moreover the limit is $\tilde{g}(x_0)$.

To see this, we take a point x_0 where $\tilde{g}$ is continuous, and fix $\varepsilon > 0$ (wow shocking). By continuity of $\tilde{g}$ at x_0, we may find some rational q_i slightly less than x_0 for which $\tilde{g}(x_0) - \tilde{g}(q_i) < \varepsilon$, and find some rational q_j slightly greater than x_0 for which $\tilde{g}(q_j) - \tilde{g}(x_0) < \varepsilon$.

Since f_{n_k} is increasing, we have that

$$f_{n_k}(q_i) \leq f_{n_k}(x_0) \leq f_{n_k}(q_j).$$

Sending $k \to +\infty$, and using the fact that f_{n_k} converges pointwise to $\tilde{g}$ on the rationals, we get that

$$\tilde{g}(x_0) - \varepsilon \leq \tilde{g}(q_i) \leq \liminf_{k\to\infty} f_{n_k}(x_0) \leq \limsup_{k\to\infty} f_{n_k}(x_0) \leq \tilde{g}(q_j) \leq \tilde{g}(x_0) + \varepsilon.$$

Sending $\varepsilon \to 0^+$, we conclude that the limit $\lim_{k\to\infty} f_{n_k}(x_0)$ exists and is $\tilde{g}(x_0)$.

Thus, if we define $g(x) := \tilde{g}(x)$ for $x \in I \setminus E \setminus F$, we finally have that $f_{n_k} \to g$ pointwise *everywhere* in I. □

It happens to be the case that $f \in BPV(I)$, and that moreover we can bound its pointwise variation.

Theorem 11.2

If $f_n \to f$ pointwise, then

$$\operatorname*{Var}_I f \leq \liminf_{n\to\infty} \operatorname*{Var}_I f_n.$$

Proof. Exercise. □

11.2 Sobolev Spaces

11.2.1 Motivations

One way to motivate Sobolev spaces is that it is the "right" space to do differentiation on L^p functions. Imagine, if you will, a space of functions $S^{1,p}(\Omega)$ which consists of all $C^1(\Omega)$ functions u for which both $\int_\Omega |u|^p \, dx$ and $\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx$ are finite for all i.

Previously, when studying spaces such as $C^1(\Omega)$, we desired a preservation of continuity which is why the appropriate norm was $\|\cdot\|_\infty$, which promoted uniform convergence to enforce the "right" structure. Here though, in this weird space $S^{1,p}(\Omega)$, we are motivated purely by wanting to differentiate in L^p spaces, and so the right structure and convergence that we want this $S^{1,p}(\Omega)$ space to exhibit is that of L^p. To wit, the "correct" norm would have to be

$$\|u\|_{S^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^p(\Omega)}.$$

Indeed this makes $S^{1,p}(\Omega)$ into a normed space with the desired structure we want: Convergence in this space means $L^p(\Omega)$ convergence of both the function and its derivatives. All seems well and good. However, it is not complete!

This means that $S^{1,p}(\Omega)$ *sucks* to work with. It's basically useless and it should feel bad. A nice analogue of this issue is $\mathbb{Q}$: It's a space that feels like it should be nice, but it's not complete so trying to do analysis on $\mathbb{Q}$ is dumb. The solution, then, is to pass to a *larger space* which is actually complete. In the case of $\mathbb{Q}$ this was $\mathbb{R}$. In the case of $S^{1,p}(\Omega)$, we weaken the strong derivatives to weak derivatives so that we obtain the larger, complete space, $W^{1,p}(\Omega)$.

As for why we are motivated to define weak derivatives in the way they are, I will for now simply say that "passing to integration by parts" is the most "natural" way to "generalize" differentiation, since the introduction of an integral "smooths out negligible gaps in differentiability". I will also note that it also makes good sense that we do not try to directly differentiate L^p functions, because they are not functions in the traditional sense; they can only be integrated, so it is logical to use integrals to try and define some kind of derivative for L^p functions. If you are unsatisfied, there are also some deeper motivations in the theory of *distributions* that I will not discuss here. At least, for now.

11.2.2 Weak Derivatives

Lemma 11.1 (Fundamental Lemma of the Calculus of Variations)

Let $\Omega \subseteq \mathbb{R}^N$ be open and $1 \leq p \leq \infty$. Suppose that $f \in L^p(\Omega)$ such that

$$\int_\Omega f\varphi \, dx = 0 \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Then $f = 0$.

Proof. *Motivation: Lebesgue points are nice. They are the only points at which an L^p "function" would have a "most natural" or "canonical" value.*

Associate f with one of its representatives. Take a Lebesgue point $x_0 \in \Omega$ of f and take $\varphi \in C_c^\infty(B(0,1))$ to be a mollifier. Then the mollification f_ε of f converges to $f(x_0)$ as $\varepsilon \to 0$. That is,

$$f(x_0) = \lim_{\varepsilon \to 0^+} \int_{B(x_0,\varepsilon)} f(x)\varphi_\varepsilon(x_0 - x)\, dx.$$

But $\int_{B(x_0,\varepsilon)} f(x)\varphi_\varepsilon(x_0 - x)\, dx = \int_\Omega f(x)\varphi_\varepsilon(x_0 - x)\, dx = 0$ since $x \mapsto \varphi_\varepsilon(x_0 - x)$ is a smooth function with compact support. So $f(x_0) = 0$.

But f is locally integrable (Hint: Hölder!) and so almost every point is a Lebesgue point. So $f = 0$ almost everywhere. $\square$

One consequence of this lemma is that weak derivatives are unique. Indeed, suppose that some $f \in L^p(\Omega)$ admits two weak derivatives g_1 and g_2. Then

$$\int_\Omega g_1\varphi \, dx = -\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = \int_\Omega g_2 \varphi \, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. Subtracting gives

$$\int_\Omega (g_1 - g_2)\varphi \, dx = 0$$

for all such φ, which then entail that $g_1 - g_2 = 0$.

Now let us look at some examples of weak derivatives.

Example 11.1: Let $I = (-1, 1)$ and let $f(x) = |x|$. Then we may view f as a function in $L^p(I)$.
f has a weak derivative: It is $\operatorname{sgn} x$. To prove this, we must show that

$$\int_{-1}^{1} |x|\varphi'(x)\,dx = -\int_{-1}^{1} (\operatorname{sgn} x)\varphi(x)\,dx$$

for all $\varphi \in C_c^\infty(-1, 1)$. Indeed, we may write

$$\begin{aligned}\int_{-1}^{1} |x|\varphi'(x)\,dx &= \int_{-1}^{0} -x\varphi'(x)\,dx + \int_{0}^{1} x\varphi'(x)\,dx \\ &= \left[0\cdot\varphi(0) - (-1)\cdot\varphi(-1) - \int_{-1}^{0} -\varphi(x)\,dx\right] \\ &\quad + \left[1\cdot\varphi(1) - 0\cdot\varphi(0) - \int_{0}^{1} \varphi(x)\,dx\right] \\ &= -\int_{-1}^{1} (\operatorname{sgn} x)\varphi(x)\,dx.\end{aligned}$$

Since $f \in L^p(I)$ and $f' \in L^p(I)$ for all $1 \le p \le \infty$, we may say $f \in W^{1,p}(I)$.

Example 11.2: Let $f : I \to \mathbb{R}$ be AC where $I \subseteq \mathbb{R}$. Then f has a weak derivative, and the weak derivative is the *strong* derivative f'. Indeed, f being AC is enough for the integration by parts to hold, so this is unsurprising.

Example 11.3: In fact, in higher dimensions, most "nice" functions have a weak derivative. For example, it is true that $f \in C^1(B(0, 1))$ admits first-order weak derivatives, and so any $f \in C^1(\overline{B(0, 1)})$ will be in the Sobolev space $W^{1,p}(B(0, 1))$. Moreover the "strong" derivatives agree with the weak derivatives.

Example 11.4: The function $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \end{cases}$ does *not* have a weak derivative. You will show something like this on the homework.

The connection between regularity and being Sobolev will be made concrete by the *Absolute Continuity on Lines (ACL)* condition.

Sobolev functions need not be bounded.

Example 11.5: Let us consider $f(x) := \frac{1}{\|x\|^\varepsilon}$, and suppose $0 < \varepsilon < 1, 1 \leq p < N, N$ are such that $(1+\varepsilon)p < N$. Then $f \in W^{1,p}(B(0,1))$. Indeed, we have

$$\int_{B(0,1)} |f|^p \, dx = \int_{B(0,1)} \frac{1}{\|x\|^{\varepsilon p}} \, dx < \infty$$

because $\varepsilon p < N$, so that $f \in L^p(B(0,1))$, and

$$\begin{aligned}\int_{B(0,1)} \left|\frac{\partial f}{\partial x_i}\right|^p \, dx &= C_1 \int_{B(0,1)} \left|\frac{1}{\|x\|^{1+\varepsilon}} \cdot \frac{x_i}{\|x\|}\right|^p \, dx \\ &\leq C_1 \int_{B(0,1)} \frac{1}{\|x\|^{p+\varepsilon p}} \, dx,\end{aligned}$$

which is finite when $p + \varepsilon p < N$.

Example 11.6: Let $f \in W^{1,p}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$. Then $f\varphi \in W^{1,p}(\Omega)$.

Indeed, we claim that the weak derivative of $f\varphi$ in the x_i direction is $\frac{\partial f}{\partial x_i}\varphi + f\frac{\partial \varphi}{\partial x_i}$. This is because for all $\psi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega \left(\frac{\partial f}{\partial x_i}\varphi + f\frac{\partial \varphi}{\partial x_i}\right)\psi \, dx &= \int_\Omega \frac{\partial f}{\partial x_i}\varphi\psi \, dx + \int_\Omega f\frac{\partial \varphi}{\partial x_i}\psi \, dx \\ &= -\int_\Omega f(\frac{\partial \varphi}{\partial x_i}\psi + \varphi\frac{\partial \psi}{\partial x_i}) \, dx + \int_\Omega f\frac{\partial \varphi}{\partial x_i}\psi \, dx \\ &= -\int_\Omega f\varphi\frac{\partial \psi}{\partial x_i} \, dx.\end{aligned}$$

It's easy to see that $f\varphi \in L^p(\Omega)$ and $\frac{\partial f}{\partial x_i}\varphi + f\frac{\partial \varphi}{\partial x_i} \in L^p(\Omega)$, thus $f\varphi \in W^{1,p}(\Omega)$.

Remark: Using the exact same proof, we have that if $f \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\varphi \in C_b^\infty(\Omega)$, then $f\varphi \in W^{1,p}(\Omega)$ and its weak derivative is what you think it is.

12 More on Sobolev Spaces

12.1 Compactness

Theorem 12.1 (Rellich-Kondrachov for Kindergarteners)

Let $\Omega := (a, b) \subseteq \mathbb{R}$ be a bounded open interval. Then $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

Proof. Obviously $\|u\|_{L^p(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$ for all $u \in W^{1,p}(\Omega)$, so $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Now take a bounded sequence in $u_n \in W^{1,p}(\Omega)$. Then u_n is bounded in $L^p(\Omega)$ and u_n' is bounded in $L^p(\Omega)$. Find some large M so that $\|u_n\|_{L^p(\Omega)} \leq M$ and $\|u_n'\|_{L^p(\Omega)} \leq M$ for all n.

Let $\tilde{u}_n$ be the AC representative of u_n. Observe that

$$\operatorname*{Var}_{\Omega} \tilde{u}_n = \int_\Omega |\tilde{u}_n'|\, dx \leq \|u_n\|_{L^p(\Omega)} \mathcal{L}^1(\Omega)^{1/p'} \leq M \mathcal{L}^1(\Omega)^{1/p'}$$

for all n, thus $\operatorname{Var}_\Omega \tilde{u}_n$ is bounded.

Moreover we claim that $\{\tilde{u}_n\}_n$ is uniformly bounded (in the $\|\cdot\|_\infty$ sense). If not, then for every K we can find n and $x \in \Omega$ for which $|\tilde{u}_n(x)| \geq K$. Since the variation of $\tilde{u}_n$ is bounded by $M\mathcal{L}^1(\Omega)^{1/p'}$, it follows that $|\tilde{u}_n(y)| \geq K - M\mathcal{L}^1(\Omega)^{1/p'}$ for all $y \in \Omega$. So

$$M^p \geq \int_\Omega |u_n|^p\, dy \geq \mathcal{L}^1(\Omega) \left(K - M\mathcal{L}^1(\Omega)^{1/p'} \right)^p.$$

This holds for all large enough K by assumption, which is clearly impossible. Contradiction.

Therefore, by Helly's Selection Theorem we may extract a subsequence $\tilde{u}_{n_k}$ which converges pointwise to some function $\tilde{u}$. But $\{\tilde{u}_{n_k}\}_k$ is uniformly bounded in $\|\cdot\|_\infty$ so by a domination argument, $\int_\Omega |\tilde{u}_{n_k} - \tilde{u}|^p\, dx \to 0$. In particular $u \in L^p(\Omega)$ where $\tilde{u}$ is a representative of u. So $u_{n_k} \to u$ in $L^p(\Omega)$. $\square$

12.2 Product Rule and Chain Rule

Theorem 12.2 (Product Rule)

Let $1 \leq p < \infty$. Suppose $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $fg \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}$$

(almost everywhere).

There *may* be a way to do this well by using the following lemma, but I couldn't quite get the convergence to work.

Lemma 12.1

For $f \in W^{1,p}(\Omega)$, we have that $f_\varepsilon \to f$ in $W^{1,p}_{\text{loc}}(\Omega)$. That is, we have $W^{1,p}$ convergence on compact subsets of Ω.

Anyways, in the end it's easy to show the product rule by simply using the ACL condition.

Proof. To use the ACL condition on fg, we first need to verify that $fg \in L^p(\Omega)$. This is easy, since

$$\|fg\|_{L^p(\Omega)} \leq \|g\|_{L^\infty(\Omega)}\|f\|_{L^p(\Omega)} < \infty.$$

With that settled, we now must show that on almost every slice, the product fg is AC with strong derivative in L^p.

We start by extracting good slices.

Since $f, g \in W^{1,p}(\Omega)$, we have by the ACL condition applied to both f and g that on almost every slice $\Omega_{x_i'}$, we have that $f(\cdot, x_i')$ and $g(\cdot, x_i')$ are AC (and moreover $\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \in L^p(\Omega)$). These slices are good, but I want them to be even better! Particularly, I want to make sure they are nice and bounded on the slices, for later.

We get this additional niceness by a Fubini argument. Since $f \in L^\infty(\Omega)$, we have that there is some $M < \infty$ for which $|f|, |g| \leq M$ almost everywhere. If we let $E = \{|f| \leq M\} \subseteq \Omega$, then $\mathcal{L}^N(\Omega \setminus E) = 0$. Thus

$$0 = \int_{\Omega \setminus E} 1\,dx = \int_{x_i' \in \mathbb{R}^{N-1}} \int_{x_i \in (\Omega \setminus E)_{x_i'}} 1\,dx_i\,dx_i' \geq 0,$$

which can only be possible if

$$\int_{x_i \in (\Omega \setminus E)_{x_i'}} 1\,dx = 0$$

for almost every $x_i' \in \mathbb{R}^{N-1}$. For all such x_i', we have $|f| \leq M$ for $\mathcal{L}^1$-a.e. $x_i \in \Omega_{x_i'}$, by definition of E! Repeating the argument for g, we have for almost every x_i' that $|f|, |g| \leq M$ $\mathcal{L}^1$-a.e. in $\Omega_{x_i'}$.

Bringing these two excursions together: We have for almost every $x_i' \in \mathbb{R}^{N-1}$ that $f(\cdot, x_i')$ and $g(\cdot, x_i')$ are AC, and $|f|, |g| \leq M$ for $\mathcal{L}^1$-a.e. $x_i \in (\Omega \setminus E)_{x_i'}$. In particular, $f(\cdot, x_i')$ and $g(\cdot, x_i')$ are *bounded by* M on this slice (since they is AC)!

For every such nice x_i' slice, we claim that $f(\cdot, x_i')g(\cdot, x_i')$ is AC. Indeed, both $f(\cdot, x_i')$ and $g(\cdot, x_i')$ are AC and bounded, so a recitation from a very long time ago lets us conclude that the product is AC.

It follows that we may apply the AC chain rule to get

$$\frac{\partial(fg)}{\partial x_i}(\cdot, x_i') = \frac{\partial f}{\partial x_i}(\cdot, x_i')g(\cdot, x_i') + f(\cdot, x_i')\frac{\partial g}{\partial x_i}(\cdot, x_i') \qquad (*)$$

almost everywhere in the slice.

Since we have the formula $(*)$ for a.e. $x_i' \in \mathbb{R}^{N-1}$, we actually have it a.e. on the whole space. That is,

$$\frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i}gc + f\frac{\partial g}{\partial x_i}$$

almost everywhere. The last thing we need to do is show that this strong derivative, $\frac{\partial(fg)}{\partial x_i}$, is in $L^p(\Omega)$. This follows because

$$\int_\Omega \left|\frac{\partial(fg)}{\partial x_i}\right|^p dx = \int_\Omega \left|\frac{\partial f}{\partial x_i}gc + f\frac{\partial g}{\partial x_i}\right|^p dx \leq C\int_\Omega \left|\frac{\partial f}{\partial x_i}\right|^p |g|^p\, dx + C\int_\Omega |f|^p \left|\frac{\partial g}{\partial x_i}\right|^p dx$$

$$\leq CM^p \int_\Omega \left|\frac{\partial f}{\partial x_i}\right|^p dx + CM^p \left|\frac{\partial g}{\partial x_i}\right|^p < \infty.$$

To wrap up: We have found that $fg \in L^p(\Omega)$ is AC on a.e. slice, and its strong derivative is in $L^p(\Omega)$. Thus by the ACL condition, $fg \in W^{1,p}(\Omega)$, and in fact $fg \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ since clearly $|fg| \leq M^2$ a.e., and the product rule holds as we found in $(*)$. $\square$

Theorem 12.3 (Chain Rule)

Let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded, and let $1 \leq p < \infty$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz and $u \in W^{1,p}(\Omega)$. Then $f(u) \in W^{1,p}(\Omega)$ and $\frac{\partial(f\circ u)}{\partial x_i}(x) = f'(\tilde{u}(x))\frac{\partial u}{\partial x_i}(x)$ a.e., where $\tilde{u}$ is the absolutely continuous representative of u, and we take $f'(\tilde{u}(x))\frac{\partial u}{\partial x_i}(x)$ to be 0 whenever $\frac{\partial u}{\partial x_i}(x)$.

This follows basically immediately from the chain rule for AC functions:

Theorem 12.4 (AC Chain Rule)

Let $I, J \subseteq \mathbb{R}$ be intervals and let $f : J \to \mathbb{R}$ and $g : I \to J$ such that $f, g, f \circ g$ are differentiable a.e.. If f has the Lusin N property, then

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

a.e., where $f'(g(x))g'(x)$ is taken to be zero whenever $g'(x) = 0$.

Let us prove the chain rule for Sobolev spaces.

Proof. Fix $x_i' \in \mathbb{R}^{N-1}$ for which $t \mapsto u(x_i', t)$ is AC with (strong) derivative in $L^p(\Omega_{x_i'})$. Then in particular, $u(x_i', \cdot)$ is strongly differentiable a.e., and moreover f is Lipschitz so f is differentiable a.e., satisfies the Lusin N property, and $f(u)$ is AC (and hence differentiable a.e.). It follows by the AC chain rule that

$$\frac{d}{dt}f(u(x_i',t)) = f'(\tilde{u}(x_i',t))\frac{\partial u}{\partial x_i}(x_i',t)$$

for a.e. t. It remains to prove that $f(u) \in L^p(\Omega)$ and $f'(\tilde{u})\frac{\partial u}{\partial x_i} \in L^p(\Omega)$.

For the fisrt, fix $x_0 \in \Omega$ and write

$$\int_\Omega |f(u)|^p\,dx \le C_p \int_\Omega |f(\tilde{u}(x)) - f(\tilde{u}(x_0))|^p\,dx + |f(\tilde{u}(x_0))|^p\,dx$$

$$\le C_p L^p \int_\Omega |\tilde{u}(x) - \tilde{u}(x_0)|^p\,dx + \mathcal{L}^N(\Omega)|f(\tilde{u}(x_0))|^p$$

$$\le C_p^2 L^p \int_\Omega |\tilde{u}(x)|^p\,dx + C_p^2 L^p \int_\Omega |\tilde{u}(x_0)|^p\,dx + \mathcal{L}^N(\Omega)|f(\tilde{u}(x_0))|^p$$

$$\le C_p^2 L^p \int_\Omega |\tilde{u}(x)|^p\,dx + C_p^2 L^p \mathcal{L}^N(\Omega)|\tilde{u}(x_0)|^p + \mathcal{L}^N(\Omega)|f(\tilde{u}(x_0))|^p < \infty.$$

For the second, we simply use $f' \le L$ and use the fact that $\frac{\partial u}{\partial x_i} \in L^p(\Omega)$. $\square$

13 Embeddings of Sobolev Spaces

13.1 Trivial Embeddings

Theorem 13.1

- For E finite measure, we have $L^p(E) \hookrightarrow L^q(E)$ for $q < p$. That is, higher-power L^p spaces embed into lower powers.
- $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$
- For Ω finite measure, we have $W^{1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ for $q < p$.

Proof. Easy. □

The point here is that higher-powers should be thought of as generally more "restrictive". So, we should not expect a priori that the L^p or $W^{1,p}$ space should embed into spaces with higher power.

But in a crazy plot twist, $W^{1,p}$ defies this intuition.

13.2 The Sobolev Embedding Theorem

Being in L^p is nothing to write home about. But if your *derivatives* are in L^p, i.e. if you're in $W^{1,p}$, we often get two *really nice* benefits:

1. It can *boost integrability*. That is, if $f \in L^p$ and $\nabla f \in L^p$, you often can conclude that $f \in L^{\text{more than } p}$!
2. It can give *compactness*. A sequence in L^p need not have a subsequence converging in L^q. But a sequence in $W^{1,p}$ often does.

I say "often" and not "always" because real analysis counterexamples exist. Ignoring the subtlties, let's state the Sobolev Embedding Theorem, which characterizes the first benefit of *increasing integrability*... and even *increasing regularity*!

Theorem 13.2 (Sobolev Embedding Theorem)

Let $\Omega \subseteq \mathbb{R}^N$ be open and "nice enough". Then:

$$\begin{cases} W^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega), & p < N \\ W^{1,N}(\Omega) \hookrightarrow L^q(\Omega) \text{ for all } 1 \le q < \infty, & p = N \\ W^{1,p}(\Omega) \hookrightarrow C^{0,1-N/p}(\overline{\Omega}), & p > N \end{cases}$$

What is "nice enough"? I say that Ω is "nice enough" if it is an *extension domain.* A sufficient condition for this to occur is if $\partial\Omega$ is Lipschitz.

This has some crazy consequences. Sobolev embeddings not only boost integrability and regularity: Since the embeddings are continuous, they boost *convergence* as well!

Integrablity and Regularity Boosting

Example 13.1: Let's suppose that $u \in W^{1,1}((0,1)^2)$. Then in fact, $u \in L^2((0,1)^2)$, because $1^* = \frac{2\cdot 1}{2-1} = 2$.

Example 13.2: Let's suppose that $u \in W^{1,2}((0,1)^2)$. Then u has pretty much unlimited integrability. For example, $u \in L^{9999999999}((0,1)^2)$.

Example 13.3: Let's suppose that $u \in W^{1,3}((0,1)^2)$. Then u is uniformly continuous.

Convergence Boosting

Example 13.4: Suppose $u_n, u \in C^1([0,1])$ and that $u_n \to u$ in $L^2([0,1])$ and $u_n' \to u'$ in $L^2([0,1])$. Then actually, $u_n \to u$ **uniformly!!!**
This is because $W^{1,2}((0,1)) \hookrightarrow C^{0,1/2}(\Omega)$, and any convergence in a Hölder space is uniform.

13.3 Rellich-Kondrachov Compactness

If we "loosen" any of the Sobolev embeddings, we get compactness!

Theorem 13.3 (Rellich-Kondrachov Compactness)

Let $\Omega \subseteq \mathbb{R}^N$ be open, bounded, and "nice enough". Then:

$$\begin{cases} W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact for all } 1 \leq q < \frac{Np}{N-p}, & p < N \\ W^{1,N}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact for all } 1 \leq q < \infty, & p = N \\ W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \text{ is compact for all } 0 < \alpha < 1 - N/p, & p > N \end{cases}$$

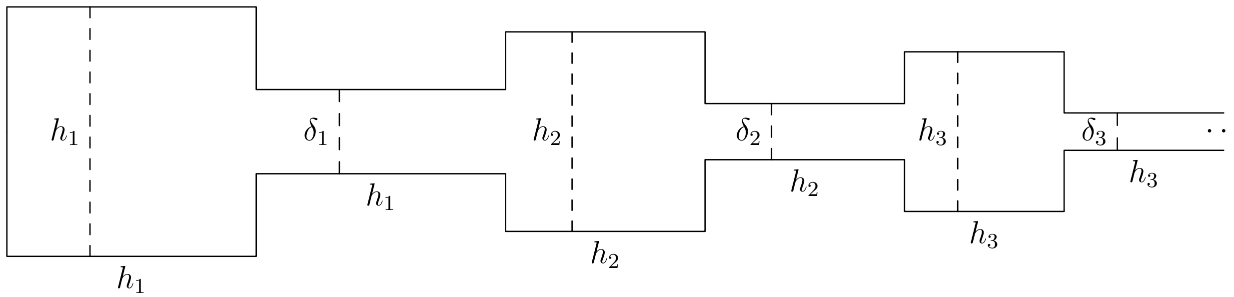
Remark: If Ω is not bounded, you'd still get compactness results, it's just that the subsequence you extract would only converge over compact sets, i.e. the convergence is local.

Example 13.5: Let $f_n \in C^1(0,1)$ be a family of functions for which $\int_0^1 |f_n|^2\,dx \leq M$ and $\int_0^1 |f_n'|^2\,dx \leq M$ for all n, for some $M > 0$.

Then f_n is bounded in $W^{1,2}(\Omega)$. $2 > 1$, so by Rellich-Kondrachov we may extract a subsequence f_{n_k} which converges to some $f \in C^{0,1/3}(\overline{\Omega})$ in $\|\cdot\|_{C^{0,1/3}(\overline{\Omega})}$. In particular, the convergence $f_{n_k} \to f$ is *uniform*!

(By the way, could we have reached a similar conclusion using Ascoli-Arzela?)

13.4 Rooms and Passages



The *Rooms and Passages* is a methodology for creating terrible sets that serve as counterexamples to both the Sobolev Embedding Theorem and Rellich-Kondrachov. They show that these theorems cannot be applied necessarily to sets with highly irregular boundary.

I refuse to explicitly describe the set, so all you get is the above picture. The "rooms" are the $h_n \times h_n$ squares, and the "passages" are the thin $h_1 \times \delta_1$ hallways between the rooms.

For this set to be a more powerful counterexample, we will take it to be bounded. So we would like

$$\boxed{\sum_{n=1}^{\infty} h_n < \infty}. \qquad (1)$$

Now we shall build a function f on this weird set, that we shall call Ω. Let the nth room be R_n, and let the nth passage be P_n. Then our function f shall be defined as follows:

- f shall take the value $a_n > 0$ in room R_n.
- In P_n, f will linearly interpolate between the values a_n and a_{n+1}. That is, f will be such that $\frac{\partial f}{\partial x} = \frac{a_{n+1} - a_n}{h_n}$ and $\frac{\partial f}{\partial y} = 0$ in P_n.

That's it! Let's make some counterexamples.

13.4.1 Killing Sobolev Embedding Theorem

Let us generate $f \in W^{1,2}(\Omega)$ such that $f \notin L^q(\Omega)$ for *any* $q > 2$. This will disprove Sobolev Embedding's $p = N$ case.

For this, let us require that the sequence of values a_n is increasing. This lets us compute bounds for $\|f\|_{L^p(\Omega)}$, $\|\partial_x f\|_{L^p(\Omega)}$, and $\|\partial_y f\|_{L^p(\Omega)}$ more easily.

In R_n, we have that $\int_{R_n} |f|^p \, d(x,y) = h_n^2 a_n^p$. Thus

$$\int_\Omega |f|^p \, d(x,y) \geq \int_{\bigcup_{n=1}^\infty R_n} |f|^p \, d(x,y) = \sum_{n=1}^\infty h_n^2 a_n^p.$$

To ensure that $f \notin L^q(\Omega)$ for all $q > 2$, it suffices to have

$$\boxed{\sum_{n=1}^\infty h_n^2 a_n^q = +\infty} \qquad (2)$$

for all $q > 2$. However, we will definitely require

$$\boxed{\sum_{n=1}^\infty h_n^2 a_n^2 < \infty} \qquad (3)$$

if we want $f \in L^2(\Omega)$.

To finish ensuring $f \in L^2(\Omega)$, we may write the bound $f(x,y) \leq a_{n+1}$ in passageway P_n to see that

$$\int_{P_n} |f|^2 \leq a_{n+1}^2 h_n \delta_n.$$

Thus

$$\int_{\bigcup_{n=1}^\infty P_n} |f|^2 \, d(x,y) \leq \sum_{n=1}^\infty \delta_n h_n a_{n+1}^2.$$

So we need

$$\boxed{\sum_{n=1}^\infty \delta_n h_n a_{n+1}^2 < \infty}. \qquad (4)$$

It remains to ensure that $\partial_x f \in L^2(\Omega)$, since $\partial_y f = 0$. In P_n, f has a "slope" of $\frac{a_{n+1} - a_n}{h_n}$. So

$$\int_{P_n} |\partial_x f|^2 \, d(x,y) = \delta_n h_n \cdot \frac{(a_{n+1} - a_n)^2}{h_n^2} = \frac{\delta_n (a_{n+1}^2 - 2a_n a_{n+1} + a_{n+1}^2)}{h_n}.$$

So the final condition we require is

$$\boxed{\sum_{n=1}^\infty \frac{\delta_n (a_{n+1} - a_n)^2}{h_n} < \infty}. \qquad (5)$$

The trickiest criteria to satisfy are (2) and (3). The only "nice" way to get this sort of behavior for the domain of convergence is the include a log. This is because taking $a_n = n^t$

for some power t simply isn't flexible enough to somehow get convergence at 2 but divergence for all greater powers. To wit, the insight is to choose $\boxed{a_n = \dfrac{n}{\log n}}$.

Now to get both (2) and (3), we choose $\boxed{h_n = \dfrac{1}{n^{3/2}}}$. Indeed, we are using the fact that $\sum_{n=1}^{\infty} \frac{1}{n \log^2 n}$ converges but $\sum_{n=1}^{\infty} \frac{1}{n^{1-\varepsilon} \log^k n}$ does not.

Our choice of h_n also ensures (1). Let us now get (5) to converge. $(a_{n+1} - a_n)^2$ is kinda small. Whatever its asymptotics are, it surely can't be better than, say, n^2. So the sum converges if $\sum_{n=1}^{\infty} \delta_n n^{3/2+2}$ converges. Noticing that there is no penalty in any of the other conditions for choosing a ridiculously small δ_n, we may simply take $\boxed{\delta_n = \dfrac{1}{n^{1000}}}$. This forces (5) to converge, and (4) obviously converges as well because this is so darn small.

13.4.2 Killing Rellich-Kondrachov

We will construct a sequence $f_n \in W^{1,2}(\Omega)$ which is bounded in $W^{1,2}(\Omega)$, but such that it has no subsequence converging in $L^2(\Omega)$.

The idea is pretty simple: We will let f_n take value $a_n > 0$ in room R_n, have it linearly slope downwards to 0 in the nearby hallways P_{n-1} and P_n, and then $f_n = 0$ in all other rooms and passages.

Let's ensure that $\|f_n\|_{W^{1,2}(\Omega)}$ is bounded by enforcing that $\|f_n\|_{L^p(\Omega)}$ is bounded and $\|\partial_x f_n\|_{L^p(\Omega)}$ is bounded. To do this, it suffices to enforce

$$\boxed{\int_\Omega |f_n|^2 \, dx \le (h_{n-1}\delta_{n-1} + h_n^2 + h_n\delta_n)a_n^2 \le 3} \qquad (1)$$

and

$$\boxed{\delta_{n-1}h_{n-1} \cdot \left(\frac{a_{n-1}}{h_{n-1}}\right)^2 + \delta_n h_n \cdot \left(\frac{a_n}{h_n}\right)^2 \le 2}. \qquad (2)$$

To ensure that there is no subsequence converging in $L^2(\Omega)$, we can just ensure that the L^2-difference between f_n and f_m is large, for $|n - m| \ge 42$. This ensures that there cannot exist a subsequence that is Cauchy in $L^2(\Omega)$.

Well, when n and m are far apart, then the rooms they are supported in are far apart, so their L^2 difference will certainly be at least $\int_{R_n} |f_n|^2 \, dx$. So we just need to ensure something like

$$\boxed{\int_{R_n} |f_n|^2 \, dx = h_n^2 a_n^2 \ge 1}. \qquad (3)$$

A quick glance shows that δ_n only appears in (1) and (2), which is really helpful (this demonstrates the power of this template for making counterexamples!). So we can start by satisfying (3). We can do this by taking something like $\boxed{h_n = \frac{1}{2^n}}$ and $\boxed{a_n = 2^n}$.

Next, we can satisfy (1), which reduces to satisfying $h_{n-1}\delta_{n_1}a_n^2 + h_n\delta_n a_n^2 \leq 2$. Evidently, a choice of δ_n such as $\boxed{\delta_n = \frac{1}{9999 \cdot 100^n}}$ surely will be enough. A quick glance shows that this choice of δ_n will also satisfy (2). Thus we have generated a counterexample to Rellich-Kondrachov compactness in the case that the domain's boundary is bad.

14 The Last Recitation

Example 14.1: Let $\Omega := \{(x,y) : 0 < x < 1, 0 < y < x^5\}$. Prove that:

- If $u : \Omega \to \mathbb{R}$ is defined as $u(x,y) := x^{-4.999}$, then $u \in W^{1,1}(\Omega)$ but u cannot be extended to a function in $W^{1,1}(\mathbb{R}^2)$.
- If $u : \Omega \to \mathbb{R}$ is defined as $u(x,y) := x^{-1.999}$, then $u \in W^{1,2}(\Omega)$ but u cannot be extended to a function in $W^{1,2}(\mathbb{R}^2)$.
- If $u : \Omega \to \mathbb{R}$ is defined as $u(x,y) := x^{-0.999}$, then $u \in W^{1,3}(\Omega)$ but u cannot be extended to a function in $W^{1,3}(\mathbb{R}^2)$.

Proof. We'll more or less handle all three cases at once.

We first show that if $u(x,y) = x^{1-\frac{6}{p}+\delta}$ then $u \in W^{1,p}(\Omega)$. Indeed, we have that

$$\int_\Omega |u|^p \, d(x,y) = \int_0^1 \int_0^{x^5} x^{(1+\delta)p-6} \, dy \, dx = \int_0^1 x^{(1+\delta)p-1} \, dx < \infty$$

and

$$\int_\Omega |\partial_x u|^p \, d(x,y) = C \int_0^1 \int_0^{x^5} x^{(1+\delta)p-7} \, dy \, dx = \int_0^1 x^{(1+\delta)p-2} \, dx < \infty$$

because $-1 < (1+\delta)p - 2$. This is enough to conclude that $u \in W^{1,p}(\Omega)$ (why?).

Now suppose for contradiction that u can be extended to a function in $W^{1,p}(\mathbb{R}^2)$ (with u not relabelled).

- For the case $p = 1$, we have by SGN that $W^{1,1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, so $u \in L^2(\mathbb{R}^2)$. But

  $$\int_\Omega |u|^2 \, d(x,y) = \int_0^1 \int_0^{x^5} x^{2(-5+\delta)} \, dy \, dx = \int_0^1 x^{-5+2\delta} \, dx = +\infty$$

  since δ is small.
- For the case $p = 2$, we have that $W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{100}(\mathbb{R}^2)$, so $u \in L^{100}(\mathbb{R}^2)$. But

  $$\int_\Omega |u|^{100} \, d(x,y) = \int_0^1 \int_0^{x^5} x^{100(-2+\delta)} \, dy \, dx = \int_0^1 x^{-200+100\delta} \, dx = +\infty$$

  since δ is small.
- For the case $p = 3$, we have by Morrey that $W^{1,3}(\mathbb{R}^2) \hookrightarrow C^{0,1/3}(\mathbb{R}^2)$, so $u \in C^{0,1/3}(\mathbb{R}^2)$. Of course, one could conclude that $u \in L^{100}(\mathbb{R}^2)$ to get a contradiction as before, but we can also observe that u being Holder continuous would imply in particular that it is continuous at $(0,0)$, and evidently $x^{-0.999}$ cannot be extended as such.

□

14.1 Calculus of Variations: Tonelli's Direct Method

hi leoni if you're reading this what happened was i completely forgot what to teach after that example so uh yeah

You won't need to know any of this for the final exam. The reason why I'm showing you all of this is to demonstrate that analysis isn't just abstract nonsense — all the tools you have learned thus far can be used to obtain extremely powerful results that have genuinely important consequences and real-world applications. The theory I'm showcasing here pretty much ties up content over your entire Leoni analysis experience.

Recall a quick definition:

Definition 14.1 (Sequentially lower-semicontinuous)

Let (X, τ) be a topological space. A function $f : X \to \overline{\mathbb{R}}$ is *sequentially lower-semicontinuous* (slsc) if for every $x_0 \in X$ and every $x_n \xrightarrow{\tau} x_0$, we have the inequality

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n).$$

(*Is it true in topological spaces that f is slsc iff f is lsc? Why or why not? What about in metric spaces?*)

Let us attempt to minimize the functional

$$F(u) := \int_0^1 |u|^2 \, dx$$

over $u \in L^2(0,1)$. Yes, it's obvious, but uh pretend for a moment that it isn't. First, we observe that there exists an infimum $I := \inf_{u\in L^2} F(u)$, so there exists a sequence u_n with $F(u_n) \to I$. We call this a *minimizing sequence.*

Now, here's the idea. We want to first use a compactness argument to find a subsequence $u_{n_k} \to u$ under some notion of convergence τ. τ needs to be *weak* enough for such a subsequence to exist. Now, if F is *sequentially lower-semicontinuous ("slsc")* with respect to τ, then we may conclude that

$$I \leq F(u) \leq \liminf_k F(u_{n_k}) = I,$$

so that $F(u) = I$ and u solves the minimization problem. τ must be *strong* enough to have sequential lower-semicontinuity (why does stronger convergence make it easier to have sequential lower-semicontinuity?).

So here we have a weird little game that we must play: Pick a good convergence τ that is weak enough to have compactness results, but is strong enough to have sequential lower-semicontinuity. This whole scheme is called *the Direct Method of Calculus of Variations.*

Let's try some notions of convergence and see what happens.

- What if we take τ to be the topology in which only eventually-constant sequences converge?

  We do not get compactness. A sequence u_n being bounded in F (i.e. $F(u_n)$ bounded, i.e. u_n is bounded in $L^2(0,1)$) need not have a subsequence that is eventually constant.

  We get the slsc. Clearly F (or any functional) will be slsc under such a topology, since $u_n \to u_0$ only when $u_n = u_0$ forever eventually, in which case it is clear that $f(u_0) \leq \liminf_{n\to\infty} F(u_n)$.

  Thus, this convergence is *too strong*.

- What if we take τ to be the topology in which all sequences converge?

  We get compactness. Clearly any sequence of u_n's has a subsequence, and this subsequence converges under this dumb topology.

  We do not get the slsc. If e.g. $u_0 \equiv 1$ and $u_n \equiv 0$, then u_n would converge to u under this topology, and this sequence contradicts F being slsc.

  Thus, this convergence is *too weak*.

- What if we take τ to be $L^2(0,1)$ convergence?

  We do not get compactness. I leave it to you to verify that there is a sequence bounded in $L^2(0,1)$ that does not admit a subsequence converging in $L^2(0,1)$.

  We get the slsc. To see this, suppose that $u_n \to u_0$ in $L^2(0,1)$. Letting $L := \liminf_{n\to\infty} F(u_n)$, we wish to prove that $F(u_0) \leq F(u_n)$.

  By properties of the liminf, there exists a subsequence u_{n_k} for which

  $$L = \liminf_{n\to\infty} F(u_n) = \lim_{k\to\infty} F(u_{n_k}).$$

  Next, since $u_{n_k} \to u_0$ in $L^2(0,1)$, we know that there exists a subsequence $u_{n_{k_j}}$ converging to u_0 a.e.. Thus the conclusion follows from Fatou:

  $$\begin{aligned} F(u_0) = \int_0^1 |u_0|^2\,dx &= \int_0^1 \liminf_{j\to\infty} |u_{n_{k_j}}|^2\,dx \\ &\leq \liminf_{j\to\infty} \int_0^1 |u_{n_{k_j}}|^2\,dx = \lim_{j\to\infty} F(u_{n_{k_j}}) = L. \end{aligned}$$

  This shows that $L^2(0,1)$ convergence is *too strong*.

14.2 Weak Convergence

If $L^2(0,1)$ convergence is too strong, then what is something weaker that is not too weak? Here is the miracle answer:

Definition 14.2 (Weak Convergence)

Let $1 \leq p < \infty$. We say that $f_n \in L^p(E)$ converges *weakly in* $L^p(E)$ to some $f \in L^p(E)$ if

$$\int_E f_n g\, dx \to \int_E f g\, dx$$

for all $g \in L^q(E)$ where q is the Hölder conjugate, and we write "$f_n \rightharpoonup f$ in $L^p(E)$".

(We have a similar definition for $p = +\infty$, and we call it "weak star" convergence instead, for a subtle reason that you shall learn in Functional Analysis.)

Exercise: *Show that "strong" $L^p(E)$ convergence implies weak $L^p(E)$ convergence. Show that the converse does not hold by proving that* $\sin(nx) \rightharpoonup -1$ *in* $L^p(-1,1)$ *as* $n \to +\infty$.

With some mental gymnastics, one can justify looking at such a definition by being inspired by category theory. Specifically, instead of studying functions in $L^p(E)$, we can study mathematical constructs that *probe* such functions, i.e. continuous linear functions on $L^p(E)$. It turns out that any $T : L^p(E) \to \mathbb{R}$ that is continuous will take the form $T(f) := \int_E fg\, dx$ for some $g \in L^q(E)$. This is called the Riesz-Representation Theorem.

Theorem 14.1 (RRT in Lp spaces)

Let $1 < p < \infty$ and q be the Hölder conjugate.

- For every $g \in L^q(E)$, the map

$$T_g : f \in L^p(E) \mapsto \int_E fg\, dx \in \mathbb{R}$$

  is a continuous linear functional on $L^p(E)$.

- Conversely (!), for every continuous linear $T : L^p(E) \to \mathbb{R}$, there is a unique $g \in L^q(E)$ for which

$$Tf = \int_E fg\, dx \qquad \forall f \in L^p(E).$$

We remark that the first bullet is pretty easy, because by Hölder,

$$|Tf| \leq \int_E fg\,dx \leq \|f\|_p \|g\|_q < \infty.$$

Weak convergence in L^p is weak enough to get compactness.

Theorem 14.2

If $\{f_n\}_n \in L^p(E)$ is bounded in $L^p(E)$, then there is a subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \rightharpoonup f$ in $L^p(E)$ for some $f \in L^p(E)$.

Proof. This is hard. If you assume the RRT for L^p spaces, you can find a proof on my blog. Otherwise, take Functional Analysis. □

But is weak $L^p(E)$ convergence too weak for sequential lower-semicontinuity? Nope! We can prove it is strong enough as follows.

Lemma 14.1

Let $(X, \|\cdot\|)$ be a normed space and $f : X \to \mathbb{R}$. Then f is convex iff f is the sup of a family $\mathcal{F}$ of continuous affine maps (i.e. continuous linear functions on X plus a constant).

Prototypical picture to have in your head: $f(x) = x^2$ is convex, and it is the sup of all its tangent lines.

Proof. ($\Longleftarrow$) Trivial.

($\Longrightarrow$) Google "Hahn-Banach". □

Theorem 14.3

$F : L^2(0,1) \to \mathbb{R}$ defined by $F(u) := \int_0^1 u^2\,dx$ is weakly sequentially lower semi-continuous ("wslsc").

Proof. F is convex, so it is the sup of some family $\mathcal{F}$ of affine maps. Take some $A \in \mathcal{F}$, so that $A(u) = \int_E uv\,dx + c$ for some continuous linear $u \mapsto \int_E uv\,dx$ (for some $v \in L^q(E)$) and some $c \in \mathbb{R}$.

Let $u_n \rightharpoonup u$ in $L^2(E)$. Then by definition of weak convergence,

$$A(u) = \int_E uv\,dx + c = \lim_{n\to\infty} \int_E u_n v\,dx + c = \lim_{n\to\infty} A(u_n).$$

On the RHS, we go up with $F(u)$ to get

$$A(u) \leq \liminf_{n\to\infty} F(u_n).$$

On the LHS, we take the sup over all $A \in \mathcal{F}$ to get

$$F(u) \leq \liminf_{n \to \infty} F(u_n).$$

□

With this, we can conclude by the direct method that F has a minimizer.

14.3 Back to Sobolev Spaces

Now let's try something less stupid:

$$F(u) := \int_0^1 u^2 + |u'|^2 \, dx$$

for $F : W^{2,2}(0,1) \to \mathbb{R}$. (We use $W^{2,2}$ instead of $W^{1,2}$ to make things a lot easier at the end.) Let's minimize this over the set $E := \{u \in W^{2,2}(0,1) : u(0) = 0, u(1) = 1\}$. (Of course, by something like $u(0) = 0$, I mean that $\tilde{u}(0) = 0$ where $\tilde{u}$ is the uniformly continuous representative of u.)

Let us run the usual program. Take $I = \inf_{u \in E} F(u)$. Take $u_n \in E$ with $F(u_n) \to I$. Then u_n is bounded in $L^2(0,1)$ and u_n' is bounded in $L^2(0,1)$. By weak compactness on both u_n and u_n', we have that there is a subsequence such that $u_{n_k} \rightharpoonup u$ in $L^2(0,1)$ and $u_{n_k}' \rightharpoonup v$ in $L^2(0,1)$.

We claim that $u \in W^{2,1}$ with $u' = v$. Indeed, since every $\varphi \in C_c^\infty(0,1)$ satisfies $\varphi, \varphi' \in L^2(0,1)$, we have by definition of weak convergence that

$$\int_0^1 u\varphi' \, dx \xleftarrow{k \to \infty} \int_0^1 u_{n_k} \varphi' \, dx = -\int_0^1 u_{n_k}' \varphi \, dx \xrightarrow{k \to \infty} -\int_0^1 v\varphi \, dx,$$

as needed.

(I'm putting this in parentheticals because it is a subtle but annoying issue: Do we have that $u \in E$? That is, does u satisfy the boundary conditions? Weak convergence does not necessarily preserve much pointwise data, so we can't quite conclude that it does... but we can modify this argument to that the boundary conditions are indeed preserved. The way to do this is as follows: Before we extract any subsequences, extract a subsequence of u_n that converges uniformly! We can do this because of Rellich-Kondrachov: $2 > 1$ and u_n is bounded in $W^{1,2}(0,1)$. Now everything works: Once we're done extracting sequences, u is not only the $W^{1,2}$-weak limit of the u_{n_k} — it must also be the uniform limit! And uniform limits are awesome at preserving pointwise data.)

Moreover, we claim that F is slsc under this so called "weak $W^{2,1}(0,1)$ convergence". That is, we claim that $u_{n_k} \rightharpoonup u$ and $u'_{n_k} \rightharpoonup u'$ in $L^2(0,1)$ implies that

$$F(u) \le \liminf_{k\to\infty} \int_0^1 u_{n_k}^2 + |u'_{n_k}|^2\,dx.$$

The handwavy proof is that "F is a sum of two convex functions so obviously it's true". Though we *do* need to do a little bit to tame the liminf. For the actual proof: let L be the liminf, and first extract a subsequence (not relabelled) so that the limit

$$L_1 := \lim_{k\to\infty} \int_0^1 u_{n_k}^2\,dx$$

exists in $[0,\infty]$, and extract another subsequence (also not relabelled) so that the limit

$$L_2 := \lim_{k\to\infty} \int_0^1 |u'_{n_k}|^2\,dx$$

exists in $[0,\infty]$. Then $L_1 + L_2 = L$. Since $v \mapsto \int_0^1 |v|^2$ is a convex functional on $L^2(0,1)$, we then have that

$$\int_0^1 u^2\,dx \le \liminf_{k\to\infty} \int_0^1 u_{n_k}^2\,dx = L_1$$

and

$$\int_0^1 |u'|^2\,dx \le \liminf_{k\to\infty} \int_0^1 |u'_{n_k}|^2\,dx = L_2.$$

Adding, we get $F(u) \le L_1 + L_2 = L$ as needed.

From all of the above, we may now conclude that F has a minimizer in $u \in W^{2,2}(0,1)$ (note that this reasoning would still have worked in $W^{1,2}(0,1)$. The extra regularity will come in soon.) But what is it?

14.4 Hunting Down the Minimizer: The Euler-Lagrange Equation

Since u is the minimizer, we must have that $F(u+t\varphi)$ obtains a relative minimum at $t=0$, where $\varphi \in C_c^\infty(0,1)$. That is, the directional derivative of F in the direction of φ is 0, which means that

$$0 = \lim_{t\to 0} \frac{F(u+t\varphi) - F(u)}{t} = \lim_{t\to 0} \int_0^1 2u\varphi + 2u'\varphi' + t\varphi^2 + t|\varphi'|^2\,dx = \int_0^1 2u\varphi + 2u'\varphi'\,dx,$$

so

$$\int_0^1 u\varphi + u'\varphi'\,dx = 0 \qquad \forall \varphi \in C_c^\infty.$$

How shall we proceed? First let us separate the terms and integrate by parts to get

$$0 = \int_0^1 u\varphi\,dx + \int_0^1 u'\varphi'\,dx = \int_0^1 u\varphi\,dx - \int_0^1 u''\varphi\,dx = \int_0^1 (u-u'')\varphi\,dx.$$

Note that here we needed that $u \in W^{2,2}(0,1)$! Otherwise we can't take a second derivative of u.

We now apply the Fundamental Lemma of Calculus of Variations.

Lemma 14.2

If $v \in L^1_{\text{loc}}(I)$ and $\int_I v\varphi\,dx = 0$ for all $\varphi \in C_c^\infty(I)$, then $v = 0$.

This was proven in Recitation 11.

Since $\int_0^1 (u-u'')\varphi\,dx = 0$ for all $\varphi \in C_c^\infty(0,1)$, it follows by the fundamental lemma that $u - u'' = 0$. This is a differential equation! And, since $u \in E$, it satisfies boundary data $u(0) = 0$ and $u(1) = 1$.

By the 8th recitation, we see that u must take the form $u(x) = c_1e^x + c_2e^{-x}$, and plugging in the boundary conditions gives the solution $\boxed{u(x) = \frac{e}{e^2-1}\cdot(e^x - e^{-x})}$. Note that this is the only solution, so **this is the unique minimizer**.

Thus, by using various things that we have learned throughout the semester, we have rigorously shown that the minimum

$$\min_{v\in E} F(v)$$

exists, and is given by $F(u) = \boxed{\frac{e^2+1}{e^2-1}}$.

14.5 Regularity Reduction

We see that we needed to ensure $u \in W^{2,2}(0,1)$, otherwise we could not have obtained a second-order differential equation for u. Is there hope if we instead did everything in $W^{1,2}(0,1)$?

Yes, there is. All of the logic in the past two sections will work up to obtaining the following equation:

$$\int_0^1 u\varphi + u'\varphi'\,dx = 0 \qquad \forall\varphi \in C_c^\infty.$$

If we knew only that $u \in W1,2(0,1)$, we now have to use a different trick. Let us instead integrate by parts on the *other* term! That is, let $U(x) := \int_0^x u(t)\,dt$ be a primitive for u.

Then

$$0 = \int_0^1 u\varphi\,dx + \int_0^1 u'\varphi'\,dx = -\int_0^1 U\varphi'\,dx + \int_0^1 U''\varphi'\,dx = \int_0^1 (U'' - U)\varphi'\,dx$$

for all $\varphi \in C_c^\infty(0,1)$. Now we pull out a totally different lemma.

Lemma 14.3 (Dubois-Reymond)

Suppose that $v \in L^1_{\text{loc}}(I)$ and that $\int_I v\varphi'\,dx = 0$ for all $\varphi \in C_c^\infty(I)$. Then v is constant.

Proof. See Lemma 4 in https://mccuan.math.gatech.edu/courses/7581/notes/lecture3.pdf. □

Thus $U'' - U = c$ for a constant c, and we have the boundary conditions $U'(0) = 0$, $U'(1) = 1$, and $U(0) = 0$. One way to solve this is to multiply by U' to get a separable equation. Whatever you do, you'll end up with the same solution for $u(x)$ — this time witout the assumption that $u \in W^{2,2}(0,1)$. Thus we have shown rigorously that the problem

$$\min_{u \in W^{1,2}(0,1), u(0)=0, u(1)=1} \int_0^1 |u|^2 + |u'|^2\,dx$$

admits a solution.