HOMEWORK 7
MATH-UA 0248-001 THEORY OF NUMBERS
due on Nov, 9, 2020

1. Show that if $n \equiv 3$ or $6 \pmod 9$, then n cannot be represented as a sum of two squares.

2. Write the integer $39690 = 2 \cdot 3^4 \cdot 5 \cdot 7^2$ as a sum of two squares.

3. Let a and b be relatively prime integers with $b > 1$ odd. If $b = p_1 p_2 \dots p_r$ is a decomposition of b into odd primes (not necessarily distinct), then the *Jacobi symbol* is defined by $(\frac{a}{b}) = (\frac{a}{p_1})(\frac{a}{p_1}) \cdots (\frac{a}{p_r})$, where the symbols on the right-hand side of the equality sign are Legendre symbols.

   (a) Evaluate the Jacobi symbol$(\frac{21}{221})$.

   (b) Show that if a is a quadratic residue mod b then $(\frac{a}{b}) = 1$, but that the converse is false.

   (c) Let a, b, a', b' be integers with $(aa', bb') = 1$. Prove that

   $$(\frac{aa'}{b}) = (\frac{a}{b}) \cdot (\frac{a'}{b}) \text{ and } (\frac{a}{bb'}) = (\frac{a}{b}) \cdot (\frac{a}{b'}).$$

   (d) Prove that $(\frac{a^2}{b}) = 1$ and that $(\frac{a}{b^2}) = 1$.

   (e) Prove that $(\frac{-1}{b}) = (-1)^{\frac{b-1}{2}}$ (Hint: If u and v are odd integers, then $\frac{u-1}{2} + \frac{v-1}{2} \equiv \frac{uv-1}{2} \pmod 2$).

   (f) Prove the Generalized Quadratic Reciprocity Low: if a, b are odd integers, each greater than 1, with $(a, b) = 1$, then

   $$(\frac{a}{b})(\frac{b}{a}) = (-1)^{\frac{a-1}{2} \cdot \frac{b-1}{2}}.$$

   (use the same hint as in the previous question)