Algebra I. Homework 6. Due on November 5, 2020.

1. Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

2. Let K be a field, let $P \in K[x]$ be a polynomial of degree 3, and let L be the splitting field of L over K. What are the possible values of $[L : K]$? Give a proof that you list all possible values, and for each value give an example.

3. Let $K = \mathbb{Q}(x)$ be the field of rational functions in one variable x over $\mathbb{Q}$. Let $K_1 \subset K$ be the field $K_1 = \mathbb{Q}(x^2)$ (of rational functions in x^2) and let $K_2 = \mathbb{Q}(x^2 - x)$.

   (a) Show that K is algebraic over K_1 and K_2 (write explicitly the minimal polynomial of $x \in K$ over these fields).

   (b) Show that K is not algebraic over $K_1 \cap K_2$ (compute $K_1 \cap K_2$).

4. Let $\mathbb{F}_{p^n}$ be a finite field with p^n elements. Write $\mathbb{F}_9$ as a splitting field of an irreducible polynomial over $\mathbb{F}_3$.

5. Let k be a field. Let $J = (x^2 + y^2 - 1, y - 1) \subset k[x, y]$ be the ideal generated by $x^2 + y^2 - 1$ and $y - 1$.

   (a) Find $V(J)$;

   (b) Find a function $f \in I(V(J))$ such that f is not in J.

   (c) Describe $I(V(J))$.