MATH-GA 2420.006 : Homework 1, due by Februry 8 morning (before 10am); send the solutions to pirutka@cims.nyu.edu

1. Show that the cubic curve $Y^2Z = X^3+AXZ^2+BZ^3$ is smooth iff $4A^3+27B^2 \neq 0$.

2. Let k be an algebraically closed field.

   (a) Let E be an elliptic curve over k given by the equation $y^2 = x^3+Ax+B$. Show that $\phi_1 : E(k) \to E(k), (x, y) \mapsto (x, -y)$ is is a group morphism.

   (b) Let E be an elliptic curve over k given by the equation $y^2 = x^3+B$. Show that $\phi_2 : E(k) \to E(k), (x, y) \to (\zeta x, -y)$, where $\zeta^3 = 1$ is a primitive root of unity is a group morphism.

   (c) Let E be an elliptic curve over k given by the equation $y^2 = x^3 + Ax$. Show that $\phi_3 : E(k) \to E(k), (x, y) \mapsto (-x, iy)$ is a group morphism.

3. **[j-invariant]** Let k be an algebraically closed field of characteristic different from 2 or 3. Let E be an elliptic curve given by an equation $y^2 = x^3+Ax+B$. Define the j-invariant of E by the formula

$$j = j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}.$$

   (a) Let $E_i, i = 1, 2$ be two elliptic curves given by the equations $y^2 = x^3 + A_ix + B_i$. Show that if $j(E_1) = j(E_2)$, then there exists $\mu \in K$, $\mu \neq 0$ such that $A_2 = \mu^4 A_1$ et $B_2 = \mu^6 B_1$.

   (b) Deduce that the map $x_2 = \mu^2 x_1$, $y_2 = \mu^3 y_1$ is a group isomorphism between $E_1(k)$ and $E_2(k)$.