These notes provide a short summary of the key concepts of electrostatics. This serves two purposes. First, it will help you make physical sense of some of the mathematical questions we will address in this course. Second, it will explain a good part of the terminology used in the field of integral equations, which is largely inspired from concepts in electrostatics.

All the derivations are inspired from J.P. Freidberg's notes on Electrostatics for the NSE class *22.105 Electromagnetic Interactions* at MIT.

Electrostatics is the field of electrodynamics that describes phenomena involving *time-independent* distributions of charges and fields.

1 Force between charged bodies – Coulomb's Law

1.1 Coulomb's law for two charges

The starting point for these notes is a law that cannot be proved mathematically but that is instead the result of a very large number of experiments and has never been proved wrong.

Coulomb's law : The force $\mathbf{F}$ on a point charge q_1 located at $\mathbf{r_1}$ due to another point charge q_2 located at $\mathbf{r_2}$ is given by

$$\mathbf{F}_{2\to 1} = k q_1 q_2 \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|^3} \tag{1}$$

k is a constant of proportionality that depends on the system of units used. In all my notes, I will use the SI system, in which $\mathbf{F}$ is written as

$$\mathbf{F}_{2\to 1} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|^3} \tag{2}$$

ϵ_0 is called the permittivity of free space, and its value is $\epsilon_0 \approx 8.854 \times 10^{-12}$ farad per meter. The SI unit of charge is the coulomb (C).

- As can be seen in Equation (2), for two charges of the same sign the force $\mathbf{F}$ is repulsive while for two charges of opposite sign, the force is attractive.
- We can also readily see that $|\mathbf{F}_{2\to 1}| = |\mathbf{F}_{1\to 2}|$: the magnitude of the force q_1 exerts on q_2 is equal to the magnitude of the force q_2 exerts on q_1, in agreement with Newton's third law of motion.

1.2 Coulomb's law for several charges

It is also experimentally verified that the total force produced on one charge by a number of other charges is the *vector sum* of the individual two-body forces of Coulomb. In particular, if there are three charges q_1, q_2 and q_3, the force on q_1 due to q_2 and q_3 is

$$\mathbf{F} = \mathbf{F}_{2\to 1} + \mathbf{F}_{3\to 1} = q_1 \left(\frac{q_2}{4\pi\epsilon_0} \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|^3} + \frac{q_3}{4\pi\epsilon_0} \frac{\mathbf{r_1} - \mathbf{r_3}}{|\mathbf{r_1} - \mathbf{r_3}|^3} \right) \tag{3}$$

Note that the reason why in Equation (3) we chose the particular factorization in which q_1 is taken outside the parenthesis but not $1/4\pi\epsilon_0$ will be apparent in the next section.

2 Electric field

A useful way to think about the Coulomb force is to imagine that each charge produces a "Coulomb force field" which acts on any charges present. For the situation given in Equation (2) we can say that the electric field at any point $\mathbf{r}$ due to the charge q_2 is

$$\mathbf{E}(\mathbf{r}) = \frac{q_2}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r_2}}{|\mathbf{r} - \mathbf{r_2}|^3} \tag{4}$$

The force on a point charge q located at $\mathbf{r}$ due to that electric field then is

$$\mathbf{F} = q\mathbf{E}(\mathbf{r}) \tag{5}$$

In particular, for the charge q_1 considered in Equation (2) we have

$$\mathbf{F}_{2\to 1} = q_1\mathbf{E}(\mathbf{r_1}) = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r_1}-\mathbf{r_2}}{|\mathbf{r_1}-\mathbf{r_2}|^3}$$

The introduction of $\mathbf{E}$ would not be particularly useful if we only considered situations with two charges. It becomes very convenient when we consider a collection of charges. Then, the property of vector addition for the Coulomb force means that the electric field due to a collection of charges is the sum of the electric fields due to each charge separately :

$$\mathbf{E}(\mathbf{r}) = \sum_k \frac{q_k}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r_k}}{|\mathbf{r}-\mathbf{r_k}|^3} \tag{6}$$

The particular factorization chosen in Equation (3) now makes sense. The electric field at any point $\mathbf{r}$ due to the presence of the charges q_2 and q_3 is

$$\mathbf{E}(\mathbf{r}) = \frac{q_2}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r_2}}{|\mathbf{r}-\mathbf{r_2}|^3} + \frac{q_3}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r_3}}{|\mathbf{r}-\mathbf{r_3}|^3} \tag{7}$$

3 Continuous charge distributions

3.1 Electric field for a continuous charge distribution

In many situations of physical interest, we study macroscopic systems for which the number of charged particles is of the order of 10^{20} or more, and one does not have detailed knowledge of the charge strength and location at the microscopic level, but only so at the mesoscopic level. In these situations, a sound approach is to replace the 10^{20} charges with a smooth, smeared out, continuous distribution of charge, called the charge density, often written ρ, with the units Coulomb$/m^3$.

When we replace discrete charges with a smeared-out charge distribution, the following transformations have to be made to calculate the electric field

$$\sum_k \quad \rightarrow \quad \int$$

$$q_k \quad \rightarrow \quad \rho(\mathbf{x'})d\mathbf{x'}$$

so that the electric field for a continuous charge distribution is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}^{'})\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}d\mathbf{r}^{'} \tag{8}$$

Note that a set of discrete charges as discussed in the previous section can also be described with a charge density ρ by means of delta functions. For example, for the three charges q_1, q_2 and q_3 discussed above, at the locations $\mathbf{r_1}$, $\mathbf{r_2}$ and $\mathbf{r_3}$, we can write

$$\rho(\mathbf{r}) = \sum_{k=1}^{3} q_k\delta(\mathbf{r}-\mathbf{r_k})$$

3.2 The Scalar Potential

Note that

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}^{'}|}\right) = -\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3} \tag{9}$$

This means that Eq.(8) can be written as :

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}^{'})\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}^{'}|}\right)d\mathbf{r}^{'}$$

Since the gradient operates on $\mathbf{r}$ but not on the integration variable $\mathbf{r}^{'}$, we can take it out of the integral.

$$\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\nabla\int\frac{\rho(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}d\mathbf{r}^{'} \tag{10}$$

Eq.(10) then allows us to define the scalar potential ϕ

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}d\mathbf{r}^{'} \tag{11}$$

such that

$$\mathbf{E} = -\nabla\phi \tag{12}$$

Eq.(12) obviously implies that the vector $\mathbf{E}$ satisfies

$$\nabla\times\mathbf{E} = \mathbf{0} \tag{13}$$

4 Differential form for the equations of electrostatics

If, for computational reasons, we wanted to avoid the integral expressions given above, and solve for $\mathbf{E}$ by means of differential relations, Eq.(13) would not be enough to determine the electric field. We now show that an additional relation for $\mathbf{E}$ can easily be obtained from Eq.(8).

We start by observing that, for $\mathbf{r}\neq\mathbf{r}^{'}$,

$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right) = \frac{3}{|\mathbf{r}-\mathbf{r}^{'}|^3} - 2\frac{3}{2}\frac{\left(\mathbf{r}-\mathbf{r}^{'}\right)\cdot\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^5} = 0 \tag{14}$$

Using this information, one might at first think that $\nabla\cdot\mathbf{E} = 0$. This is incorrect, because Eq.(14) is ill-defined when $\mathbf{r} = \mathbf{r}^{'}$. In order to evaluate $\nabla\cdot\mathbf{E}$ properly, we have to use the divergence theorem, and the concept of delta functions. The key idea is to calculate the following integral :

$$\int\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)d\mathbf{r}^{'}$$

Because of Eq.(14), there will be no contribution to the integral anywhere except when $\mathbf{r} = \mathbf{r}^{'}$. Thus, we can calculate this integral over a sphere centered in $\mathbf{r}$, of arbitrary radius R, and use the spherical coordinates $(u^{'}, \theta, \phi)$, where $u^{'} = |\mathbf{r}-\mathbf{r}^{'}|$. According to the divergence theorem, we have :

$$\int\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)d\mathbf{r}^{'} = -\int\nabla^{'}\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)d\mathbf{r}^{'} = -\int\mathbf{n}^{'}\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)dS^{'} \tag{15}$$

where $\mathbf{n}^{'}$ represents the local unit normal vector to the sphere surface. In the spherical coordinates we introduced, we have

$$\int\mathbf{n}^{'}\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)dS^{'} = -\int_0^{\pi}\int_0^{2\pi}\frac{u^{'}}{u^{'3}}u^{'2}\sin\theta d\theta d\phi = -4\pi \tag{16}$$

So that

$$\int\mathbf{n}^{'}\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)dS^{'} = 4\pi \tag{17}$$

Combining Eq.(14) and Eq.(17), we conclude :

$$\nabla\cdot\left(\frac{\mathbf{r}-\mathbf{r}^{'}}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right) = 4\pi\delta(\mathbf{r}-\mathbf{r}^{'}) \tag{18}$$

or, using Eq.(9),

$$\nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{19}$$

where δ is the Dirac delta function.

Using Eq.(18), we obtain the desired differential form of Gauss's law for $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = \frac{4\pi}{4\pi\epsilon_0} \int \rho(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')d\mathbf{r}' = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{20}$$

And finally Eq.(20) can be expressed as an equation for the electrostatic potential :

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{21}$$

5 Summary of electrostatics

The goal in electrostatics is to determine the potential ϕ or equivalently the electric field $\mathbf{E}$ due to a distribution of fixed charges that do not vary in time. The relationship between $\mathbf{E}$ and ϕ is $\mathbf{E} = -\nabla\phi$, and $\mathbf{E}$ and ϕ are determined in terms of the charge distribution ρ according to the following formulas

— **Integral formulation**

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'$$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

— **Differential formulation**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

— **Boundary conditions**

The formulations described above are not strictly equivalent, because of boundary conditions one may give on $\mathbf{E}$ of ϕ.

By construction, the integral expression only applies to the calculation of the electric field and electrostatic potential for *free space boundary conditions*. Other boundary conditions would require a modification of the formulation, in which the solution ϕ^{tot} would be written as the sum of the free space solution ϕ^{fs} and of a homeogenous solution ϕ^{h} satisfying $\nabla^2\phi = 0$ and constructed such that ϕ^{tot} satisfies the desired boundary conditions.

In contrast, the differential formulation applies locally, and is therefore correct for any boundary condition, which need to be specified for the solution to Poisson's equation to be unique.

The boundary conditions depend on the physics problem of interest. A common boundary condition is that of a *perfect conductor*. The electric field inside a perfect conductor is identically zero, which means that ϕ is a constant inside a perfect conductor. Thus, in the presence of a perfect conductor, one wants to solve the Dirichlet boundary value problem :

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \quad \text{in } \Omega \qquad , \qquad \phi = K \text{ on } \partial\Omega$$

where K is a constant.

6 Free space Green's function for Laplace's equation

In the light of Eq.(19) and Eq.(11), we see that we may write the electrostatic potential as the convolution

$$\phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int \rho(\mathbf{r}')G_L(\mathbf{r}, \mathbf{r}')d\mathbf{r}' \tag{22}$$

where

$$G_L(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{23}$$

is the free space Green's function for Laplace's equation.

The convolution in Eq.(22) takes a general form, which we will use for free-space solutions of other partial differential equations. What will change, for these other partial differential equations, is the form of the Green's function G.

Note that for Poisson's equation in two-dimensions, we may write

$$\phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int \rho(\mathbf{r}')G_L(\mathbf{r}, \mathbf{r}')d\mathbf{r}' \quad , \quad G_L(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln(|\mathbf{r} - \mathbf{r}'|) \quad \text{Two-dimensional case} \tag{24}$$

7 Layer potentials

7.1 Single layer potential

Consider the expression (22), and consider the particular case in which the charge distribution is only nonzero on the surface $\partial\Omega$ of a domain Ω. In that case, the potential is given by the surface integral

$$\phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_{\partial\Omega} \rho(\mathbf{r}')G_L(\mathbf{r}, \mathbf{r}')dS(\mathbf{r}') \tag{25}$$

This layer potential is called a *single-layer potential*, an expression which we will frequently use in a much wider context in this course. We will write the expression above in the following concise manner

$$\phi(\mathbf{r}) = S_L\left[\rho\right] := \frac{1}{\epsilon_0} \int_{\partial\Omega} \rho(\mathbf{r}')G_L(\mathbf{r}, \mathbf{r}')dS(\mathbf{r}') \tag{26}$$

7.2 Double layer potential

Suppose now that the charge distribution is on two parallel surfaces separated by a distance δ, and that the distribution on the first surface is $\rho(\mathbf{r}')/\delta$ and the distribution on the parallel surface is $-\rho(\mathbf{r}')/\delta$, forming a dipole configuration. For any δ, the electrostatic potential due to this charge distribution is

$$\phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_{\partial\Omega} \rho(\mathbf{r}') \frac{[G_L(\mathbf{r}, \mathbf{r}') - G_L(\mathbf{r}, \mathbf{r}' + \mathbf{n}(\mathbf{r}')\delta)]}{\delta} dS(\mathbf{r}') \tag{27}$$

where $\mathbf{n}$ is the normal vector to the first surface. Taking the limit $\delta \to 0$, we obtain what we call the *double-layer potential*

$$\phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int_{\partial\Omega} \rho(\mathbf{r}')\mathbf{n}(\mathbf{r}') \cdot \nabla G_L(\mathbf{r}, \mathbf{r}')dS(\mathbf{r}') \tag{28}$$

which in this class we will write concisely as

$$\phi(\mathbf{r}) = D_L\left[\rho\right] := \frac{1}{\epsilon_0} \int_{\partial\Omega} \rho(\mathbf{r}')\mathbf{n}(\mathbf{r}') \cdot \nabla G_L(\mathbf{r}, \mathbf{r}')dS(\mathbf{r}') \tag{29}$$