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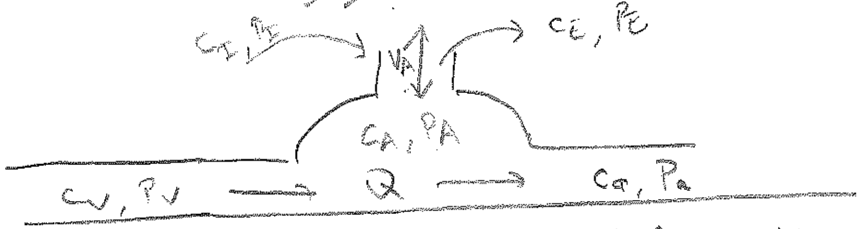
(Optional) gas transport in the lung

Recall our model of gas exchange inside an alveolus



Last time we wrote down model equations and derived

$$c_A = \frac{r c_I + c_v}{r + \sigma k T}$$

$c_a = \sigma k T c_A$ where $r = V_A / Q$ is the so called "ventilation to perfusion ratio".

What changes when we consider multiple alveoli?

Model inputs: $c_I, c_v, P_I, P_v, \boxed{V_A}, \boxed{Q}$

Effects of gravity: less blood reaches higher parts of lung. Lower part of lung contract more when breathing out

Let $r_i = (V_A)_i / Q_i$, where i indexes i^{th} alveolus

Then analogously

$$(c_A)_i = \frac{r_i c_I + c_v}{r_i + \sigma k T} \qquad (c_a)_i = \sigma k T (c_A)_i$$

Net transport (flux) of the gas

$$f_i = Q_i((c_a)_i - c_v) = (V_A)_i(c_I - (c_A)_i)$$

$$= (V_A)_i\left(-\frac{r_i c_I + c_v}{r_i + \sigma k T} + \cancel{\frac{c_I r_i}{r_i + \sigma k T}} + \frac{c_I \sigma k T}{r_i + \sigma k T}\right)$$

$$= (V_A)_i\left(\frac{-c_v + c_I \sigma k T}{r_i + \sigma k T}\right)$$

$$= (V_A)_i\left(\frac{-\sigma P_v + c_I \sigma k T}{r_i + \sigma k T}\right)$$

$$= (V_A)_i \sigma \frac{(-P_v + c_I k T)}{r_i + \sigma k T}$$

$$= Q_i r_i \sigma \frac{(P_I - P_v)}{r_i + \sigma k T}$$

The total flux is therefore

$$f = \sum_i f_i = \sigma(P_I - P_v)\sum_i\left(\frac{Q_i r_i}{r_i + \sigma k T}\right)$$

Suppose that all the venous blood directly contacted the inspired air then what would be the transport of gas?

$$\underset{c_v \quad Q_0 \quad c_a}{\text{venous blood}} \qquad P_a = P_I \to \quad P_a = kTc_I$$
$$c_a = \sigma P_a = \sigma kT c_I$$

$$f_{max} = Q_0(c_a - c_v) = Q_0(\sigma P_a - \sigma P_v)$$
$$= Q_0(\sigma P_I - \sigma P_v) = Q_0\sigma(P_I - P_v).$$

Let $Q_0 = \sum_i Q_i \qquad (\dot{V}_A)_0 = \sum (\dot{V}_A)_i$

Then define an "efficiency"

$$E = \frac{f}{f_{max}}$$

$$E = \frac{\sum_i f_i}{} = \frac{\cancel{\sigma(P_I - P_v)} \sum_i \frac{Q_i r_i}{(\sigma kT + r_i)}}{\cancel{\sigma(P_I - P_v)} Q_0}$$

$$= \frac{1}{Q_0} \sum_i \frac{Q_i r_i}{(\sigma kT + r_i)}$$

Claim: E is maximal when $r_i = r_0 = \frac{\sum (\dot{V}_A)_i}{\sum Q_i} = \frac{(\dot{V}_A)_0}{Q_0}$

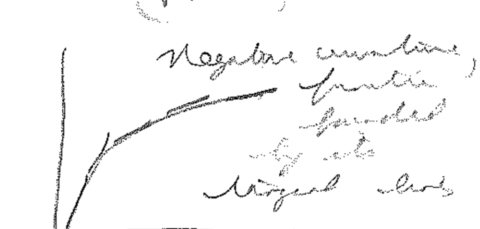
Proof: Consider the function

$$g(r) = \frac{r}{r + \sigma kT}$$

$$g'(r) = \frac{r + \sigma kT - r}{(r + \sigma kT)^2} = \frac{\sigma kT}{(r + \sigma kT)^2} \qquad g''(r) = \frac{-2\sigma kT}{(r + \sigma kT)^3} < 0$$

So, because curvature is negative

$$g(r_i) \le g(r_0) + (r_i - r_0)\, g'(r_0).$$



thus

$$E = \frac{1}{Q_0} \sum_i Q_i\, g(r_i)$$

$$\le \frac{1}{Q_0} \sum_i g(r_0) + \frac{1}{Q_0} \sum_i Q_i (r_i - r_0)\, g'(r_0).$$

Exercise: show that $\sum_i Q_i (r_i - r_0) = 0$.

$$\sum_i (\dot{V}_A)_i - r_0 Q_0 \quad ✓.$$

Therefore, we have shown that

$E \le E(r_0)$; i.e. efficiency is maximal when all alveoli have the same ventilation to perfusion ratio.

In other words, inhomogeneities in the lungs degrade gas transport.

Another interesting manifestation of this: think about equilibrium between mean arterial blood and mean alveolar air. Define mean arterial blood concentration

Question: if you drive 60 mph for 4 hrs, and 30 mph for 1 hr, what is your average speed? $60 \cdot \frac{4}{5} + 30 \cdot \frac{1}{5} = \frac{270}{5} = 54$ mph

In exactly the same way:

$$\langle c_a \rangle = \frac{1}{\sum Q_i} \sum_i (c_a)_i Q_i = \frac{1}{Q_0} \sum_i (c_a)_i Q_i$$

$$\langle c_A \rangle = \frac{1}{(\dot V_A)_0} \sum_i (\dot V_A)_i (c_A)_i = \frac{1}{Q_0 r_0} \sum_i r_i Q_i (c_A)_i$$

$$\langle P_a \rangle = \langle c_a \rangle / \sigma$$

$$\langle P_A \rangle = \langle c_A \rangle kT.$$

In a single alveolus, $P_A = P_a$. Is this true for the mean pressure also?

$$\langle P_A \rangle - \langle P_a \rangle = \frac{kT}{r_0 Q_0} \sum_i r_i Q_i (c_A)_i - \frac{1}{\sigma Q_0} \sum_i (c_a)_i Q_i$$

$$= \frac{1}{Q_0} \left[\sum_i \frac{r_i}{r_0} Q_i kT (c_A)_i - \frac{1}{\sigma} \sum_i (c_a)_i Q_i \right]$$

Exercise: Substitute $(c_A)_i = \frac{r_i c_I + c_v}{r_i + \sigma kT}$ $\quad (c_a)_i = \sigma kT (c_A)_i$

and simplify:

$$= \frac{kT}{Q_0} \sum_i Q_i (c_A)_i \left(\frac{r_i}{r_0} - 1 \right)$$

$$= \frac{kT}{Q_0} \sum_i Q_i \left(\frac{r_i c_I + c_v}{r_i + \sigma kT} \right) \left(\frac{r_i}{r_0} - 1 \right).$$

$\boxed{\text{HINT:}}$ $\frac{r_i c_I + c_v}{r_i + \sigma kT} = \frac{r_0 c_I + c_v}{r_0 + \sigma kT} + (r_i - r_0) \frac{\sigma kT c_I - c_v}{(r_i + \sigma kT)(r_0 + \sigma kT)}$

$$\langle P_A \rangle - \langle P_a \rangle = \cancel{\frac{kT}{\mathcal{D}_0} \sum_i \mathcal{D}_i \left(\frac{r_0 c_I + c_V}{r_0 + \sigma kT} \right)} \left(\frac{r_i}{r_0} - 1 \right)$$

$$+ \frac{kT}{\mathcal{D}_0} \sum_i (r_i - r_0) \left(\frac{r_i}{r_0} - 1 \right) \left[\frac{\sigma kT c_I - c_V}{(r_i + \sigma kT)(r_0 + \sigma kT)} \right]$$

First line $\dfrac{\sum \mathcal{D}_i r_i}{r_0} = 1$.

$$\langle P_A \rangle - \langle P_a \rangle = \frac{kT}{r_0 \mathcal{D}_0} \sum_i (r_i - r_0)^2 \left(\frac{\sigma kT c_I - c_V}{(r_i + \sigma kT)(r_0 + \sigma kT)} \right)$$

≥ 0 unless $r_i = r_0$

Thus $\langle P_A \rangle = \langle P_a \rangle$ only if $r_i = r_0$. Otherwise, $\langle P_A \rangle - \langle P_a \rangle$ has same sign as $c_I - c_V$. It's not all getting into the blood!