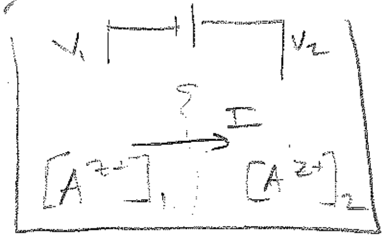
10/27/2022

Adding electrical effects to cell volume model

In reality, the cell has a voltage because of the trapped negative ions. This voltage produces a current, which changes the concentration of the ions.

To model this, let's consider a simplified situation where a battery is used to drive current:



$V = V_1 - V_2$
= work per unit charge to move ion from $1 \to 2$.

The relationship between I and V is what we will consider here. As before, we will use energy balance.

1) Work performed by battery

$$\dot{W}_{battery} = I_A (V_1 - V_2) = I_A V.$$

Definition of potential (voltage) = work / unit charge.

2) Rate at which work is required to drive current through membrane is

$$\dot{W}_{membrane} = R_A I_A^2,$$ where R_A is the resistance of the membrane to current.

3) Suppose we had an ideal gas. What is rate of work required to change concentration?

$$\dot{W}_{conc} = -\int_{①}^{②} P \, d\dot{V} = -\int_{①}^{②} ckT \, d\left(\frac{\dot{n}}{c}\right) = +\int_{①}^{②} \frac{kT\dot{n}}{c} \, dc.$$

$$\dot{W}_{conc} = \dot{n} kT \ln\left(\frac{c_2}{c_1}\right) = \left(\frac{I_A}{zq}\right) kT \ln\left(\frac{[A^{z+}]_2}{[A^{z+}]_1}\right)$$

By conservation of energy

$$\dot{W}_{battery} = \dot{W}_{membrane} + \dot{W}_{conc}$$

$$I_A V = R_A I_A^2 + \frac{I_A}{zq} kT \ln\left(\frac{[A^{z+}]_2}{[A^{z+}]_1}\right)$$

Exercise: Solve for I_A and find the voltage at which it is 0.

$$I_A = \underbrace{\frac{1}{R_A}}_{g_A} \left(V - \frac{kT}{zq} \ln\left(\frac{[A^{z+}]_2}{[A^{z+}]_1}\right)\right)$$

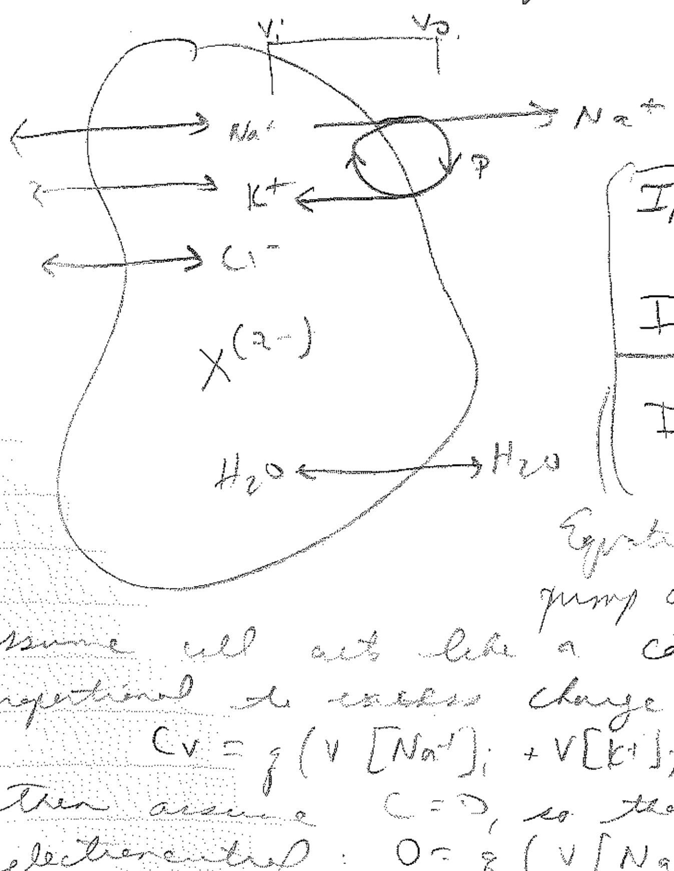
Current = 0 when

$$E_A = \frac{kT}{zq} \ln\left(\frac{[A^{z+}]_2}{[A^{z+}]_1}\right).$$

which is called the equilibrium potential. It is the voltage at which there is no current.
Idea: if there are more + ions in 2 than 1 and $V=0$, current flows in opposite direction to equilibrate. Need a voltage to counteract this which is the same sign. If $V>0$, current wants to go other way.

Applying to cell volume model

Now we make a new model where 1 is the inside and 2 is the outside of the cell.



Electrical effects add Cl⁻ ion to model.

$$I_{Na} = g_{Na}\left(V - \frac{kT}{q} \ln\left(\frac{[Na^+]_o}{[Na^+]_i}\right)\right) + pq$$

$$I_K = g_K\left(V - \frac{kT}{q} \ln\left(\frac{[K^+]_o}{[K^+]_i}\right)\right) - pq$$

$$I_{Cl} = g_{Cl}\left(V + \frac{kT}{q} \ln\left(\frac{[Cl^-]_o}{[Cl^-]_i}\right)\right)$$

Equations for before accounting for pump current. (assuming 1:1 pump)

Assume cell acts like a capacitor - membrane potential proportional to excess charge

$$Cv = q\left(v[Na^+]_i + v[K^+]_i - v[Cl^-]_i - xz\right)$$

Then assume $C=0$, so that the interior of the cell is electroneutral: $0 = q\left(v[Na^+]_i + v[K^+]_i - v[Cl^-]_i - xz\right)$

Osmotic flux of water

$$R_{H_2O} Q = kT\left([Na^+]_o - [Na^+]_i + [K^+]_o - [K^+]_i + [Cl^-]_o - [Cl^-]_i - \frac{X}{v}\right)$$

SKIP

Let N = total number of negative charges. Take limit $x \to 0$, with $z \to \infty$, for fixed N. Steady state eqns are

$$0 = g_{Na}\left(v - (kT/q)\ln\left([Na^+]_o/[Na^+]_i\right)\right) + pq$$

$$0 = g_K\left(v - (kT/q)\ln\left([K^+]_o/[K^+]_i\right)\right) - pq$$

$$0 = g_{Cl}\left(v + (kT/q)\ln\left([Cl^-]_o/[Cl^-]_i\right)\right)$$

$$0 = [Na^+]_i + [K^+]_i - [Cl^-]_i - N/V$$

$$0 = [Na^+]_i + [Cl^-]_i + [K^+]_i - [Na^+]_o - [K^+]_o - [Cl^-]_o$$

5 eqns in 5 unknowns: $[Na^+]_i$, $[K^+]_i$, $[Cl^-]_i$, V, v.

External concentrations ~~known, but external~~ solution must be electroneutral $\boxed{[Na^+]_o + [K^+]_o = [Cl^-]_o}$

The solution of these equations is

$$V = \frac{N}{2[Cl^-]_o\sqrt{1-\beta}}$$

$$v = \left(\frac{kT}{q}\right)\ln\left(1 - \sqrt{1-\beta}\right)$$

where $\beta = \dfrac{\beta_{Na}[Na^+]_o + \beta_K [K^+]_o}{[Na^+]_o + [K^+]_o}$, $\quad \beta_{Na} = \exp\left(\dfrac{-pq^2}{g_{Na}kT}\right)$, $\quad \beta_K = \exp\left(\dfrac{pq^2}{g_K kT}\right)$.

THE TAKEAWAY

$v < 0$: meaning current flows into the cell (positive charge).