Homework 1 Solution

(a) $$E = \frac{1}{Q_0} \sum_i \frac{Q_i r_i}{r_i + \sigma k_T} = \frac{1}{Q_1 + Q_2}\left(\frac{Q_1 r_1}{r_1 + \sigma k_T} + \frac{Q_2 r_2}{r_2 + \sigma k_T}\right)$$

$$= \frac{1}{2}\left(\frac{r_1}{r_1 + \sigma k_T} + \frac{r_2}{r_2 + \sigma k_T}\right)$$

(b) $r_1 = r_2$.

(c) $$c_a^* = \sigma k_T \left(\frac{r_1 c_I + c_V}{r_1 + \sigma k_T}\right)$$

(d) $$c_a = \sigma k_T \left(\frac{r_2 c_I + c_a^*}{r_2 + \sigma k_T}\right)$$

$$= \sigma k_T \frac{r_2 c_I}{r_2 + \sigma k_T} + (\sigma k_T)^2 \frac{r_1 c_I + c_V}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}$$

$$= \frac{\sigma k_T r_2 c_I (r_1 + \sigma k_T) + (\sigma k_T)^2 r_1 c_I + c_V}{(r_2 + \sigma k_T)(r_1 + \sigma k_T)}$$

$$= \frac{\sigma k_T r_2 r_1 c_I + (\sigma k_T)^2 (r_2 c_I + r_1 c_I) + c_V (\sigma k_T)^2}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}$$

(e) $$c_a - c_V$$

$$= c_a - \frac{(r_1 + \sigma k_T)(r_2 + \sigma k_T) c_V}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}$$

$$= \frac{r_2 r_1 (\sigma k_T c_I - c_V) + \sigma k_T (r_2 + r_1)(\sigma k_T c_I - c_V)}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}$$

$$= \sigma \left[\frac{r_2 r_1 (P_I - P_V) + \sigma k_T (r_2 + r_1)(P_I - P_V)}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}\right]$$

$$f = Q \sigma (P_I - P_V)\left[\frac{r_2 r_1 + \sigma k_T (r_2 + r_1)}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}\right]$$

(f) $$E = \frac{r_1 r_2 + \sigma k_T (r_2 + r_1)}{(r_1 + \sigma k_T)(r_2 + \sigma k_T)}$$

$$E_{series} = \frac{r_1 r_2 + \sigma kT(r_1 + r_2)}{(r_1 + \sigma kT)(r_2 + \sigma kT)}$$

Now $r_1 = r_1^{(1)} = \frac{1}{2} r^{(p)}$, $r_2 = r_2^{(1)} = \frac{1}{2} r^{(p)}$, so:

$$E_{series} = \frac{\frac{1}{4} r_1^{(p)} r_2^{(p)} + \frac{1}{2} \sigma kT (r_1^{(p)} + r_2^{(p)})}{\left(\frac{1}{2} r_1^{(p)} + \sigma kT\right)\left(\frac{1}{2} r_2^{(p)} + \sigma kT\right)}$$

$$E_{series} = \frac{r_1^{(p)} r_2^{(p)} + 2\sigma kT (r_1^{(p)} + r_2^{(p)})}{(r_1^{(p)} + 2\sigma kT)(r_2^{(p)} + 2\sigma kT)}$$

$$E_{parallel} = \frac{1}{2}\left(\frac{r_1}{r_1 + \sigma kT} + \frac{r_2}{r_2 + \sigma kT}\right)$$

$$= \frac{1}{2}\left[\frac{r_1 r_2 + r_1 \sigma kT + r_1 r_2 + r_2 \sigma kT}{(r_1 + \sigma kT)(r_2 + \sigma kT)}\right]$$

$$E_{parallel} = \frac{2 r_1 r_2 + \sigma kT (r_1 + r_2)}{(2 r_1 + 2\sigma kT)(r_2 + \sigma kT)}$$

$2 = r_1 = r_2$, $\sigma kT = 1$