What are SRB measures, and which dynamical systems have them?

To David and Yasha

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This is a slightly expanded version of the text of a lecture I gave in a conference at Rutgers University in honor of David Ruelle and Yasha Sinai. In this lecture I reported on some of the main results surrounding an invariant measure introduced by Sinai, Ruelle and Bowen in the 1970s. *SRB measures*, as these objects are called, play an important role in the ergodic theory of dissipative dynamical systems with chaotic behavior. Roughly speaking,

- *SRB measures are the invariant measures most compatible with volume when volume is not preserved;*
- *they provide a mechanism for explaining how local instability on attractors can produce coherent statistics for orbits starting from large sets in the basin.*

An outline of this paper is as follows.

The original work of Sinai, Ruelle and Bowen was carried out in the context of Anosov and Axiom A systems. For these dynamical systems they identified and constructed an invariant measure which is uniquely important from several different points of view. These pioneering works are reviewed in Section 1.

Subsequently, a nonuniform, almost-everywhere notion of hyperbolicity expressed in terms of Lyapunov exponents was developed. This notion provided a new framework for the ideas in the last paragraph. While not all of the previous characterizations are equivalent in this broader setting, the central ideas have remained intact, leading to a more general notion of SRB measures. This is discussed in Section 2.

The extension above opened the door to the possibility that the dynamics on many attractors are described by SRB measures. Determining if this is (or is not) the case, however, let alone proving it, has turned out to be very challenging. No genuinely nonuniformly hyperbolic examples were known until the early 1990s, when SRB measures were constructed for certain Hénon maps. Today we still do not have a good understanding of which dynamical systems admit SRB measures, but some progress has been made; a sample of it is reported in Section 3.

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It is now clear that away from Axiom A systems, the dynamical picture is necessarily very complex. This and other observations form my concluding remarks.

Finally, I close the introduction by stating the obvious, namely that this short note is in no way intended to be a complete treatment of SRB measures. A number of topics have been omitted. I mention in particular [PS2], [SW], [D2] and [R4], which contain promising ideas.

1 SRB measures for Axiom A attractors

We introduce the setting in which SRB measures were first conceived.

Definition 1.1 (a) *Let $f : M \to M$ be a diffeomorphism of a compact Riemannian manifold onto itself. We say f is an* **Anosov diffeomorphism** *if the tangent space at every $x \in M$ is split into $E^u(x) \oplus E^s(x)$ where E^u and E^s are Df-invariant subspaces, $Df|_{E^u}$ is uniformly expanding and $Df|_{E^s}$ is uniformly contracting.*

(b) *A compact f-invariant set $\Lambda \subset M$ is called an* **attractor** *if there is a neighborhood U of Λ called its* **basin** *such that $f^n x \to \Lambda$ for every $x \in U$. Λ is called an* **Axiom A attractor** *if the tangent bundle over Λ is split into $E^u \oplus E^s$ as above.*

To be precise, by the *uniform expanding* property of $Df|_{E^u}$, we mean there is a constant $\lambda > 1$ such that $||Df(v)|| \geq \lambda ||v||$ for every $v \in E^u$. Anosov diffeomorphims can be viewed as special cases of Axiom A attractors with $\Lambda = U = M$. In the discussion to follow, we will assume that E^u is nontrivial for the attractors in question, so that there is sensitive dependence on initial conditions or *chaos*. We will also assume that the attractor is *irreducible*, meaning Λ cannot be written as the union of two disjoint attractors. The main results on SRB measures for Axiom A attractors can be summarized as follows:

Theorem 1 ([S2], [R1], [R2], [B2], [K1]) *Let f be a C^2 diffeomorphism with an Axiom A attractor Λ. Then there is a unique f-invariant Borel probability measure μ on Λ that is characterized by each of the following (equivalent) conditions:*

(i) *μ has absolutely continuous conditional measures on unstable manifolds;*

(ii)
$$h_\mu(f) = \int |\det(Df|_{E^u})| d\mu$$
where $h_\mu(f)$ is the metric entropy of f;

(iii) *there is a set $V \subset U$ having full Lebesgue measure such that for every continuous observable $\varphi : U \to \mathbb{R}$, we have, for every $x \in V$,*
$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i x) \ \to \ \int \varphi d\mu \ ;$$

(iv) μ *is the zero-noise limit of small random perturbations of* f.

The invariant measure μ in Theorem 1 is called the **Sinai-Ruelle-Bowen measure** or **SRB measure** of f. There are analogous results for flows but for simplicity we will limit our exposition to the discrete-time case. We wish now to explain informally the meanings of (i)–(iv), postponing technical definitions to Section 2 where a broader class of dynamical systems is treated. More detailed attributions are given after these explanations.

In a neighborhood of an attractor, the defining map is often volume decreasing, ruling out the possibility of an invariant measure equivalent to volume. The meaning of (i) is that every Axiom A attractor admits an invariant measure with a density in the unstable direction. (In the stable direction, this measure is usually singular with respect to Lebesgue measure.)

(ii) expresses the following variational principle: *Metric entropy* or *Kolmogorov-Sinai entropy* is a measure of dynamical complexity for measure-preserving transformations. For differentiable maps, there is another measure of dynamical complexity, namely the expanding part of the derivative. For Axiom A attractors it was shown that the first of these quantities is always dominated by the second, and μ is the unique invariant measure for which these two quantities coincide.

If one adopts the view that positive Lebesgue measure sets correspond to observable events, then (iii) expresses the fact that μ can be "observed", for it governs the asymptotic distributions of orbits starting from Lebesgue-a.e. initial condition. This property does not follow from the Birkhoff Ergodic Theorem: The support of μ is contained in Λ, which for a genuinely volume decreasing attractor has Lebesgue measure zero, yet it controls the behavior of orbits starting from an open set.

Here is what we mean by "**zero-noise limit**": Let $P^\varepsilon(\cdot|\cdot)$, $\varepsilon > 0$, be a family of Markov chains whose transition probabilities have densities with some regularity properties and which satisfy $P^\varepsilon(\cdot|x) \to \delta_{fx}$, the Dirac measure at fx, as $\varepsilon \to 0$. (iv) says that the stationary measures for P^ε converge to μ as $\varepsilon \to 0$. If one accepts that the world is intrinsically a little noisy, then zero-noise limits are the observable invariant measures.

SRB measures have their origins in statistical mechanics. Here is a very brief account of how (i)–(iv) came about. In 1968, Sinai constructed for Anosov diffeomorphisms certain partitions called *Markov partitions* [S1]. These partitions enabled him to identify points in the phase space with configurations in one-dimensional lattice systems. In [S2], he developed for Anosov systems a Gibbs theory analogous to that in statistical mechanics. SRB measures are special cases of Gibbs measures. They are defined by the potential $-\log|\det(Df|_{E^u})|$ or, equivalently, by the fact that they have smooth conditional measures on unstable manifolds. This point of view is reflected in (i). At about the same time, Bowen extended the construction of Markov partitions to Axiom A attractors [B1]. Ruelle brought earlier works in statistical mechanics to bear on Axiom A theory, notably the variational principle and the notion of equilibrium states [R1]. He also emphasized the connection of SRB measures to Lebesgue measure on the ambient manifold [R2], [BR]. These points of view are reflected in (ii) and (iii); see also [B2] for an exposition. The idea that zero-noise limits represent measures that are truly physically observable was expressed by Kolmogorov. That this property is equivalent to (i) is proved formally by Kifer [Ki]; see also [Y1].

2 A more general notion of SRB measures

In this section we discuss extensions of the four properties in Theorem 1 to more general dynamical systems. As before, we will limit ourselves to the discrete-time case. Also, to avoid technicalities, we consider only diffeomorphisms, i.e. differentiable maps with differentiable inverses, leaving aside noninvertible maps, maps with singularities etc. for which some versions of the results discussed also hold. Throughout this section, let $f : M \to M$ be a C^2 diffeomorphism of a compact Riemannian manifold to itself.

2.1 Conditional measures on W^u and the entropy formula

Recall the definition of Lyapunov exponents: For $x \in M$ and $v \in T_xM$, let

$$\lambda(x, v) \;=\; \lim_{n\to\pm\infty} \frac{1}{n} \log \|DT^n_x(v)\|$$

if these limits exist and coincide. A theorem due to Oseledec [O] states that if μ is an f-invariant probability on M, then there exist measurable functions λ_i such that at μ-a.e. x, the tangent space $T_xM = \oplus E_i(x)$ where $\lambda(x, v) = \lambda_i(x)$ for $v \in E_i(x)$. The λ_i are called the **Lyapunov exponents** of (f, μ). We remark that since $Df(E_i(x)) = E_i(fx)$, the functions $x \mapsto \lambda_i(x)$ and $\dim E_i(x)$ are constant μ-a.e. if (f, μ) is ergodic.

Suppose now that (f, μ) has a positive (respectively negative) Lyapunov exponent a.e. Then unstable (resp. stable) manifolds are well defined a.e. More precisely, for $x \in M$, let

$$W^u(x) \;:=\; \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) < 0\}.$$

A well known fact from nonuniform hyperbolic theory [P1] states that at μ-a.e. x, $W^u(x)$ is an immersed submanifold tangent at x to $\oplus_{\lambda_i>0} E_i(x)$. It is called the **unstable manifold** at x. The **stable manifold** at x, denoted $W^s(x)$, is defined analogously with $d(f^{-n}x, f^{-n}y)$ in the definition of W^u replaced by $d(f^nx, f^ny)$. Stable and unstable manifolds are invariant, i.e. $f(W^u(x)) = W^u(fx)$, $f(W^s(x)) = W^s(fx)$.

A measurable partition ζ of M is said to be *subordinate to* W^u if for μ-a.e. x, $\zeta(x)$, the element of ζ containing x, is contained in $W^u(x)$. We focus on the following two families of measures on the elements of ζ: $\{\mu_x^\zeta\}$, the conditional measures of μ, and m_x^ζ, the restriction of the Riemannian measure on $W^u(x)$ to $\zeta(x)$.

Definition 2.1 *An f-invariant Borel probability measure μ is said to have* **absolutely continuous conditional measures on unstable manifolds** *if (f,μ) has a positive Lyapunov exponents a.e., and for every measurable partition ζ subordinate to W^u, we have $\mu_x^\zeta << m_x^\zeta$ at μ-a.e. x.*

As in Section 1, let $h_\mu(f)$ denote the metric entropy of f with respect to μ. Theorem 2 states that properties (i) and (ii) in Theorem 1 are, in fact, equivalent under very general conditions. (Unlike Theorem 1, however, it does not assert the existence of a measure with these properties.)

Theorem 2 ([R3], [P2], [LS], [LY1]) *Let f be an arbitrary diffeomorphism and μ an f-invariant Borel probability measure with a positive Lyapunov exponent a.e.. Then μ has absolutely continuous conditional measures on W^u if and only if*

$$h_\mu(f) \;=\; \int \sum_{\lambda_i>0} \lambda_i \;\dim E_i \; d\mu \;.$$

Without the absolute continuity assumption on μ, "$=$" above is replaced by "$\leq$".

This result has the following interpretation: It suggests that entropy is created by the exponential divergence of nearby orbits. A strict inequality results when some of the expansion is "wasted" due to leakage or dissipation from the system under forward iterations; by the same token equality holds when no leakage takes place. Thus measures with absolutely continuous conditional measures on W^u can be seen as the counterpart of Lebesgue or Liouville measure in systems that are "conservative in positive time". (Note that in the presence of an attractor, the direction of time is never ambiguous: if time is reversed, the attractor becomes a repellor.)

Theorem 2 is a summary of results from several papers combined together:

(1) The general inequality is due to Ruelle [R3], and is valid for all C^1 maps.

(2) For μ equivalent to Lebesgue, the displayed equality is proved by Pesin [P2]; this result is extended to the case of measures having absolutely continuous conditional measures on W^u by Ledrappier and Strelcyn [LS].

(3) The reverse implication is proved by Ledrappier [L] in the case where none of the Lyapunov exponents is zero, and is extended to complete generality by Ledrappier and Young [LY1].

Definition 2.2 *Let f be a C^2 diffeomorphism of a compact Riemannian manifold. An f-invariant Borel probability measure μ is called an* **SRB measure** *if (f,μ) has a positive Lyapunov exponent a.e. and μ has absolutely continuous conditional measures on unstable manifolds.*

This terminology was first formally introduced in the review article by Eckmann and Ruelle [ER]. The concept itself had been identified and studied a couple of years earlier (see e.g. [LS], [L], [LY1], [Y2]).

2.2 Absolute continuity of W^s and physical measures

We now explain the relation between SRB measures and property (iii) in Theorem 1.

Definition 2.3 *Let $f : M \to M$ be an arbitrary map and μ an invariant probability measure. We call μ a* **physical measure** *if there is a positive Lebesgue measure set $V \subset M$ such that for every continuous observable $\varphi : M \to \mathbb{R}$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \ \to \ \int \varphi d\mu$$

for every $x \in V$.

We say a point $x \in M$ is *μ-generic* if the time averages of continuous observables along the trajectory of x converge to their space averages with respect to μ. Hence a measure is physical if its set of generic points has positive Lebesgue measure. From the point of view of physical observations, it is not clear if this is a more or less realistic notion of observability than zero-noise-limits (see Section 1 for a definition), but since the latter notion already has a suggestive name, we will, following [ER], use the definition of "physical" in the sense of Definition 2.3.

What allows us to go from Lebesgue measure on W^u-leaves to Lebesgue measure in the phase space is the following regularity property of the stable "foliation". The technical definition of this property is a little complicated. We give a version which is adequate for our purposes, and refer the reader to e.g. [PS1] for more details.

Definition 2.4 *Let (f, μ) be ergodic, and assume it has a negative Lyapunov exponent a.e. We say its W^s-foliation is* absolutely continuous *if the following holds: Let Σ and Σ' be embedded disks having complementary dimension to W^s, and let $\{D^s_\alpha\}$ be a positive μ-measure set of local stable manifolds such that each D^s_α meets both Σ and Σ' transversally. Let $\Phi : (\cup D^s_\alpha) \cap \Sigma \to \Sigma'$ be the holomony map[2], and let m_Σ and $m_{\Sigma'}$ denote Lebesgue measure on Σ and Σ' respectively. Then for $E \subset (\cup D^s_\alpha) \cap \Sigma$, $m_\Sigma(E) > 0$ if and only if $m_{\Sigma'}(\Phi(E)) > 0$.*

Theorem 3 ([P2], [KS], [PS1]) *Assume (f, μ) has a negative Lyapunov exponent a.e. Then its W^s-foliation is absolutely continuous. It follows that every ergodic SRB measure with no zero Lyapunov exponents is a physical measure.*

[2] $\Phi(x)$ is the unique point in $D^s_\alpha(x) \cap \Sigma'$ where $D^s_\alpha(x)$ is the disk containing x.

In the context of nonuniformly hyperbolic systems, this property was first proved in the volume-preserving case by Pesin [P2], and later on by Katok and Strelcyn [KS], who also allowed singularities; the general case is due to Pugh and Shub [PS1]. We explain why the second assertion follows from the first. Recall that if μ is an SRB measure, then by definition, there are W^u-leaves on which Lebesgue-a.e. point is μ-generic. In the absence of zero Lyapunov exponents, transversal to W^u-leaves are families of W^s-leaves, forming a kind of local "coordinate system". Since all points on a W^s-leaf have asymptotically the same future, one obtains, by "integrating out" along stable manifolds and appealing to the absolute continuity of the W^s-foliation, that the set of μ-generic points has positive Lebesgue measure in the phase space.

We record below two important facts about SRB measures. The proofs of these facts rely heavily on the structures of stable and unstable manifolds and their absolute continuity properties.

Theorem 4 (a) [P2] *In the presence of positive Lyapunov exponents, invariant measures equivalent to Riemannian volume are special cases of SRB measures.*

(b) [P2], [L] *Let μ be an SRB measure with no zero Lyapunov exponents. Then*

(i) *μ has at most a countable number of ergodic components, and*

(ii) *on each ergodic component, f permutes k ($k \geq 1$) disjoint sets restricted to each one of which (f^k, μ) is mixing.*

The following result, due to Tsujii, gives a partial converse to Theorem 3.

Theorem 5 [T] *Let $f : M \to M$ be a diffeomorphism, and suppose there is a positive Lebesgue measure set $R \subset M$ such that the following hold for every $x \in R$:*
(i) $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x}$ converges weakly to an ergodic measure which we call μ_x;
(ii) the Lyapunov exponents at x as $n \to +\infty$ coincide with those of μ_x;
(iii) μ_x has no zero and at least one positive Lyapunov exponent.
Then μ_x is an SRB measure for Lebesgue-a.e. x in R.

The conclusion of Theorem 5 is not valid if condition (ii) above is omitted. To see this, consider the "figure-eight attractor": Figure 1 shows a flow with a stationary saddle point p whose stable and unstable manifolds coincide. Let f be the time-one-map, and assume that $|\det Df(p)| < 1$. Then every point in the basin of this attractor is generic with respect to its unique invariant measure δ_p, which is not an SRB measure.

Remarks on the relation between SRB measures and physical measures. We have seen in Theorem 3 that ergodic SRB measures with no zero Lyapunov exponents are physical measures. Ergodicity is needed because physical measures, by definition, are ergodic. SRB measures with zero Lyapunov exponents may or may not be physical. Conversely, not all physical measures are SRB. The simplest counterexamples are point masses on attractive fixed points and periodic orbits. The figure-eight attractor has a physical measure with a positive Lyapunov exponent which is not an SRB measure.

To sum up then, ergodic SRB measures with no zero Lyapunov exponents are *a special kind of physical measures*: they describe chaotic behavior, are accompanied by rich geometric and dynamical structures (see Theorems 2, 3 and 4), and carry special meanings (such as the entropy formula). Physical measures, on the other hand, focus on a single property. It is a very important property, but by focusing on it alone, they carry little additional information. For example, physical measures do not distinguish between chaotic attractors and simple equilibria.[3]

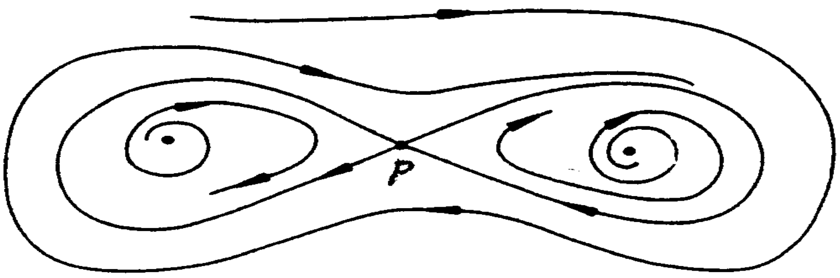

Fig. 1 The figure-eight attractor

2.3 Randomly perturbed dynamical systems

Turning now to zero-noise limits, it is conceivable that under suitable conditions, SRB measures are *stochastically stable*, i.e. they can be realized as zero-noise limits, but as of now there is no general result to that effect. Evidently, not all zero-noise limits are SRB; point masses at sinks and on the figure-eight attractor, for example, are zero-noise limits. The relation between physical measures and zero-noise limits is also unclear.

We finish with a brief discussion of *random dynamical systems*. Consider a family of small perturbations P^ε defined by random diffeomorphisms, i.e. for $\varepsilon > 0$, let ν_ε be a probability measure on $\mathrm{Diff}(M)$, the space of diffeomorphisms of M, and let $P^\varepsilon(E|x) = \nu_\varepsilon\{g, g(x) \in E\}$. Not all Markov chains can be represented by random diffeomorphisms, but some can be, including solutions to stochastic differential equations [Ku]. For this type of Markov chains, there are two kinds of invariant measures: μ_ε, the marginal of the stationary measure of the process, and $\{\mu_\omega^\varepsilon\}$, sample measures corresponding to individual realizations. More precisely, $\mu_\varepsilon = \int g_* \mu_\varepsilon d\nu_\varepsilon(g)$, and

[3]We caution the reader that recently some authors have chosen to call physical measures "SRB measures". We discourage this change in terminology, for in addition to introducing confusion into the literature, it takes away the multifaceted meanings of SRB measures.

associated with a.e. *iid* sequence $\omega = (g_i)_{i=-\infty}^{\infty}$ with law ν_ε, we have a probability measure μ_ω^ε satisfying $\mu_\varepsilon = \int \mu_\omega^\varepsilon d\nu_\varepsilon^{\mathbb{Z}}$ and $(g_0)_*\mu_\omega^\varepsilon = \mu_{\sigma\omega}^\varepsilon$ where σ is the shift operator. Finally, we assume that as $\varepsilon \to 0$, ν_ε tends to the Dirac measure at f.

Extending slightly the ideas for a single map, Lyapunov exponents and entropy are defined for a.e. ω and are nonrandom. (See [LY2] or [K2] for more detail.) We let λ_i^ε denote the Lyapunov exponents and h_ε the entropy of the random maps at noise level ε. Stable and unstable manifolds make sense for individual realizations.

Theorem 6 [LY2] *Assume for $\varepsilon > 0$ that μ_ε has a density with respect to Lebesgue measure. Then for a.e. ω, μ_ω^ε has absolutely conditional measures on unstable manifolds and*

$$h_\varepsilon \; = \; \int \sum_{\lambda_i^\varepsilon > 0} \lambda_i^\varepsilon \; \dim E_i^\varepsilon \; d\mu_\omega^\varepsilon \; .$$

For obvious reasons, μ_ω^ε with the properties above are called **random SRB measures**. Theorem 6 expresses the fact that in noisy systems, chaos is synonymous with the invariant measure being SRB, lending some hope to the possibility that zero-noise limits may, under fairly general conditions, continue to be SRB measures (although we know this is not always true). We remark also that while the formula above resembles that in Theorem 2, the mechanisms responsible for them are really quite different: the one here comes not from special properties of the maps but from randomness of the noise, which in turn is guaranteed by the densities of the transition probabilities.

3 Which dynamical systems have SRB measures?

In Section 2 we explained how the idea of SRB measures can be generalized to arbitrary diffeomorphisms. We note that the theorems there are "abstract", meaning they contain no assertions of *existence*, not to mention *prevalence*, and do not tell us in general how to determine if a concrete dynamical system admits an SRB measure. These questions are, in fact, very difficult, and the first results are only beginning to come in.

To help conceptualize this emerging information, I find it useful to distinguish between the following two approaches, which account for most (but not all) of the results I know. The first, which I call the "axiomatic approach", seeks to relax the conditions that define Axiom A in the hope of systematically enlarging the set of maps with SRB measures. The second approach is aimed at modeling concrete examples of dynamical behaviors (not necessarily related to Axiom A). Here one seeks to identify dynamical phenomena, mechanisms and underlying characteristics conducive to having SRB measures. For lack of a better name, I will call this the "phenomenological approach". Samples of results from these two approaches are presented in Sects. 3.2 and 3.3.

3.1 Construction of SRB measures on Axiom A attractors

To better understand the recent results, we think it is instructive to first explain how SRB measures are constructed for Axiom A attractors. The construction below was not exactly the one used by Sinai, Ruelle or Bowen in their original works, but it is not far from Sinai's construction; see also [PS]. Among the proofs of existence I know, this one generalizes most easily.

Let $f : M \to M$ be a C^2 diffeomorphism of a compact Riemannian manifold, and let $\Lambda \subset M$ be an attractor. We pick an arbitrary piece of local unstable manifold $\gamma \subset \Lambda$, and let m_γ denote Lebesgue measure on γ. Now use f^i to transport m_γ forward, and call the image measures $f^i_*(m_\gamma),\ i = 1, 2, \cdots$. We claim that any limit point μ of

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} f^i_*(m_\gamma) \right\}_{n=1,2,\cdots}$$

is an SRB measure. Invariance is obvious because an average is taken. Since $f^i|_\gamma$ expands distances uniformly with bounded distortion, $f^i_*(m_\gamma)$ has a density with respect to Lebesgue measure on $f^i(\gamma)$. These densities have uniform upper and lower bounds independent of i. Since the W^u-leaves are roughly parallel, these bounds are passed on to μ. (For a formal proof, see e.g. [Y3].)

3.2 Existence of SRB measures: axiomatic approach

Three sets of results will be reported representing three different ways of relaxing the Axiom A condition. I have chosen to organize these results in a conceptually convenient way rather than to follow chronological order. Let f and Λ be as above.

The first set of results assumes *a priori* a uniform **invariant dominated splitting** everywhere on the attractor. This means that the tangent bundle is decomposed into Df-invariant subbundles $E \oplus F$, with $\min\{||Df(u)||/\|u\|,\ u \in E\} \geq \lambda \max\{||Df(v)||/\|v\|, v \in F\}$ for some $\lambda > 1$ (abbreviated notation: $Df|_E > Df|_F$). In addition to having a dominated splitting, an Axiom A map is uniformly expanding in E and uniformly contracting in F. Theorem 7 keeps the invariant dominated splitting and explores several ways to relax the other two conditions.

Theorem 7 *Assume in each of the results below that the tangent bundle over Λ is decomposed into $E \oplus F$ with $Df|_E > Df|_F$.*

(a) [PS], [BnV] *If $Df|_E$ is uniformly expanding and $Df|_F$ is (nonuniformly) contracting or neutral (see remark below), then f has an SRB measure.*

(b) [HY] *Assume $dim(M) = 2$, $Df|_F$ is uniformly contracting, and $Df|_E > 1$ except at a fixed point p where $Df|_E(p) = 1$. Then $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \to \delta_p$ for Lebesgue-a.e. x. In particular, f does not admit a (finite) SRB measure.*

(c) [ABV] *Assume that the invariant splitting* $E \oplus F$ *extends to a neighborhood* U *of* Λ, $Df|_F$ *is uniformly contracting, and there is a positive Lebesgue measure set* $V \subset U$ *such that* $\forall x \in V$,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log\ \min\{||Df(f^i x)u||, u \in E(f^i x),\ \|u\| = 1\}\ > 0\ .$$

Then f *has an SRB measure.*

(d) [CY] *Suppose* f *has a zero-noise limit* μ *such that* μ*-a.e. the Lyapunov exponents* $\lambda(\cdot, u)$ *are* ≥ 0 $\forall u \in E$ *and* ≤ 0 $\forall u \in F$. *Then* μ *satisfies the entropy formula in Theorem 2. In particular, for a.e. ergodic component* μ_e *of* μ, *either all the Lyapunove exponents of* μ_e *are* ≤ 0 *or it is an SRB measure. (This result requires that* f *be* C^∞*.)*

We remark that (a) follows immediately from the construction in Sect. 3.1, since what happens in the subbundle F is immaterial. Part (b) contains a *nonexistence* result; it gives a sense of the delicateness of the existence question. Since the maps in (b) lie on the boundary of the set of Anosov diffeomorphisms, it also shows how SRB measures can "disappear suddenly". See also [Hu]. Part (d) is obtained by letting $\varepsilon \to 0$ in Theorem 6.

Our second set of results has to do with **partially hyperbolic systems**. Following [PS2], we say $f|_\Lambda$ is *partially hyperbolic* if $T\Lambda = E \oplus C \oplus F$ where E, C and F are Df-invariant subbundles, $Df|_E > Df|_C > Df|_F$, $Df|_E$ is uniformly expanding, and $Df|_F$ is uniformly contracting. Theorem 8 treats some situations when the behavior in the "central subbundle" is relatively simple.

Theorem 8 (a) [CY] *If dim* C*=1, then* f *has an SRB measure.*

(b) [D1] *Assume* $\Lambda = M$. *If* μ *is the unique invariant measure with absolutely continuous conditional measures on the invariant manifolds tangent to* E, *then it is a physical measure.*

The third group of results is intended for systems which are **"predominantly hyperbolic"**, a descriptive, nontechnical term I like to use to express the fact that the system in question has a great deal of expansion and contraction on large parts of its phase space. No global invariant splittings of the type in the two previous theorems are assumed. Our goal here is to identify structures in the phase space the presence of which is sufficient for the construction of SRB measures. Theorem 9 contains two such sets of sufficient conditions.

To avoid getting technical, I will give only the main ideas, referring the reader to the cited papers for precise formulations. In part (b), for example, I will leave it to the reader to imagine what a "generalized horseshoe" looks like, given that we think of the standard horseshoe (due to Smale [Sm]) as having "2 branches both of which return at time 1".

Theorem 9 (a) [BY1] *Suppose there is a W^u-leaf γ, a number $\delta > 0$, a continuous stack of unstable disks $\{D^u_\alpha\}$, and a sequence of times $n_1 < n_2 < \cdots$ for which the following hold: for each i, there exists $\gamma^{(n_i)} = \cup \gamma_j^{(n_i)} \subset \gamma$ with $m_\gamma(\gamma^{(n_i)}) \geq \delta$ such that f^{n_i} maps each $\gamma_j^{(n_i)}$ onto one of the D^u_α-disks with uniformly bounded distortion. Then f has an SRB measure.*

(b) [Y4] *Suppose that embedded in the attractor Λ is a "generalized horseshoe" Ω with infinitely many branches returning at variable times. We assume also that Ω has positive measure in the unstable direction, i.e. $m_\gamma(\Omega \cap \gamma) > 0$ for some unstable leaf γ. Let $R : \Omega \to \mathbb{Z}^+$ denote the return time function. If $\int R \, dm_\gamma < \infty$, then f has an SRB measure.*

Theorem 9 has been used in various applications (including the ones in Theorems 11 and 12). The structures required in (a) are essentially the "minimum" needed for the construction in Sect. 3.1 to go through. The advantage of the additional structures in (b) is that the tail behavior of R, i.e. the rate at which $m_\gamma\{R > n\}$ decreases, contains a great deal of information on the statistical properties of f with respect to its SRB measure (see [Y4], [Y5]).

3.3 Analysis of a class of strange attractors

As an example of what I called the "phenomenological approach", we focus in this subsection on a body of work surrounding the analysis of a particular class of strange attractors. Leaving precise definitions for later, we first give a rough description of the maps in this class:

- The defining maps are strongly dissipative, i.e. $|\det(Df)| << 1$;
- the attractors are chaotic with a single direction of instability, and
- some unstable curves have "folds" similar to those in the Hénon maps (see below).

Natural examples of attractors that fit this general description will be discussed later. We begin with the results which inspired this study.

Theorem 10 ([J], [GS], [L1], [L2]) *The following hold for the logistic family $Q_a(x) = 1 - ax^2$, $x \in [-1, 1]$, $a \in [0, 2]$:*
(i) *There is an open and dense set $\mathcal{A}$ in parameter space such that for all $a \in \mathcal{A}$, Q_a has a periodic sink to which the orbit of Lebesgue-a.e. point converges.*
(ii) *There is a positive Lebesgue measure set of parameters $\mathcal{B}$ such that for $a \in \mathcal{B}$, Q_a has an invariant measure absolutely continuous wrt Lebesgue measure.*
The union of $\mathcal{A}$ and $\mathcal{B}$ has full measure in parameter space.

Part (ii) of Theorem 9 is, in many ways, the precursor to the results on SRB measures in this subsection. This important theorem is due to Jakobson [J]. The density of $\mathcal{A}$ was first announced by Graczyk and Swiatek (see [GS] and [L1]), and the fact that $\mathcal{A} \cup \mathcal{B}$ has full Lebesgue measure is due to Lyubich [L2].

In 1977, Hénon carried out numerical studies of a family of maps and demonstrated that for certain ranges of parameters there were chaotic attractors [He]. The Hénon family is given by

$$T_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$$

Observe that when $b = 0$, $T_{a,b}$ maps the x-axis to itself and reduces to the logistic family. By continuity, it follows that given any $a \in (0, 2)$, $T_{a,b}$ maps a rectangle into itself and has an attractor provided b is sufficiently small.

Benedicks and Carleson studied small values of b for $a \approx 2$, treating $T_{a,b}$ as small perturbations of Q_a. In a work that is an analytic *tour de force* [BC], they developed techniques for tracking the growth of derivatives along certain crucial orbits. Building on the analysis in [BC], Benedicks and I obtained the following result:

Theorem 11 [BY1] *For every $b > 0$ that is sufficiently small, there is a positive Lebesgue measure set $\Delta_b \subset (2 - \varepsilon, 2)$ such that for each $a \in \Delta_b$, $T_{a,b}$ admits a unique SRB measure.*

To my knowledge this is the first time SRB measures were constructed for nonuniformly hyperbolic attractors. Theorem 9(a) is used in this proof.

A few years ago, Qiudong Wang and I began to think that (i) even though [BC] and [BY] treated only the Hénon family, the type of analysis involved can probably be carried out for the class of maps described at the beginning of this subsection, and (ii) there is a number of naturally occurring attractors fitting this general description. To achieve (i), we would have to replace the computation-based arguments in [BC] by a more conceptual approach, and to replace the formulas of the Hénon maps (which are used explicitly in [BC]) by qualitative, geometric conditions. If our results are to be applicable to the situations in (ii), these conditions would have to be sufficiently inclusive, and checkable.

The current status of this project is that we have isolated a class of attractors for which we have developed a dynamical theory, and have begun to apply it to some simple situations. The ultimate usefulness of this project remains to be seen. I report below on our results thus far, focusing on the parts related to SRB measures.

To give a sense of the type of conditions used, I will state them in full although perhaps a little tersely, referring the reader to [WY1] or [WY2] for further clarification if necessary. Let $M = N \times D_{m-1}$ where $N = S^1$ or $[0, 1]$ and D_{m-1} is the unit disk in $\mathbb{R}^{m-1}$, $m \geq 2$. Points in M are denoted by $(x, \mathbf{y})$ where $x \in N$ and $\mathbf{y} = (y^1, \cdots, y^{m-1}) \in D_{m-1}$.

(C1) *Existence of singular limits.* We assume that the maps of interest can be embedded in a 2-parameter family $T_{a,b} : M \to M$, $b > 0$, of the form

$$T_{a,b} : \begin{pmatrix} x \\ \mathbf{y} \end{pmatrix} \mapsto \begin{pmatrix} F_a(x,\mathbf{y}) \ + \ bu(x,\mathbf{y};a,b) \\ b\ \mathbf{v}(x,\mathbf{y};a,b) \end{pmatrix}$$

where $(x, y, a) \mapsto F_a(x,\mathbf{y}),\ u(x,\mathbf{y};a,b)$ and $\mathbf{v}(x,\mathbf{y};a,b)$ have C^3-norms bounded above by a constant independent of b. We assume also that $T_{a,b}$ is one-to-one, and that for each (a, b), $\det(DT_{a,b})$ at different points in M are roughly comparable.

The maps $F_a : M \to N$ and $f_a : N \to N$ defined by $f_a(x) = F_a(x, \mathbf{0})$ are called the **singular limits** of $T_{a,b}$. The rest of the conditions are on these singular limits.

(C2) *Existence of a map in $\{f_a\}$ which is sufficiently expanding.* We assume there exists a^* such that f_{a^*} satisfies the Misiurewicz condition [M].[4]

(C3) *Transversality with respect to parameters.* Let C denote the critical set of f_{a^*}. For $c \in C$ and $p = f_{a^*}(c)$, let $c(a)$ and $p(a)$ [5] denote the smooth continuations of c and p respectively for a near a^*. Then

$$\frac{d}{da} f_a(c(a)) \neq \frac{d}{da} p(a) \qquad \text{at } \ a = a^*.$$

(C4) *Nondegeneracy at critical points.* For every $c \in C$, there exists j such that

$$\frac{\partial F_{a^*}(c, \mathbf{0})}{\partial y^j} \neq 0.$$

Theorem 12 ([WY1], [WY2]*)* *Assume* **(C1)**– **(C4)**. *Then for every sufficiently small b, there is a positive measure set of a for which $T = T_{a,b}$ has the following dynamical description:*

(i) *T admits r ergodic SRB measures where $1 \le r \le$ the number of critical points of the map f_{a^*} in* **(C2)**. *Let us call these measures $\mu_1, \cdots, \mu_r$.*

(ii) *The μ_i are physical measures, and if $\mathcal{B}(\mu_i)$ denotes the set of generic points of μ_i, then $\cup_i \mathcal{B}(\mu_i)$ has full Lebesgue measure in M.*

(iii) *Let $\varphi : M \to \mathbb{R}$ be a Hölder continuous observable. Then on the mixing components of each μ_i, the sequence $\varphi,\ \varphi \circ T,\ \varphi \circ T^2,\ \cdots,\ \varphi \circ T^n,\ \cdots$ satisfies the Central Limit Theorem and has exponential decay of correlations.*

[4]This means essentially that f_{a^*} has no stable or neutral periodic orbits, that all of its critical points are nondegenerate, and the positive iterates of the critical points stay a bounded distance away from the critical set.

[5]$p(a)$ refers to the point with the same itinerary with respect to f_a as p does with respect to f_{a^*}. It is not to be confused with $f_a(c(a))$.

The 2-dimensional case of this theorem is proved in [WY1]. The general case, which has more complicated geometry, is treated in [WY2]. See Section 2 for some of the definitions.

Applications of Theorem 12

(1) *The Hénon family* [WY1]. **(C1)–(C4)** hold with a^* equal to any of the uncountably many Misiurewicz points in the logistic family. (The case $a^* = 2$ was treated earlier: building on [BC], (i) was proved in [BY1], (ii) in [BeV] and (iii) in [BY2].)

(2) *Hénon-like attractors arising from homoclinic bifurcations* [WY1]. From [PT], we know that the maps defining these attractors have the form in **(C1)** with the same F_a as the Hénon maps, i.e. they have the same singular limits. Thus **(C2)–(C4)**, which depend only on these limits, follow immediately. (Direct adaptations of [BC] to these settings were carried out earlier by Viana et. al. in [MV] and [V].)

(3) *A simple mechanical model with periodic forcing* [WY3]. Consider a linear second order equation describing the motion of a particle in a circle, externally forced so that it tends to uniform motion. If attraction to the resulting limit cycle is weak, then with additional periodic "kicks", the system has, for a positive measure set of parameters, a strange attractor with an SRB measure. (Models of this type were studied earlier in the physics literature; see [Z].)

(4) *Strange attractors emerging from Hopf bifurcations* [WY4]. We show under reasonable conditions that if a system undergoing a Hopf bifurcation is subjected to certain types of periodic kick-forces, then for a positive measure set of parameters, what comes off the newly unstable fixed point is not an invariant circle but a strange attractor with an SRB measure.

These are the main applications that have been worked out so far. We are hopeful there will be others.

4 Concluding remarks

The theory of finite-dimensional dynamical systems, in my opinion, has provided excellent models or paradigms for understanding chaos. Axiom A theory, and the idea of SRB measures in particular, were remarkable breakthroughs. The fact that these ideas go beyond Axiom A (as we now know) makes them all the more powerful. The purpose of this article is to report on the main developments related to SRB measures since their original conception. With regard to the two questions in the title of this paper, I hope Section 2 gives at least a partial understanding of what SRB measures are in the context of all finite-dimensional systems. I have tried, in Section 3, to convey the directions of recent research and to report on some of the results. I would like to attempt now to put things in perspective, and to explain why I think we are still very much in search of an answer to the question "which dynamical systems admit SRB measures".

By focusing on positive results in Section 3, I may have given the impression that we have gone far beyond Axiom A. I would like to discuss in these closing paragraphs some of the first hurdles which stand between current techniques and "the general dynamical picture", whatever the latter means.

To varying degrees, some of the results from the "axiomatic approach" take for granted *a priori* knowledge of the expanding and contracting directions of the map. Various assumptions are made to the effect that these directions vary continuously and are uniformly separated, meaning the respective subspaces are bounded away from each other. While these properties are natural within certain classes of maps, they are quite special among all dynamical systems. In general, almost everywhere with respect to an invariant measure, the expected picture is that stable and unstable directions vary measurably but not continuously, and the angles between them can be arbitrarily small in places. [6]

The description above is for (f, μ) where f is a map and μ is a known invariant measure. If only f is given – which is the case at hand since we are looking for our invariant measure – then we lose the benefit of statistical arguments and the situation becomes even more dire: Along the orbits of arbitrary points, there is no reason why a map cannot be sometimes expanding and sometimes contracting in a given direction. In other words, the quantities $\frac{1}{n}\log\|Df^n(x)v\|$ may oscillate arbitrarily and indefinitely with n. Stable and unstable directions, if and where they are defined, are difficult to identify because that requires information on an infinite number of iterates. These are some of the problems we face when attempting to determine if an arbitrary dynamical system has an SRB measure.

The results in Sect. 3.3 do not presume *a priori* knowledge of expanding and contracting directions. Indeed, the attractors in question have some of the properties of full-blown nonuniformly hyperbolic systems. But the analysis there rely on other circumstances which are, in my opinion, quite special. Let me mention two of them.

The first is the strong codimension-one contraction on most parts of the phase space. Locally, this gives the attractor a one-dimensional character, allowing us to borrow techniques from the theory of 1-D maps. We have a well developed theory of 1-D endomorphisms with nonuniformly expanding properties. This cannot be said about n-D maps for $n \geq 2$; a good understanding of the geometry of n-dimensional

[6] Results specialized to the C^1-generic picture (e.g. [BcV]) are not pertinent to this discussion. In most of the ergodic theory of hyperbolic systems, the map is assumed to be $C^{1+\alpha}$ for some $\alpha > 0$. In particular, the definition and properties of SRB measures use in a crucial way the regularity of Df, without which even the existence of stable and unstable manifolds is not guaranteed.

unstable manifolds is also lacking. These are some of the hurdles one probably has to overcome to go beyond a theory with *one* direction of instability.

For 1-D maps with critical points, the critical set is an obvious source of non-expansion. For the attractors in Sect. 3.3, it is shown that there are small fractal sets (near the critical points of the corresponding 1-D maps) which play essentially the same role, i.e. away from these sets the map is uniformly hyperbolic, and upon approaching them, the stable and unstable directions of an orbit are confused. In other words, the maps in question have the following special feature: they are not uniformly hyperbolic, but all the problems are caused by certain well-defined, localized "bad sets" or sources of nonhyperbolicity. As a first generalization of Axiom A, maps which are predominantly hyperbolic with identifiable "bad sets" are excellent models to consider, but I doubt that all dynamical systems have this property.

Returning to the question "which dynamical systems have SRB measures", I hope that the preceding paragraphs, read in conjunction with Section 3, will enable the reader to assess more accurately the progress to date and the many challenges ahead.

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