Geostrophic and Quasi-Geostrophic Balances

Qiyu Xiao

June 19, 2018

1 Introduction

Understanding how the atmosphere and ocean behave is important to our everyday lives. Techniques such as weather forecasting will not be possible without this understanding. For example, if we observe that a snowstorm is forming near the city, we would like to know if it is going to reach the city. Even if we have satellite observational data as comprehensive as we want, we need to know the pattern of atmospheric motion to make predictions. To develop a reliable mathematical model for weather forecasting and other large scale atmospheric and oceanic problems, scientists use Navier-Stokes equation as the starting point, since air and water are fluids. Because wind and ocean currents are motions relative to the Earth's rotation, atmosphere and ocean scientists usually use the Navier-Stokes equation in a rotating framework instead of an inertial one. The Earth's rotation makes the fluid dynamics of the atmosphere and ocean different from the traditional one. We will show how to transform the equations from the inertial to the rotating frame in Section 2.

Given that in most cases we can't solve Navier-Stokes equation directly, we need to approximate the equation by simplified ones to better understand the property of solutions. Atmosphere and ocean scientists construct approximations by estimating the magnitude of each term in the Navier-Stokes equation using observational data. This note is intended to introduce two important concepts used in standard approximations, geostrophic balance and quasi-geostrophic (QG) balance. These concepts were developed more than half a century ago and are widely applied in atmospheric and oceanic studies, such as weather prediction and ocean circulation research. For example, they are essential for us to understand a wind's direction. We will show the math derivation and some applications of geostrophic balance in Section 3. Then a brief introduction about quasi-geostrophic balance is in Section 4.

2 Basic Equations

A fluid element is an infinitesimal, indivisible piece of fluid, effectively a very small fluid parcel of fixed mass. The material derivative is the rate of change of a property (such as temperature or momentum) of a particular fluid element instead of a specific position in space. We usually call this Lagrangian description of fluid and denote material derivative as Df/Dt. We can also describe a fluid by stating its property at certain point (x, y, z) over time, and this is called Eulerian description. The relation between the derivatives under these two framework is[4]

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u(x,y,z)\frac{\partial f}{\partial x} + v(x,y,z)\frac{\partial f}{\partial y} + w(x,y,z)\frac{\partial f}{\partial z}. \tag{1}$$

With conservation law of momentum, we can deduce the momentum equation for fluid to be

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{F}_b \tag{2}$$

with $\mathbf{v}(x, y, z)$ the three-dimensional (3D) velocity, ρ the fluid density, p the pressure, ν the kinematic viscosity and $\mathbf{F}_b$ the external body force. In the following we will neglect the last two terms in the right hand side but the gravity part, because they are small in scale and the analysis will be much easier without them.And with conservation law of mass, we can deduce the continuity equation for fluid to be

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{3}$$

Here are five unknowns in four equations. So we need one more equation to close the problem. Usually an 'equation of state' derived from thermodynamics will be used. For example, ideal gas law $p = \rho RT$ is used in studying atmosphere problem because air in the Earth's atmosphere is very close to be ideal. But this equation plays no role in our discussion, so we will not explore it further with the awareness of its existence.

These equations are valid in an inertial framework. But our earth is rotating, and wind or ocean current that we care are motions relative to the earth. Thus it's more reasonable to consider the problems in a rotating frame that shares the same angular velocity and center with our earth.

Assuming the angular velocity of the rotation frame is $\mathbf{\Omega} = \Omega \mathbf{k}$. The change of a vector $\mathbf{B}$ with respect to time t in inertial frame and rotation frame has relation:

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)_I = \left(\frac{\partial \mathbf{B}}{\partial t}\right)_R + \mathbf{\Omega} \times \mathbf{B}. \tag{4}$$

When $\mathbf{B}$ is taken to be the position $\mathbf{r}$ of our fluid parcel, we get a relation the velocity in the two coordinate system

$$(\mathbf{v})_I = (\mathbf{v})_R + \mathbf{\Omega} \times \mathbf{r}. \tag{5}$$

In rotating frame with spherical coordinate (r, θ, λ), the momentum equations and mass equation will take the form

$$\frac{Du}{Dt} - (2\Omega + \frac{u}{r\cos\theta})(v\sin\theta - w\sin\theta) = -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial \lambda}, \tag{6}$$

$$\frac{Dv}{Dt} + \frac{wv}{r} + (2\Omega + \frac{u}{r\cos\theta})u\sin\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}, \tag{7}$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g, \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\theta}\frac{\partial (u\rho)}{\partial \lambda} + \frac{1}{r\cos\theta}\frac{\partial}{\partial \theta}(v\rho\cos\theta) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w\rho) = 0. \tag{9}$$

Working in spherical coordinate is usually inconvenient. Locally, we can make Cartesian approximation with a tangent plane(figure 1) and thus write the equations in independent variables (x, y, z).

The equations will take the form[5]

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{10}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{11}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - g, \tag{12}$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{13}$$

with $f = 2\Omega\sin\theta$ and g the gravity acceleration. This set of equations are called 'the primitive equations', where most atmospheric and oceanic analysis begins from.

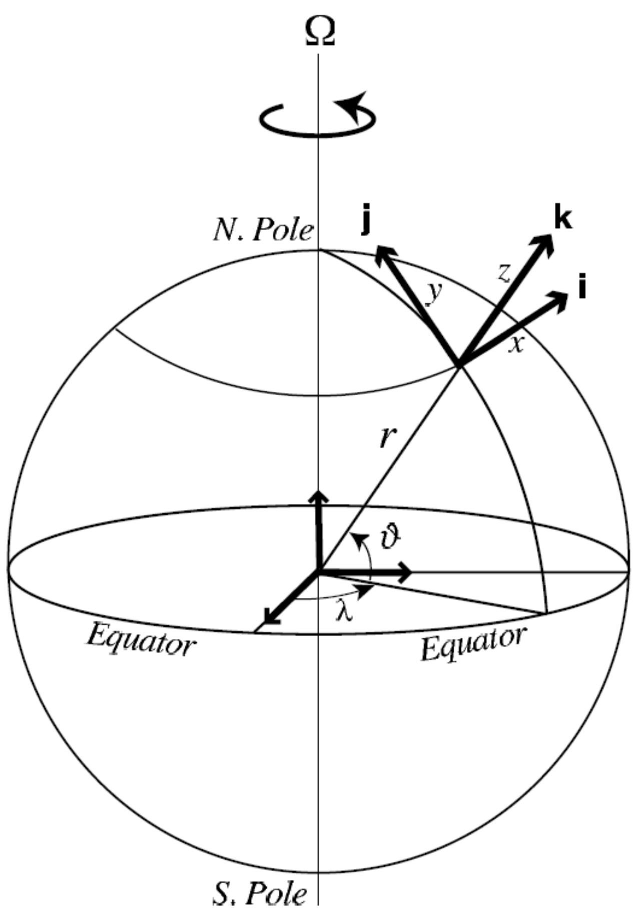

Figure 1: The spherical coordinate and the approximating Cartesian coordinate.[9]

3 Geostrophy

3.a Derivation with Scale Analysis

The equations(10)-(13) are not possible to be solved directly in most cases, largely because of the nonlinear terms in the left hand sides. Yet there're knowledge about the underlying physical problem that we can apply, that is the scale size between each terms.

Based on observations, the horizontal velocity u, v have scale $U \sim 10m/s$, vertical velocity w has scale $W \sim 1cm/s$, horizontal length scale $L \sim 10^5 m$, vertical length scale $H \sim 10^3 m$, time scale $T \sim L/U \sim 10^2 s$ and angular velocity $f \sim 10^{-4} s^{-1}$.

Then the first two terms of equations (10) and (11) have scale $\frac{U^2}{L}$ and fU. Their ratio U/fL is pretty small, in the scale of 10^{-2}. We will refer this ratio to be Rossby number from now on

$$Ro = \frac{U}{fL} \tag{14}$$

The smallness of Rossby number tells us that in our problem, the advection term $Du/Dt, Dv/Dt$ is relatively small compared with the Cariolis force fu, fv. Another way to think about the Rossby number is through time scales. Recalling that $1/f$ is the time scale of earth's motion and L/U is the time scale of fluid motion that we are interested, Rossby number is the ratio between these two time scales. When Ro is small, the fluid motion happens in a comparable time scale with our earth's self rotation.

Now that we know the Coriolis term dominates the left hand sides of equations (10) and (11), the only balance we can reach is by the pressure gradient on the right hand sides. And this balance is called 'geostrophic'

$$v = \frac{1}{f\rho} p_x, \tag{15}$$

$$u = -\frac{1}{f\rho} p_y. \tag{16}$$

This approximation can give explanations to phenomenons we observe as we will see.

3.b Applications

We will give two examples here to show the applications of geostrophic theory.

First is about direction of flows from high pressure to low pressure. With intuition, the flow should has direction align with gradient of isobars(constant pressure lines) pointing from high pressure to low pressure.

But with geostrophic relation (15)-(16), we see that the velocity is actually tangential to the pressure gradient, and thus flow are parallel to lines of constant pressure. In north hemisphere, $f > 0$, and the flow is anticlockwise around a low pressure region, clockwise around a high pressure region.(figure 2) And the opposite case for south hemisphere($f < 0$).This explains why hurricanes or storms we observe are in shapes of spiral.(figure 3)

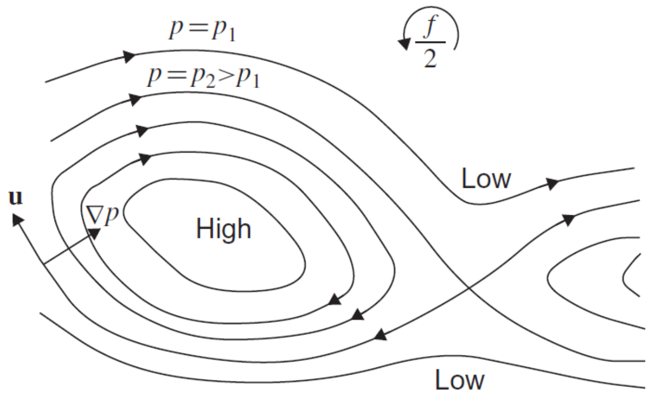

Figure 2: Geostrophic flow around high pressure region.[5]

Figure 3: NOAA's GOES East satellite captured this infrared image of Hurricane Irma in the Bahamas at 4:45 a.m. EDT. Credit: NASA/NOAA GOES Project

The second example is 'Taylor column', that is, in a rapidly rotating homogeneous incompressible fluid with flat bottom or lid, when facing an obstacle, a flow will go around it horizontally instead of lifting up to pass the obstacle.

Because the fluid is homogeneous, it has constant density ρ. And from equation (15), (16) we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho f}\frac{\partial p}{\partial x}\right) = 0. \tag{17}$$

At the same time from mass equation(13) we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{18}$$

and immediately we learn that vertical velocity w is independent of vertical position z

$$\frac{\partial w}{\partial z} = 0. \tag{19}$$

Yet because we assume the fluid has flat botton or lid, the vertical velocity is 0 either at the bottom or the top, and thus always equal to 0 inside the fluid.(figure 4)

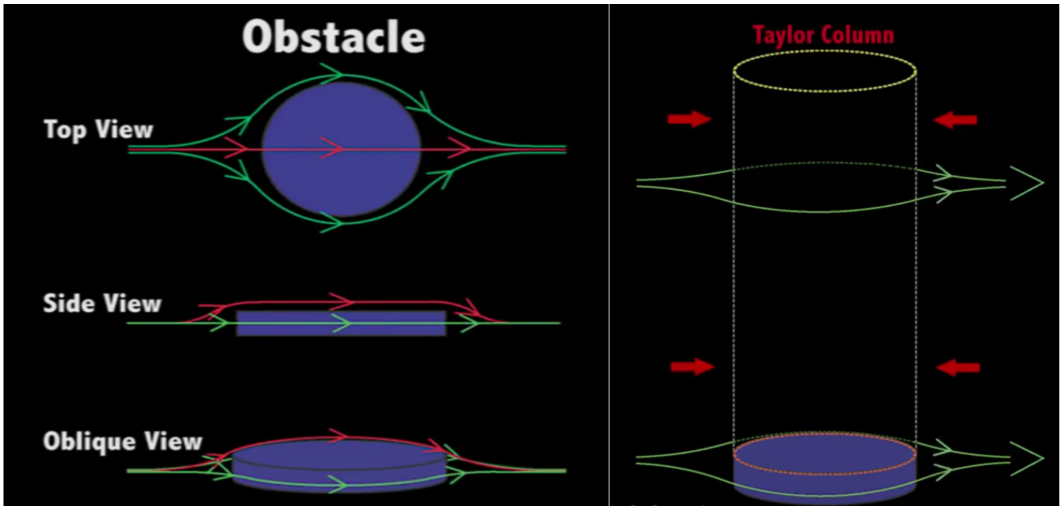

Figure 4: left:In non-rotating environment, fluid will flow past an obstacle both horizontally and vertically; right:In rapidly rotation environment, fluid will just flow past the obstacle horizontally.[8]

4 Quasi-Geostrophy (QG)

4.a Disadvantage of Geostrophy

The geostrophic approximation gives very useful insight about geophysical phenomenons, yet it's far from being perfect. First is the assumption that Coriolis force dominates(Ro small) may not hold very strongly when problem considered has smaller length scale or shorter time scale. In this case, the nonlinear terms ignored earlier need to be brought back, even in a size smaller than the Coriolis term. The second unfavorable property of geostrophic approximation is that it is a static system; the time evolution part is gone. But we can rarely observe any static system in real nature. And thus for a better approximation we need the time derivative term back. All these lead us to quasi-geostrophic model.

4.b Quasi-Geostrophic Model

Since we are having terms that are in two different size, we can try to distinguish these two parts in each variable first. For example,

$$u = u_g + u_a, \tag{20}$$
$$v = v_g + v_a \tag{21}$$

where u_g and v_g represent the geostrophic components of u, v and u_a, v_a are the ageostrophic correction in the order of Rossby number smaller than their geostrophic counterparts. Similarly, other variables can be decomposed based on observations:

$$f = f_0 + \beta y, \tag{22}$$
$$\rho = \rho_0(z) + \rho'(x,y,z,t), \tag{23}$$
$$p = p_0(z) + p'(x,y,z,t). \tag{24}$$

Substitute these into equations (10)-(13). The lower order terms give us:

$$u_g = -\frac{1}{f_0\rho_0}\frac{\partial p'}{\partial y}, \tag{25}$$
$$v_g = \frac{1}{f_0\rho_0}\frac{\partial p'}{\partial x}. \tag{26}$$

And higher order terms give us(with substitution of equation(25)-(26)

$$-\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial y \partial t} - \frac{1}{\rho_0^2 f_0^2}J\left(p', \frac{\partial p'}{\partial y}\right) - f_0 v - \frac{\beta_0}{\rho_0 f_0}y\frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{27}$$
$$\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial x \partial t} + \frac{1}{\rho_0^2 f_0^2}J\left(p', \frac{\partial p'}{\partial x}\right) + f_0 u - \frac{\beta_0}{\rho_0 f_0}y\frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \tag{28}$$

which further gives us

$$u = u_g + u_a = -\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t \partial x} - \frac{1}{\rho_0^2 f_0^3}J\left(p', \frac{\partial p'}{\partial x}\right) + \frac{\beta}{\rho_0 f_0^2}y\frac{\partial p'}{\partial y}, \tag{29}$$
$$v = v_g + v_a = \frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y} - \frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial t \partial y} - \frac{1}{\rho_0^2 f_0^3}J\left(p', \frac{\partial p'}{\partial y}\right) - \frac{\beta}{\rho_0 f_0^2}y\frac{\partial p'}{\partial x}. \tag{30}$$

Now the horizontal velocities are no longer non-divergent, and from continuity equation(13) we have for vertical velocity

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0}J\left(p', \nabla^2 p'\right) + \beta_0\frac{\partial p'}{\partial x}\right]. \tag{31}$$

This part only arises because the ageostrophic part of horizontal velocity, thus w has an order smaller than u, v. And the continuity equation(13) further gives us

$$\frac{\partial}{\partial t}\left[\nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \frac{1}{\rho_0 f_0}J\left[p', \nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \beta_0\frac{\partial p'}{\partial x} = 0. \tag{32}$$

If we let ψ to be the streamfunction of u_g and v_g, we will have a more concise version of equation(32):

$$\frac{\partial q}{\partial t} + J\left(\psi, q\right) = 0, \tag{33}$$
$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y. \tag{34}$$

Now we have a single time evolution equation about variable ψ and we can derive other variable from it at every moment t:

$$u_g = -\frac{\partial \psi}{\partial y}, \tag{35}$$

$$v_g = \frac{\partial \psi}{\partial x}, \tag{36}$$

$$u_a = -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial x} - \frac{1}{f_0} J\left(\psi, \frac{\partial \psi}{\partial x}\right) + \frac{\beta_0}{f_0} y \frac{\partial \psi}{\partial y}, \tag{37}$$

$$v_a = -\frac{1}{f_0}\frac{\partial^2 \psi}{\partial t \partial y} - \frac{1}{f_0} J\left(\psi, \frac{\partial \psi}{\partial y}\right) + \frac{\beta_0}{f_0} y \frac{\partial \psi}{\partial x}, \tag{38}$$

$$w = -\frac{f_0}{N^2}\left[\frac{\partial^2 \psi}{\partial t \partial z} + J\left(\psi, \frac{\partial \psi}{\partial z}\right)\right], \tag{39}$$

$$p' = \rho_0 f_0 \psi, \tag{40}$$

$$\rho' = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}. \tag{41}$$

4.c Property

Except for the time evolution part of quasi-geostrophic model, there are many other advantages that make this model widely used since its birth 70 years ago. Here we are going to discuss two conservation laws that come with it, the conservation of energy and potential vorticity.[7]

If we multiply equation(33) with streamfunction ψ we will have

$$\frac{d}{dt}\iiint \frac{1}{2}\rho_0 \left|\nabla^2 \psi\right|^2 dxdydz + \frac{d}{dt}\iiint \frac{1}{2}\rho_0 \frac{f_0^2}{N^2}\left(\frac{\partial \psi}{\partial z}\right)^2 dxdydz = 0. \tag{42}$$

The first term in left hand side is kinematic energy. And the second term is 'available potential energy', that is, the difference between the total potential energy and the minimum value it could have after an arbitrary adiabatic rearrangement of all the fluid particles in the ocean. This equation tells us that the total energy is conserved.

Another conserved quantity of quasi-geostrophic model is the potential vorticity. Under different assumption, this potential vorticity will have different expression. For example, if the fluid we are discussing is homogeneous, then the potential vorticity is defined as $(v_x - u_y + f)/h$ and we have:

$$\frac{D}{Dt}\left(\frac{v_x - u_y + f}{h}\right) = 0. \tag{43}$$

This tells us how vorticity $v_x - u_y$ would change when the flow is moving horizontally (f changes) or moving vertically (h changes).

5 Conclusion

By focusing on the important role played by the Coriolis force induced by the Earth's rotation, the geostrophic approximation gives very useful insights about planetary scale motion of wind and ocean currents. But this approximation is too simple to give a precise prediction about many phenomena we are interested in. In particular, the time derivative term and nonlinear terms are ignored in the geostrophic equations.

The quasi-geostrophic model does a higher order approximation, bringing back the time evolution and nonlinear terms while assuming them to be in a smaller scale. This keeps the model concise yet gives a lot of detail about the solution from the models. The time evolution part it brings back enabled Charney to develop the first successful weather forecasting code with Von Neumann's machine in 1950s based on it. The quasi-geostophic model has become the backbone of synoptic-scale weather forecasting in the middle latitudes ever since. And the smaller scale (ageostrophic) terms give rise the vertical motions, w, that is missing in the geostrophic model. This helps people better understand the ocean's circulation.

In the note we only shows the easiest QG model, and sometimes the assumptions we make may not hold. For example, the viscous effect may be important for certain problem and we need to have the viscous term in the right hand side of equation(2). As long as we keep the assumption that geostrophic balance dominates the Navier-Stokes equation, we can apply a similar derivation as in Section 4.b and give a different form of QG model. In the case of having viscous effect, the QG model we get has the form

$$\frac{\partial q}{\partial t} + J(\psi, q) = \frac{\partial}{\partial x}\left(A\frac{\partial q}{\partial x}\right) + \frac{\partial}{\partial y}\left(A\frac{\partial q}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_E\frac{\partial q}{\partial z}\right)$$

where q is defined as before.

Since QG is an approximation one order higher compared to geostrophic model, what will happen if we take an even higher order approximation? In the 1990s many people made effort to develop more advanced balance models than QG to see if that could bring us more information. For example, David Muraki(1999)[6] tried to approximate the equation(10)-(13) to second Rossby order. A series of review paper for other balance models is [1–3]. So far none of those work has gained as huge of an influence as QG has.

References

[1] JS Allen, JA Barth, and PA Newberger. On intermediate models for barotropic continental shelf and slope flow fields. part i: Formulation and comparison of exact solutions. *Journal of Physical Oceanography*, 20(7):1017–1042, 1990.

[2] JS Allen, JA Barth, and PA Newberger. On intermediate models for barotropic continental shelf and slope flow fields. part iii: Comparison of numerical model solutions in periodic channels. *Journal of Physical Oceanography*, 20(12):1949–1973, 1990.

[3] JA Barth, JS Allen, and PA Newberger. On intermediate models for barotropic continental shelf and slope flow fields. part ii: Comparison of numerical model solutions in doubly periodic domains. *Journal of Physical Oceanography*, 20(7):1044–1076, 1990.

[4] Stephen Childress. *An introduction to theoretical fluid mechanics*, volume 19. American Mathematical Soc., 2009.

[5] Benoit Cushman-Roisin and Jean-Marie Beckers. *Introduction to geophysical fluid dynamics: physical and numerical aspects*, volume 101. Academic Press, 2011.

[6] David J Muraki, Chris Snyder, and Richard Rotunno. The next-order corrections to quasi-geostrophic theory. *Journal of the atmospheric sciences*, 56(11):1547–1560, 1999.

[7] Rick Salmon. *Lectures on geophysical fluid dynamics*. Oxford University Press, 1998.

[8] ucla spinlab. Record Player Fluid Dynamics: A Taylor Column Experiment. `https://www.youtube.com/watch?v=7GGfsW7gOLI`, 2015. [Online; accessed Sep 19, 2014].

[9] Geoffrey K Vallis. *Atmospheric and oceanic fluid dynamics*. Cambridge University Press, 2017.