Incompressible Flow

CFD Spring 2013

$$\left\{\begin{array}{l} \vec{u}_t + \underbrace{\nabla \Pi}_{\substack{\text{Lagrange} \\ \text{multiplier for}}} = -\underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\text{advection}} + \underset{\substack{\uparrow \\ \text{kinematic} \\ \text{viscosity}}}{\nu} \nabla^2 \vec{u} + \text{other} \\ \\ \nabla \cdot \vec{u} = 0 \qquad \text{(incompressibility)} \end{array}\right.$$

Note $\qquad u \cdot \nabla u \equiv \nabla \cdot (u \otimes u)$

$$u_i \longrightarrow u_j \partial_j u_i \equiv \partial_j (u_i u_j)$$

$$\text{since} \qquad \partial_j (u_i u_j) = u_j \partial_j u_i + u_i \underbrace{\cancel{\partial_j u_j}}_{\text{zero}}$$

$$\begin{cases} \partial_t u_i + \partial_i p = - u_j \partial_j u_i + \nu \partial_j \partial_j u_i \\ \partial_j u_j = 0 \qquad , i = 1, 2, \dots d \quad \left(\begin{matrix} d=2 \text{ or} \\ d=3 \end{matrix} \right) \end{cases}$$

Implied summation (Einstein) convention for repeated indices

This form of the equations applies only if density is constant

$$\rho = \text{const.}$$

$$\nu = \frac{\eta}{\rho} \leftarrow \text{viscosity}$$

Vorticity

$$\omega = \nabla \times u \quad \Rightarrow \quad \nabla \cdot \omega = 0$$

$\omega = 0 \rightarrow$ potential flow

$$u \cdot \nabla u = \frac{1}{2} \nabla |u|^2 - u \times (\underbrace{\nabla \times u}_{\omega})$$

(vector identity)

Note $\quad \nabla \times (\nabla p) = 0$

$$\boxed{\partial_t u + \omega \times u + \nabla \tilde{p} = \nu \nabla^2 u + f}$$

Vorticity formulation $\quad = -\nu \nabla \times \omega + f$

where we used $\quad \nabla \times (\nabla \times v) = \nabla (\nabla \cdot v) - \nabla^2 v$

Apply a curl to equation

and use

$$\nabla \times (v \times \omega) = \omega \cdot \nabla v - v \cdot \nabla \omega$$

to get <u>vorticity equation</u>:

$$\boxed{\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v + \nu \nabla^2 \omega}$$

In two dimensions, $\vec{v} = (u, v, 0)$

$\omega \cdot \nabla v = 0$, so there is no vorticity generation:

$$\partial_t \omega + v \cdot \nabla \omega = \nu \nabla^2 \omega \quad \text{in 2D}$$

Define a <u>stream function</u> ψ

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

i.e. $u = (\nabla \psi) \times \hat{z}$

to get Poisson equation for ψ

$$\boxed{\nabla^2 \psi = -\omega} \quad \text{in } \Omega \quad (2D)$$

$$u \xrightarrow[\nabla\times]{} \omega \xrightarrow[\nabla^{-2}]{} \psi \xrightarrow[\nabla\times\hat{z}]{} u$$

$$\left.\begin{aligned} &\partial_t \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega \\ &u = -\nabla\,[(\nabla^{-2}\omega) \times \hat{z}] \end{aligned}\right\} \text{ Only vorticity appears!}$$

PSEUDO-SPECTRAL FOR INCOMPRESSIBLE FLOW

CFD SPRING 2013

Let's start with the simplest case of 2D periodic incompressible flow in the vorticity-stream formulation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega$$

Recall $\begin{cases} \nabla^2 \psi = -\omega \\ u = \nabla^\perp \psi = (\psi_y, -\psi_x) \end{cases}$

Let's handle advection explicitly and diffusion implicitly

For spatial discretization, we can use a spectral method for linear (diffusive) terms: ②

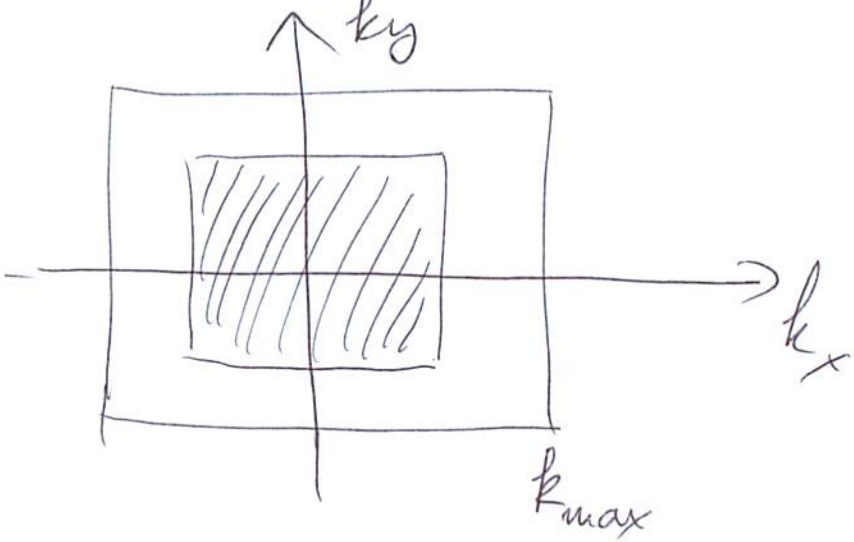
$$\partial_t \hat{\omega} + \widehat{(u \cdot \nabla \omega)} = -\nu k^2 \hat{\omega}$$

$$FT(u \cdot \nabla \omega) = \underset{\substack{\uparrow \\ \text{convolution}}}{\hat{u} \circledast} (-ik\hat{\omega})$$

Convolution is slow, so it is better to handle advection in real space → <u>pseudospectral method</u>

$$\underset{\substack{\uparrow \\ \text{UNALIASED}}}{\widetilde{FT}}(u \cdot \nabla \omega) = \begin{cases} FT(u \cdot \nabla \omega) & \text{if } (k_x, k_y) \leq \frac{2}{3} k_{max} \\ 0 & \text{otherwise} \end{cases}$$

The anti-aliasing procedure simply (3)
<u>filters</u> all wave numbers outside of
the box



There are many other (smoother) filters. Using a filter is often necessary to prevent unphysical artifacts (Gibbs phenomena)

$$\hat{N} = \widehat{(u \cdot \nabla w)} = \widetilde{FT}\,(u \cdot \nabla w) =$$
$$= \widetilde{FT}\left[u_x \nabla_x w + u_y \nabla_y w \right]$$

For temporal integrator can be Crank-Nicolson for diffusion + Adams-Bashforth for advection:

$$\left(\frac{1}{\Delta t} + \frac{\nu k^2}{2}\right)\hat{w}^{n+1} = \left(\frac{3}{2}\hat{N}^{n} - \frac{1}{2}\hat{N}^{n-1}\right) + \left(\frac{1}{\Delta t} - \frac{\nu k^2}{2}\right)\hat{w}^{n} \quad \ldots (*)$$

$N = u \cdot \nabla w$ with $\hat{N}$ filtered

ALGORITHM:

(1) $\hat{\psi}^n = \hat{w}^n / k^2$ (Solve Poisson equation)

(2) $\hat{u}^n = (ik_y, -ik_x)\,\hat{\psi}^n$; $\widehat{\nabla w}^n = (ik_x, ik_y)\,\hat{w}^n$

(3) $u^n = iFFT(\hat{u}^n)$; $\nabla w^n = iFFT(\widehat{\nabla w}^n)$

4 $\hat{N}^n = \hat{u}^n \cdot \nabla \hat{w}^n = u_x \cdot \nabla_x w + u_y \cdot \nabla_y w$

(multiplication!)

5 $\hat{N}^n = \widetilde{FT}(N^n)$ (anti-aliased)

6 Solve (*) for $\hat{w}^{n+1}$

By using Fourier basis we eliminated a Poisson solve and also turned advection into a simple multiplication and got <u>spectral accuracy</u> <u>but</u> <u>only</u> if the solution (u and w) are actually <u>smooth</u> (no high-frequency components!)