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## SPLITTING METHODS

$$u_t + \nabla\cdot(\underline{a}u) = \nabla\cdot(D\nabla u) + f(u)$$

$\nabla\cdot(\underline{a}u)$: non-stiff advection

$\nabla\cdot(D\nabla u)$: stiff diffusion

$f(u)$: very stiff chemisty

Splitting methods use different temporal integrators for different parts.

Main caveat:

$\boxed{\text{Only "easy" to do up to second order accuracy!}}$ SPLITTING ERRORS

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$$w' = Aw = (A_1 + A_2) w$$

$$w(t_{n+1}) = e^{\tau A} w(t_n)$$

$$= e^{\tau (A_1 + A_2)} w(t_n)$$

SPLITTING $\Big\{ \approx e^{\tau A_2} e^{\tau A_1} w_n$

$$\boxed{w^{n+1} = e^{\tau A_2} e^{\tau A_1} w^n} \leftarrow \text{Lie SPLITTING}$$

This is in general only <u>first-order</u> accurate even if $A_1 / A_2$ exponentials handled exactly!

3 BCH formula:

$$e^{\tau A_2} e^{\tau A_1} = e^{\tau \tilde{A}}$$

$$\tilde{A} = A + \frac{\tau}{2}[A_2, A_1] + O(\tau^2)$$

$$[A_2, A_1] = A_2 A_1 - A_1 A_2$$ commutator

$\Rightarrow$ Lie splitting exactly solves the modified equation

$$\begin{cases} w' = \tilde{A} w = Aw - \frac{\tau}{2}[A_2, A_1] w \\ \text{so error is } O(\tau) \end{cases}$$

(4) One can do <u>second-order</u> splitting by <u>symmetrizing</u> the order of splitting to eliminate odd powers of $\tau$:

$$\boxed{w^{n+1} = e^{\tau A_1/2} e^{\tau A_2} e^{\tau A_1/2} w^n}$$

STRANG splitting

Local truncation error

$$\delta^n = \frac{\tau^2}{24}\left([A_1,[A_1,A_2]] + 2[A_2,[A_1,A_2]]\right) w^{n+1/2}$$

or

$$\boxed{w^{n+1} = \frac{1}{2}\left[e^{\tau A_1} e^{\tau A_2} + e^{\tau A_2} e^{\tau A_1}\right] w^n}$$

(5) One can do similar tricks with multi-component splitting

$$A = A_1 + A_2 + A_3$$

$$w^{n+1} = e^{\tau A_1/2} e^{\tau A_2/2} e^{\tau A_3/1} e^{\tau A_2/2} e^{\tau A_1/2} w^n$$

Some results for <u>advection-diffusion</u>

(1) Advection commutes with diffusion only if $\underline{a} = \text{const}$ and $D = \text{const}$

(2) Advection commutes with local terms (reaction, forcing, etc.) if $\nabla \cdot \underline{a} = 0$ and forcing is autonomous

$$f = f(u)$$

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## Non linear case

$$u_t(\underline{x}, t) = f[x, u(\underline{x}, t)]$$

$$f = f_1 + f_2$$

$$u^{n+1} = S^2_{\tau}\left[S^1_{\tau}(u^n)\right] \leftarrow \text{Lie splitting}$$

one-step update for $u_t = f_1$ and $u_t = f_2$

Local truncation error

$$\rho_n = \frac{\tau}{2}\left[\frac{\partial f_1}{\partial u} f_2 - \frac{\partial f_2}{\partial u} f_1\right] u^n = \frac{\tau}{2}[f_1, f_2]\, u^n$$

commutator for functions

(7) **Problems** with splitting:

1) Higher-order integrators are complicated

2) For PDEs, the physical boundary conditions are given for the whole problem, NOT for each of the split problems individually (think advection-diffusion splitting)

$$w' = Aw + g(t) = (A_1 w + g_1) + (A_2 w + g_2)$$

$$w^{n+1} = e^{\tau A_2} e^{\tau A_1} w^n + e^{\tau A_2} \int_0^\tau e^{(\tau - s) A_1} g_1(s)\, ds + \int_0^\tau e^{(\tau - s) A_2} g_2(s)\, ds \neq \text{exact}$$

(8) So now we have an additional splitting error due to BCs; even if $[A_1, A_2] = 0$!

$$\text{error} \approx \int_0^\tau e^{(\tau - s)A} g(s)\,ds - \left[ e^{\tau A_2} \int_0^\tau e^{(\tau - s)A_1} g_1(s)\,ds + \int_0^\tau e^{(\tau - s)A_2} g_2(s)\,ds \right]$$

3) If $A_1$ has eigenvalues proportional to $-1/\varepsilon$, $\tau/\varepsilon \gg 1$ (very stiff problem), then even Strang splitting is first order (reduction of order)