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CFD SPRING SEMESTER A. DONEV

CONVERGENCE : SPATIAL DISCRETIZATION

STABILITY + CONSISTENCY = CONVERGENCE

$$u_t(x,t) = f(u(x,t), x, t)$$

$$w'(t) = F_h(t, w(t))$$ ← semi-discrete

$$w(t) \in \mathbb{R}^m$$

<u>Spatial error</u> :

$$\mathcal{E}(t) = u_h(t) - w(t)$$

where $u_h(t) \Leftrightarrow u(t)$ on h-grid

(e.g., finite difference or finite-volume)

<u>Global error</u>

$$\boxed{\|\mathcal{E}(t)\| = O(h^p)} \text{ for } 0 \le t \le T$$

p - order of convergence

(2) **Local error**

$$\boxed{\sigma_h(t) = u_h'(t) - F(t, u_h(t))}$$

[truncation analysis usually done using done using Taylor series assuming sufficient smoothness]

$$\|\sigma_h(t)\| = O(h^q), \quad q>0 \text{ (usually integer)}$$

$$\boxed{q = \text{consistency order}}$$

Usually $p=q$ but not always (see upcoming lecture on BCs)

$$p \geq q$$

(3) Linear PDE + linear scheme:

$$w'(t) = A\,w(t) + g(t) \quad \ldots\ldots (*)$$

$A \in \mathbb{C}^{m\times m}$ is <u>constant</u>

$(*)$ is <u>STABLE</u> if

Stability
$$\left\{ \begin{array}{l} \|e^{tA}\| \leq K e^{\omega t} \quad \text{for } 0 \leq t \leq T \\ K \geq 1 \text{ and } \omega \in \mathbb{R} \\ \underline{\text{independent}} \text{ of } h \text{ [uniform in } h] \end{array} \right.$$

$\Rightarrow$ bound on global error if consistent. This is a precise version (but not general) of

consistency + stability $\Rightarrow$ convergence

(4)

$$\varepsilon'(t) = u_h'(t) - w'(t)$$

$$= \left(\sigma_h + F(u_h)\right) - \cancel{F(w)} = A(u_h - w) + \sigma_h = A\varepsilon + \sigma_h$$

$$\Rightarrow \varepsilon(t) = e^{tA} \cdot \varepsilon(0) + \int_0^t e^{(t-s)A} \sigma_h(s)\, ds$$

$$\|\varepsilon\| \leq \|e^{tA}\| \, \|\varepsilon(0)\| + \int_0^t \|e^{(t-s)A}\| \, ds \max_{0 \leq s \leq t} \|\sigma_h\|$$

$$\|\varepsilon\| \leq \varepsilon_0 K e^{\omega t} + K \int_0^t e^{\omega \cdot (t-s)} ds \cdot \max \|\sigma_h(t)\|$$

Assume consistency of order q:

$$\|\sigma_h(t)\| \leq c h^q$$

$$\|\varepsilon_0\| \leq c_0 h^q \quad \text{(initial truncation error)}$$

(5)

$$\boxed{\|\mathcal{E}(t)\| \leq K C_0 e^{\omega t} h^q + \frac{KC}{\omega}(e^{\omega t}-1) h^q}$$

If $\omega \geq 0$ linear or exponential growth of errors in time

If $\omega < 0$ then we can bound the error for <u>all</u> $T \geq 0$

$$\boxed{\omega < 0 \Rightarrow \text{global error} \simeq \text{local error}}$$

The difficulty is finding a sharp estimate for ω, i.e., smallest possible ω

Normal A : $AA^* = A^*A$, $A = U \Lambda U^{-1}$

$$\Rightarrow e^{tA} = U e^{t\Lambda} U^{-1}$$

⑥ Aside on norms and things

Normal A: $\|e^{tA}\| \le \underset{\text{condition number} = \|U\|\|U^{-1}\|}{\mathrm{cond}(U)} \cdot \max |e^{\lambda_k t}|$

$$\Rightarrow \boxed{\omega = \max_k \mathrm{Re}(\lambda_k)}$$

For non-normal A one can use logarithmic "norm"

$$\omega = \mu(A) = \lim_{t \to 0} \frac{\|I + tA\| - 1}{t}$$

$$\begin{cases} \mu_1(A) = \max_j \left(\mathrm{Re}\, a_{jj} + \sum_{i \neq j} |a_{ij}| \right) \\ \mu_\infty(A) = \max_i \left(\mathrm{Re}\, a_{ii} + \sum_{i \neq j} |a_{ij}| \right) \end{cases}$$

7 Note that vector (error) norms here should have grid-size (cell-volume) in them:

$$
\begin{cases}
\|v\|_1 = h \sum |v_j| \approx \int_x |u(x)|\, dx \\
\|v_2\|^2 = h \sum v_j^2 \simeq \int_x u^2(x)\, dx \\
\|v\|_\infty = \max_j |v_j| \simeq \max_x u(x)
\end{cases}
$$

(8) <u>Example</u>: Adv-diff equation:

$$u_t + a u_x = d u_{xx} \quad - \quad \underline{\text{periodic}}$$

$$\varepsilon(t) = \sum_k \hat{\varepsilon}_k(t)\, \varphi_k \quad \rightarrow \text{Fourier basis}$$

$$\sigma_h(t) = \sum_k \hat{\sigma}_{h,k}(t)\, \varphi_k$$

$$\Rightarrow \quad \hat{\varepsilon}_k' = \lambda_k \hat{\varepsilon}_k(t) + \hat{\sigma}_{h,k}(t) \rightarrow \underline{\underline{\text{scalar}}} \text{ ODE}$$

For linear equations different (discrete) wavenumbers are fully <u>decoupled</u>

$$\Rightarrow \quad \left[\ \omega = \max_{k \neq 0} \mathrm{Re}(\lambda_k) \right. \leftarrow \text{exclude } k=0 \text{ due to conservation}$$

9. Going back to our eigenvalue calculation:

$$\omega = \begin{cases} -\frac{4d}{h^2}\sin^2(\pi h) \simeq -4d\pi^2 & \text{centered } 2^{nd} \text{ order} \\ -2\frac{|a|}{h}\sin^2(\pi h) \simeq -2|a|\pi^2 h & \text{~~second~~ first-order upwind} \end{cases}$$

not uniform in h
but useful for estimates

Homework: $u_t + u_x = 0$, $u(x,0) = (\sin(\pi x))^{100}$, $0 < x < 1$

Plot evolution of $\|\varepsilon(t)\|_2$ for $m = 100, 200, 400, 800$ for $0 \le t \le 10$

Make error relative to something
compare to theory

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Variable Coefficients

$$u_t + \left[a(x)\, u\right]_x = \left[d(x)\, u_x\right]_x$$

conservation law — advective flux — diffusive flux

Recall $\bar{u}_j(t) = w_j = \frac{1}{h} \int_{x_j - h/2}^{x_j + h/2} u(x,t)\, dx$

Tayler series $= u(x_j, t) + \frac{h^2}{24} u_{xx}(x_j) + \ldots$

We can choose finite-difference or finite-volume

$$\begin{cases} w_j \approx u(x_j, t) \\ w_j \approx \bar{u}_j(t) \end{cases}$$

→ Makes <u>no difference</u> up to second order

(11) But for discretization, we use <u>finite-volume</u>, which means we use flux form or conservation form:

$$w_j'(t) = \frac{1}{h}\left[F_{j-1/2} - F_{j+1/2}\right]$$

fluxes through cell boundaries

Let's take the natural

$$F_{j+1/2} = \underbrace{a(x_{j+1/2})\,w_{j+1/2}}_{\text{advective flux}} - \underbrace{d(x_{j+1/2})(w_j - w_{j+1})}_{\text{diffusive flux}}$$

What is $w_{j+1/2}$?

How to go from <u>cell centers</u> to <u>cell faces</u>?

(12) If $a(x) > 0$ for $\forall x$

$w_{j+1/2} = w_j$ is upwind (first-order)

More generally, one writes

$$\begin{cases} a(x_{j+1/2})\, w_{j+1/2} = a^{+}(x_{j+1/2})\, w_j(t) - a^{-}(x_{j+1/2})\, w_{j+1} \\ \text{where } a^{+} = \max(a,0),\ a^{-} = \min(a,0) \\ \text{for first-order upwind scheme} \end{cases}$$

or

$$\begin{cases} w_{j+1/2} = \frac{1}{2}(w_j + w_{j+1}) \\ \text{for second-order centered scheme} \end{cases}$$

DIY: Convince yourself that for constant coeff. these are the same as before

Do-It-Yourself

(13) One can prove stability for these by using, for example, the logarithmic "norm" $\mu(A)$. For <u>pure advection</u>, one gets:

$$\mu_\infty(A) = w = \frac{1}{h} \max_j \left[a(x_{j-1/2}) - a(x_{j+1/2}) \right] = O(1)$$

$\uparrow$ assume differentiable

and for <u>pure diffusion</u>

$$\mu_1(A) \le 0 \quad \text{and} \quad \mu_\infty(A) \le 0 \quad \Rightarrow$$

$$\|e^{tA}\|_1 \le 1 \quad \text{and} \quad \|e^{tA}\|_\infty \le 1$$

and the same holds for L_2 norm

(14) For <u>third-order upwind-biased</u>:

$$w_{j+1/2} = \begin{cases} \frac{1}{6}\left[-w_{j-1} + 5w_j + 2w_{j+1}\right] \\ \text{if } a(x_{j+1/2}) \geq 0, \text{ and similarly} \\ \text{flip direction for } a_{j+1/2} \leq 0 \end{cases}$$

Analytical <u>excercise</u>:
Show that the order of consistency of this scheme is $q=2$ for finite-difference interpretation, but $q=3$ for finite-volume

No stability result exists in general –
we use <u>heuristics</u> and <u>frozen coefficients</u> arguments if a is smooth