4 Godunov methods

Consider a general conservation law

$$q_t + (f(q))_x = 0$$

flux function

fundamental hyperbolic equation

$$f(q) = aq \quad \text{for advection}$$

Define cell-average:

$$q_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(t^n, x)\, dx$$

Take time derivative and commute

(5)

$$\frac{d q_i^n}{dt} = -\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{d f(q(x))}{dx} \, dx$$

$$= -\frac{1}{\Delta x} \Big(f(q(t, x_{i+1/2})) - f(q(t, x_{i-1/2})) \Big)$$

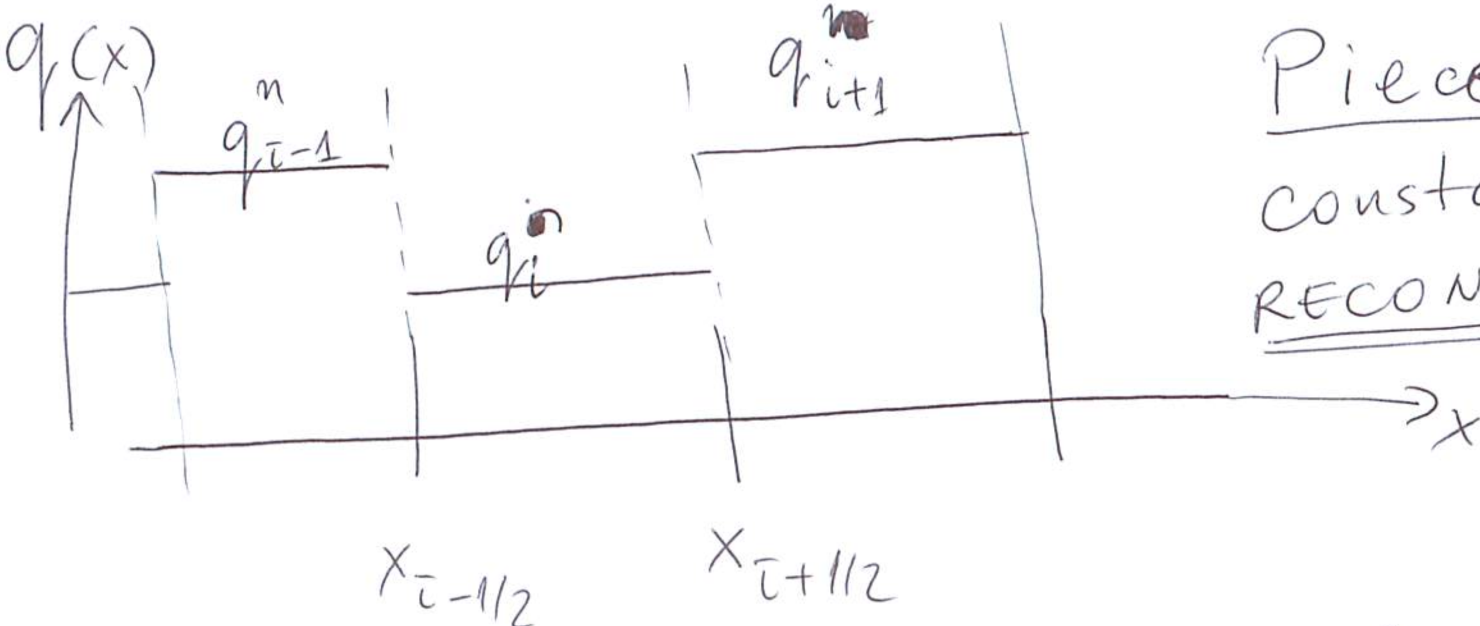
fluxes
through <u>faces</u> of
finite-volume grid

Now integrate in time

$$q_i^{n+1} = q_i^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \Big(f(q(t, x_{i+1/2})) - f(q(t, x_{i-1/2})) \Big) \, dt$$

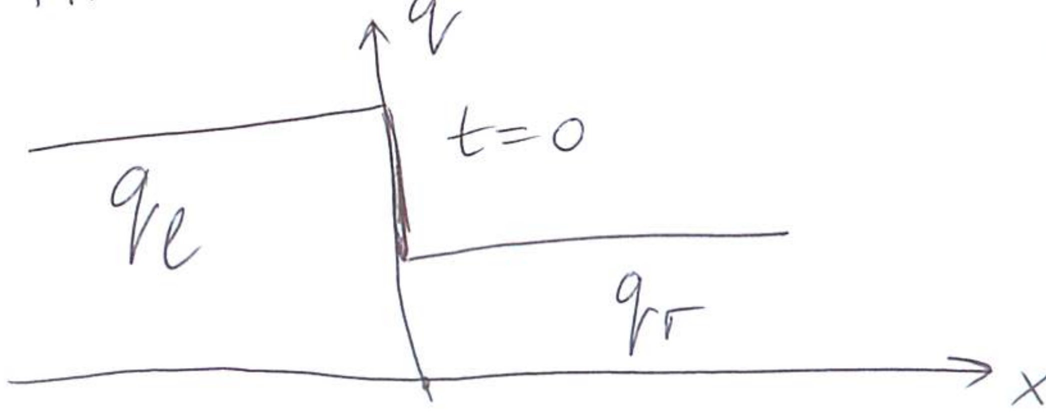
This is so far exact, now we need an approximation

⑥ Riemann problem



Piecewise constant RECONSTRUCTION

Assume that we can solve a piecewise constant problem exactly over a finite time interval



Riemann problem to compute $f^{\downarrow}(q_l, q_r)$

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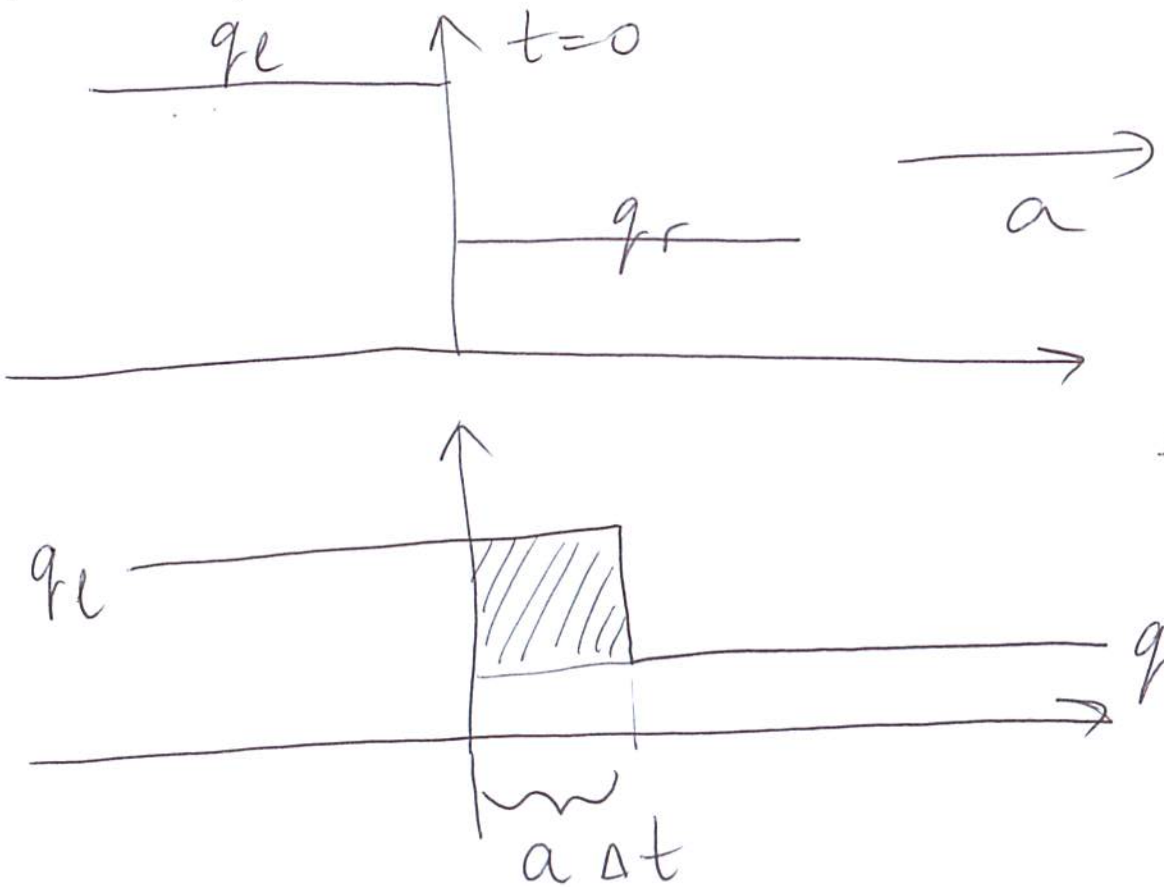
$f^{\downarrow}(q_l, q_r)$ — approximation of <u>average</u> flux in Riemann problem

For linear advection:



$$\boxed{\text{If } f'(q) > 0 \quad f^{\downarrow} \simeq f(q_l)}$$

$$f^{\downarrow} = a q_l$$

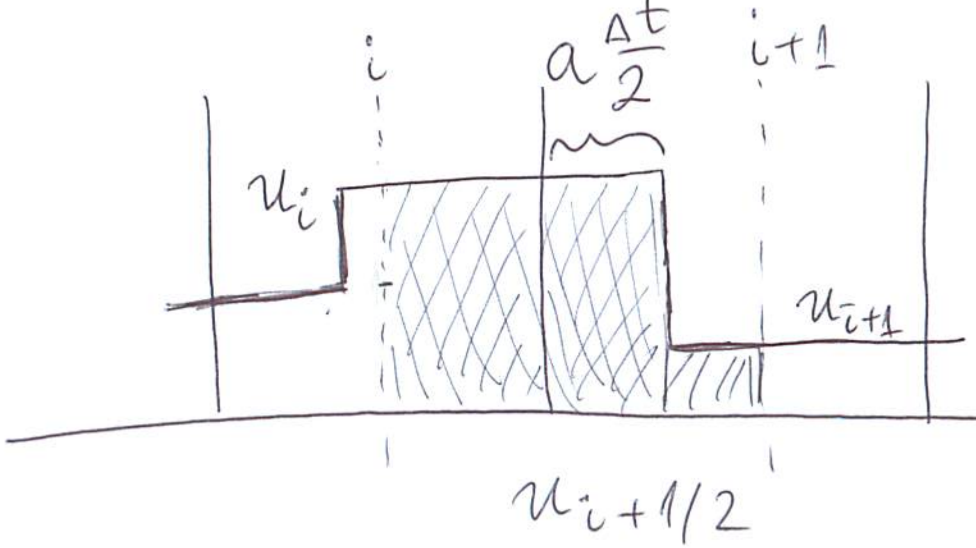
Gives simple <u>upwinding</u>

(8) How can we make this higher order?

Let's center the flux in space and time:

$$f_{i+1/2} = \frac{1}{\Delta x} \int_{-1/2\Delta x}^{1/2\Delta x} f\left[u_{i+1/2}\left(x, \frac{\Delta t}{2}\right)\right] dx$$

Solution of Riemann problem



Assume

CFL < 1

$$a\frac{\Delta t}{2} < \frac{\Delta x}{2}$$

⑨ For linear advection

$$f_{i+1/2} = a\left[\frac{1}{2}u_i + \frac{c}{2}u_i + \left(\frac{1}{2} - \frac{c}{2}\right)u_{i+1}\right]$$

$$= \frac{a}{2}\left[(1+c)u_i + (1-c)u_{i+1}\right]$$

$$f_{i-1/2} = \frac{a}{2}\left[(1+c)u_{i-1} + (1-c)u_i\right]$$

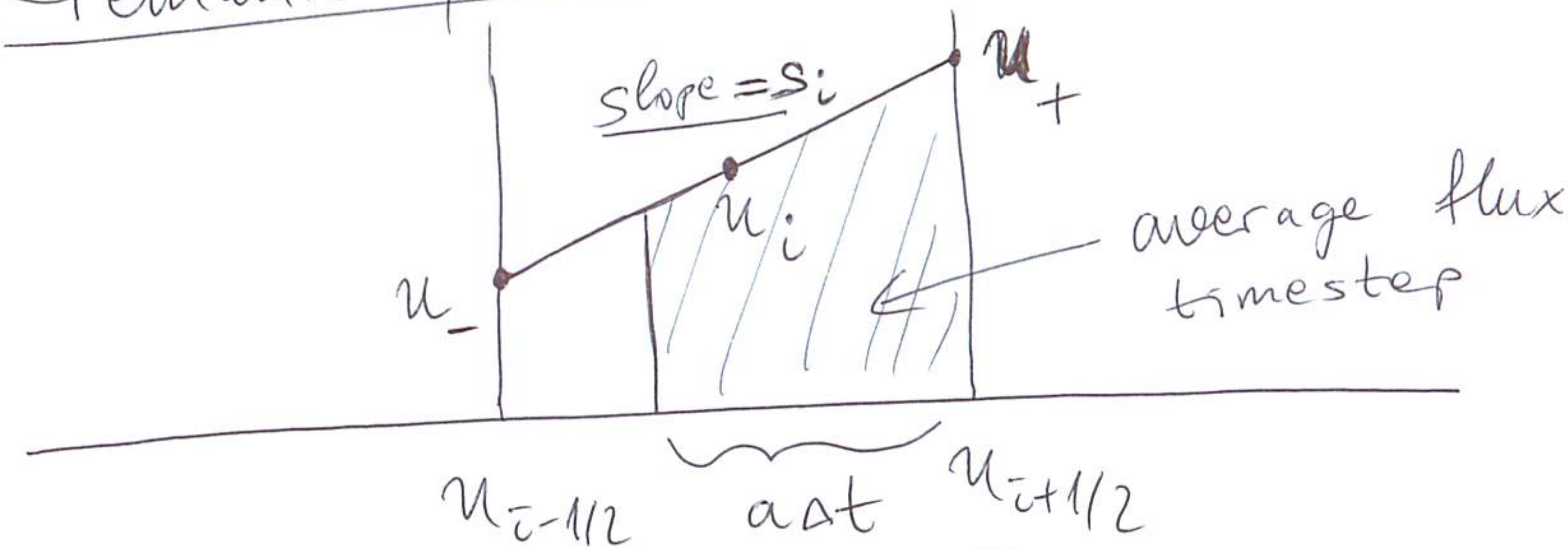
$$\frac{du_i}{dt} = -\frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = -\frac{a}{2\Delta x}\left[(1-c)u_{i+1} + 2cu_i - (1+c)u_{i-1}\right]$$

$u_i^{n+1} = u_i^n$ [redacted] $c\left[\left(\frac{1-c}{2}\right)u_{i+1} + cu_i - \left(\frac{1+c}{2}\right)u_{i-1}\right]$

$\equiv$ Lax-Wendroff!

(10) Another approach to getting higher order is to use **higher-order reconstruction**, such as linear reconstruction

Riemann problem ⟶ a



$$f^{\downarrow}_{i+1/2} = \frac{\text{area} \cdot a}{\Delta x} = \frac{a}{2}\Big[\underbrace{(u_- + u_+)}_{= 2u_i} + (1-c)(u_+ - u_-)\Big]$$

conservation in reconstruction

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$$f^{\downarrow}_{i+1/2} = a u_i + \frac{a \Delta x}{2}(1-c)\, \underset{\uparrow\ \text{slope}}{S_i}$$

Now we need to choose how to compute the slope of the <u>local reconstruction</u>

Set $\quad S_i = \dfrac{u_{i+1} - u_i}{\Delta x} \implies$ Lax-Wendroff

$$f^{\downarrow}_{i+1/2} = a u_i + \frac{a}{2}(1-c)(u_{i+1} - u_i)$$

$$= \frac{a}{2}\left[(1+c)u_i + (1-c)u_{i+1}\right]$$

just as before

But there are <u>other</u> <u>choices</u>

(12) For example, choose <u>centered</u> slopes

$$s_i = \frac{u_{i+1} - u_i}{\Delta x}$$

$$f_{i+1/2} = a u_i + \frac{a}{4}(1-c)\left(u_{i+1} - u_{i-1}\right)$$

$$u_i^{n+1} = u_i^n - c\left(u_i - u_{i-1}\right) \leftarrow \text{upwind piece}$$

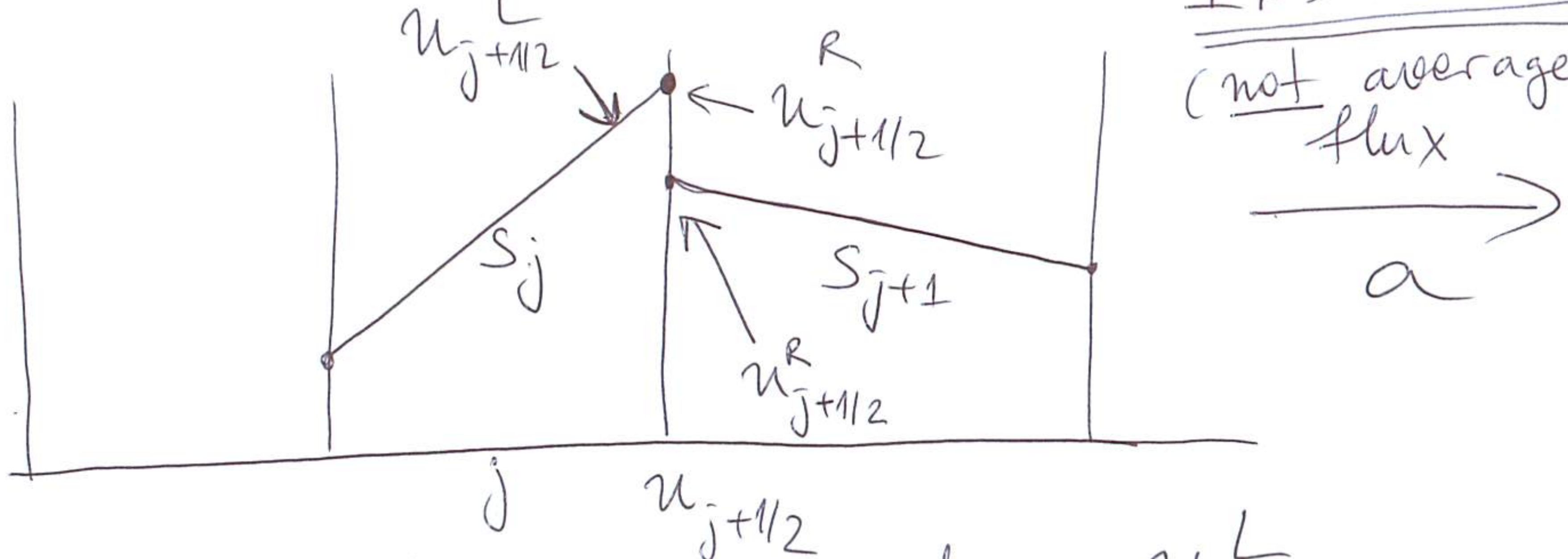
$$- \frac{c(1-c)}{4}\left(\underbrace{u_{i+1} - u_i - u_{i-1} + u_{i-2}}_{\simeq u_{xx} \cdot \Delta x^2}\right) \leftarrow \text{second order correction}$$

this is known as <u>FROMM'S METHOD</u>

15 The higher-order reconstruction can also be used with a MOL (method of lines) approach:

$$f_{j+1/2} = f^{\text{instant}}_{\text{Riemann}}\left(u^L_{j+1/2},\ u^R_{j+1/2}\right)$$

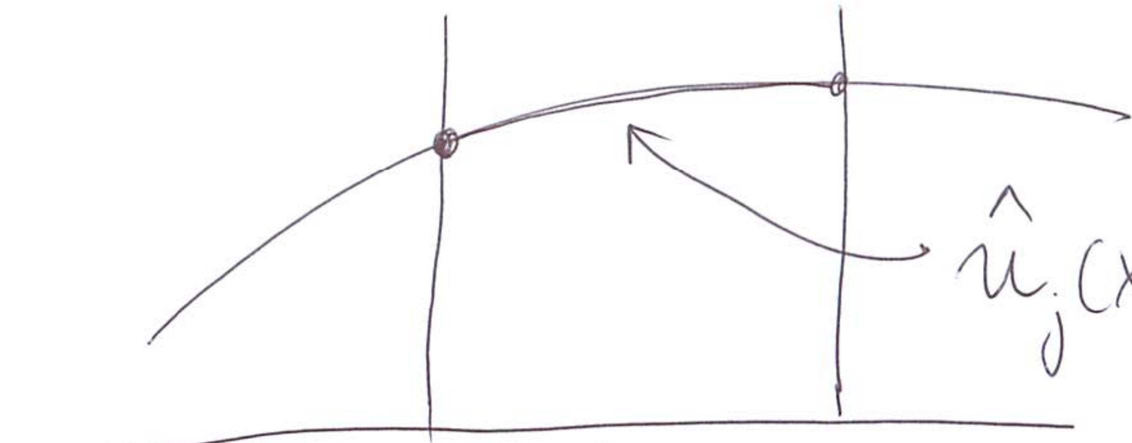
<u>INSTANTANEOUS</u> (<u>not</u> average) flux



For advection this picks $u^L_{j+1/2}$

(16) $$\frac{du_j}{dt} = -\left(\frac{f_{j+1/2} - f_{j-1/2}}{\Delta x}\right)$$

Let us consider a quadratic reconstruction



$$\hat{u}_j(x) = c_1 + c_2 x + c_2 x^2$$

cell j

Conditions for finite-volume averages:

$$\frac{1}{\Delta x}\int_{\text{cell } j} \hat{u}_j(x) = u_j$$

and same for average over cells $j-1$ and $j+1$

(17) Result:

$$\hat{u}_j(x) = u_j + \left(\frac{u_{j+1}-u_{j-1}}{2\Delta x}\right)(x-x_j)$$

$$+ \underbrace{\Bigg(\frac{u_{j+1}-2u_j+u_{j-1}}{2\Delta x^2}\Bigg)\Bigg((x-x_j)^2}_{\text{Taylor series}} - \underbrace{\frac{\Delta x^2}{12}}_{\text{finite-volume piece}}\Bigg)$$

Choose upwinding for flux!

$$f_{j+1/2} = a\,\hat{u}_j\left(x = (j+1/2)\Delta x\right)$$

$$= a\,u^L_{j+1/2}$$

18 Putting it all together gives

$$\frac{du_j}{dt} = - \frac{f_{j+1/2} - f_{j-1/2}}{\Delta x}$$

$$= - \frac{a}{6\Delta x}\left(2u_{j+1} + 3u_j - 6u_{j-1} + u_{j-2}\right)$$

which is our familiar third-order upwind biased spatial discretization.

The second-order version (linear reconstruction) is

$$\frac{du_j}{dt} = - \frac{a}{4\Delta x}\left(u_{j+1} + 3u_j - 5u_{j-1} + u_{j-2}\right)$$

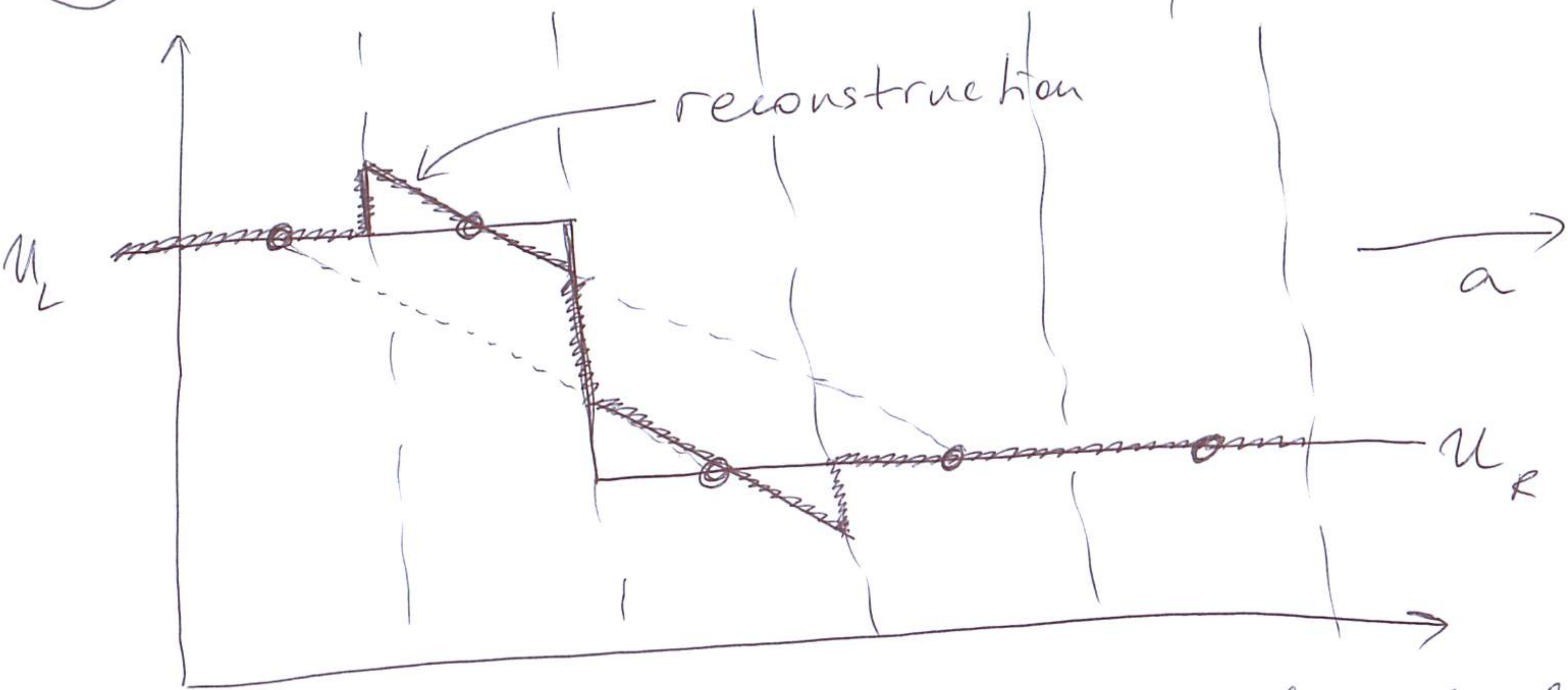
(NOT WORTH IT)

19 The idea of limiting can also be applied to reconstructions:
slope limiting vs. flux limiting
(more or less the same thing)

$u^L_{j+1/2}$ and $u^R_{j-1/2}$ should lie between u_{j-1} and u_{j+1} unless
u_j is an extremum

This is what ensures monotonicity preserving schemes (and in 1D it gives TVD schemes)

(20) Consider a discontinuous profile



Note how the reconstruction has local over and under shoots which will persist upon advection to the right

We MUST LIMIT SLOPE $\quad S_j \neq \frac{u_{j+1} - u_{j-1}}{2\Delta x}$

(21) E.g. minmod limiter:

$$S_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x} \xrightarrow[\text{limiting}]{} \underset{\text{abs}}{\min} \left\{ \frac{u_{j+1} - u_j}{\Delta x}, \frac{u_j - u_{j-1}}{\Delta x} \right\}$$

or zero if opposite sign

The van Leer limiter

$$S_j = \frac{1}{\Delta x} \left(\frac{2 \Delta_+ \Delta_-}{\Delta_+ + \Delta_-} \right) \text{ where}$$

$$\Delta_+ = u_{j+1} - u_j \qquad \Delta_- = u_j - u_{j-1}$$

If $\Delta^- \gg \Delta^+ \Rightarrow S_j \rightarrow 2 \left(\frac{u_{j+1} - u_j}{\Delta x} \right)$

$\Rightarrow u^L_{j+1/2} = u_{j+1}$ which is the max permissible

(22) These ideas can be generalized to:

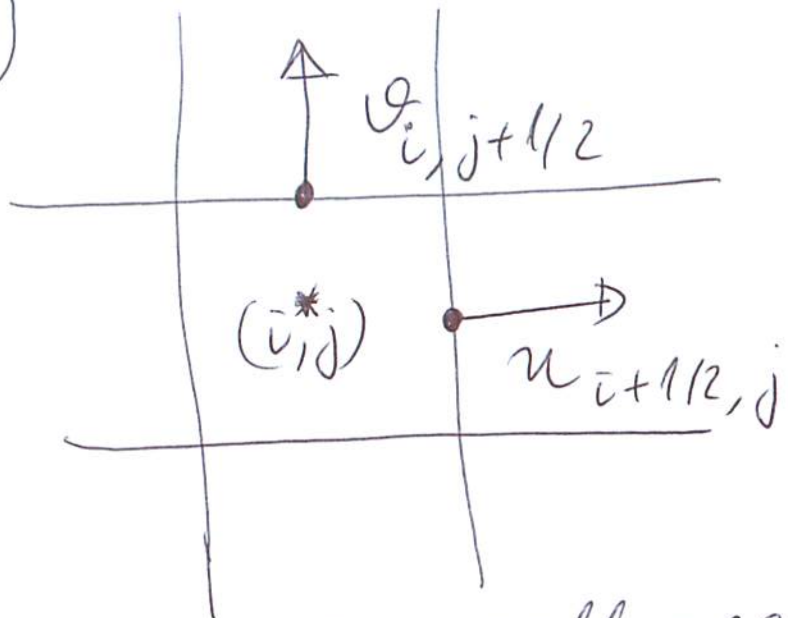
(1) Dimensions larger than 1

(2) Nonlinear hyperbolic equations (eigenvalue decomposition of local solution) ← e.g. Roe's method

(3) systems of equations

As an example, let us <u>briefly</u> summarize the high-resolution advection scheme of Bell-Dawson-Shubin (BDS) in two dimensions → see paper by May, Nonaka, Almgren, Bell linked on course webpage

(23)



$$\begin{cases} S_t + \vec{v} \cdot \vec{\nabla} S = 0 \\ \nabla \cdot \vec{v} = 0 \\ \vec{v} = (u, v) \end{cases}$$

$s \equiv$ cell-centered scalar

$\vec{v} = (u, v) \equiv$ (staggered) face-centered advection velocity field

BDS method:

(1) Construct a limited (bi-)linear polynomial reconstruction of $\hat{s}_{i,j}(x,y)$

(2) Advect reconstruction over time Δt and compute average fluxes

(3) Update solution conservatively

(24) So step (3) is:

$$S^{n}_{i,j} = S^{n}_{i,j} - \frac{\Delta t}{\Delta x}\left[u_{i+1/2,j}\, S_{i+1/2,j} - u_{i-1/2,j}\, S_{i-1/2,j}\right]$$

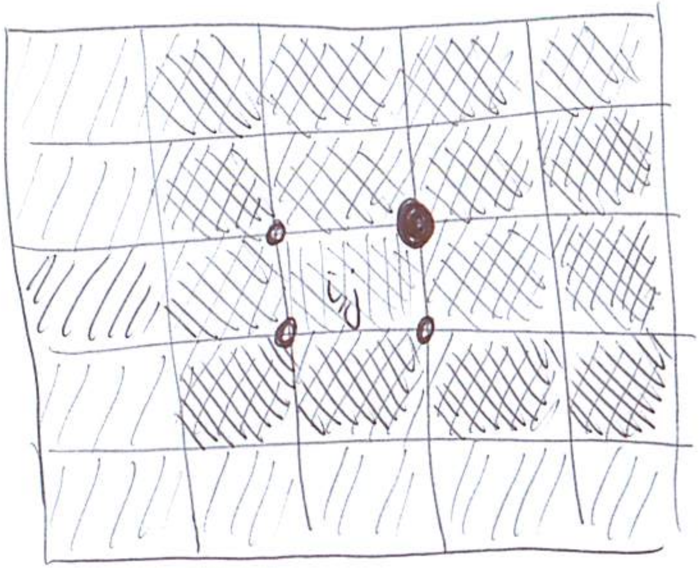
computed in step 2

$$- \frac{\Delta t}{\Delta y}\left[v_{i,j+1/2}\, S_{i,j+1/2} - v_{i,j-1/2}\, S_{i,j-1/2}\right]$$

<u>Step 1</u> first constructs an unlimited bilinear intepolant

$$P_{i,j}(x,y) \approx \hat{S}_{i,j}(x,y) = S_{xy}(x-x_i)(y-y_j) + S_x(x-x_i) + S_y(y-y_j)$$

(25) where the stencil for corner (node) $i+1/2, j+1/2$



see eq. (6) in paper

$$\frac{1}{144}\begin{bmatrix}1 & -7 & -7 & 1\\ -7 & 49 & 49 & -7\\ -7 & 49 & 49 & -7\\ 1 & -7 & -7 & 1\end{bmatrix}$$

$S_{i+1/2,\, j+1/2}$

Limiting consists of solving the local optimization problem:

$$\min \left\| \hat{S}_{i,j}(x,y) - P_{i,j}(x,y) \right\|_{L_2}$$

(26) such that

(a) $$\frac{1}{4}\sum_{\alpha,\beta=\pm 1} \hat{S}_{i,j}\left(x_{i+\alpha\cdot\frac{1}{2}},\, y_{j+\beta\cdot\frac{1}{2}}\right) = S_{i,j}$$

(conservation: average value unchanged)

(b) $\hat{S}_{i,j}\left(x_{i+\alpha\cdot\frac{1}{2}},\, y_{j+\beta\cdot\frac{1}{2}}\right)$ lie between the minimum and maximum of the 4 cell values surrounding the corner

$i, j+1$	$i+1, j+1$
i, j	$i+1, j$

$i+1/2, j+1/2$

$$\hat{S}_{i+1/2,\, j+1/2} \in \left[\begin{matrix}\min \\ \max\end{matrix}\left(S_{i,j}, S_{i+1,j+1}, S_{i,j+1}, S_{i+1,j}\right)\right]$$

(27) This is a linearly constrained quadratic optimization problem that can be solved exactly or approximately (since min L_2 norm was an arbitrary choice) [see paper for details]

Step 2 is the most involved step and can be explained first by considering constant (u, v):

$$S_t + u S_x + v S_y = 0 \,, \quad u, v > 0$$

which can be generalized ...

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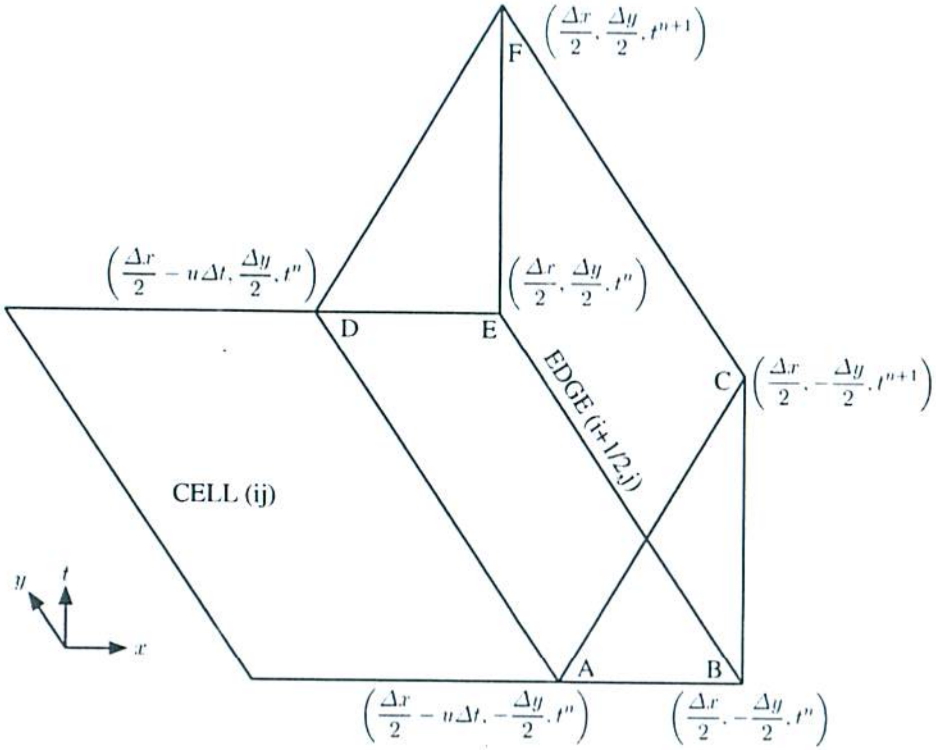
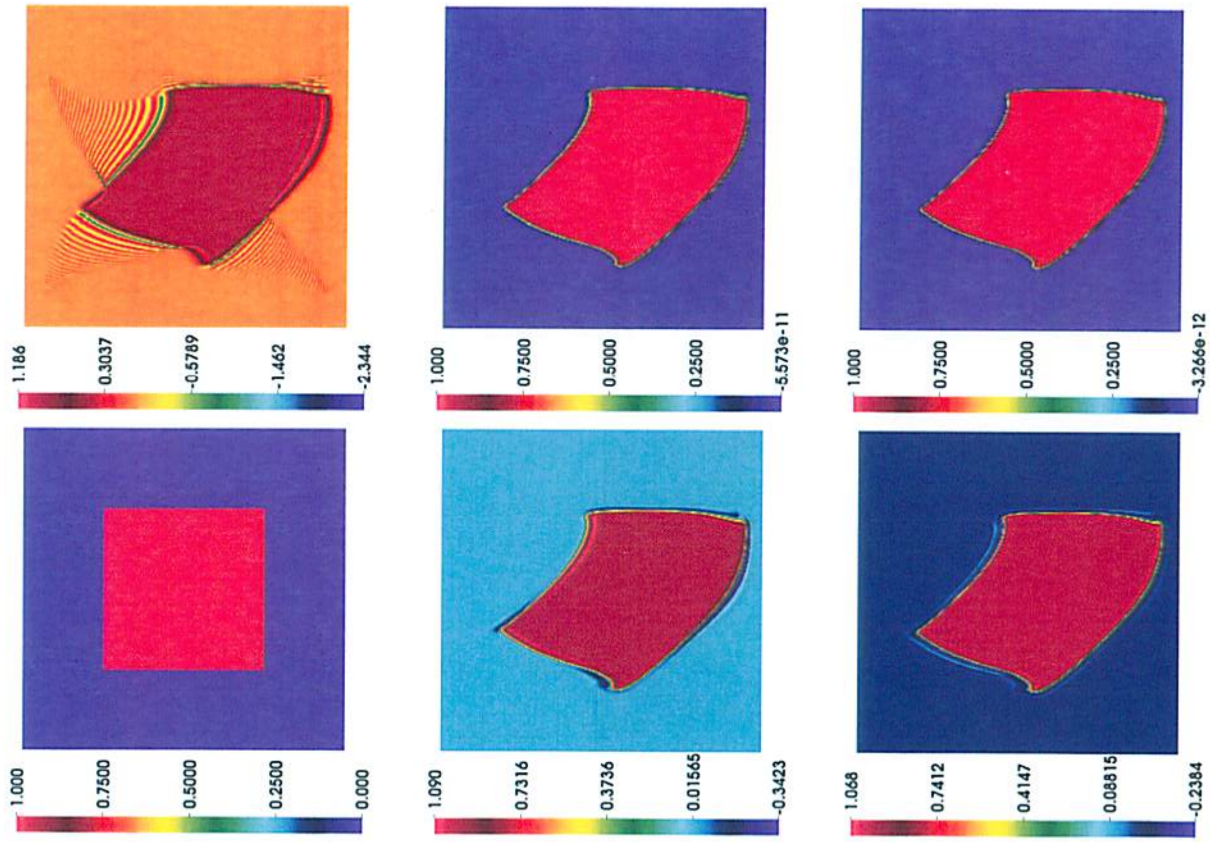


$$u\, s^L_{i+1/2,j}\, \Delta t\, \Delta y = u \iint_{BCEF} s\, dy\, dt$$
$$= \iint_{ABDE} s\, dx\, dy + v \iint_{ABC} s\, dx\, dt - v \iint_{DEF} s\, dx\, dt. \quad (13)$$

See paper for details Note that one cannot do the advection step exactly even with constant (u, v), unlike in one dimension

(29) Example BDS advection of square (A.J. Nonaka from LBL)

CENTERED LIMITED BILINEAR LIMITED QUADRATIC



t=0 UNLIMITED BILINEAR UNLIMITED QUADRATIC

FIG. 8.2. *Square profile advection with infinite Schmidt number and sharp discontinuities. Colorbars indicate the concentration. The initial condition is in the top-left; the other images represent the final ($t = 1.25$) condition: (Top-Right) centered advection, (Middle-Left) unlimited bilinear BDS, (Middle-Right) limited bilinear BDS, (Bottom-Left) unlimited quadratic BDS (Bottom-Right) limited quadratic BDS*

LID-DRIVEN CAVITY EXAMPLE