SERIES-EXPANSION METHODS

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This is based on ch. 4 in "Numerical methods for wave equations" by Durran.

We want to solve PDE

$$\frac{\partial \psi}{\partial t} + F(\psi) = 0$$

We have a numerical approximation

$$\varphi \simeq \psi$$

and want to expand/represent it in some finite basis for the functional space of interest.

$$\varphi(x,t) = \sum_{k=1}^{N} a_k(t)\, \varphi_k(x)$$

basis functions

Residual

$$R(\varphi) = \frac{\partial \varphi}{\partial t} + F(\varphi)$$

should somehow be minimized

1 Equivalent: Minimize $R(\varphi)$ in L_2 norm

or require that $R(\varphi)$ be orthogonal to the ~~finite~~ finite-dimensional subspace spanned by $\{\varphi_k(x)\}$: GALERKIN

2 Collocation: $R[\varphi(j\Delta x)] = 0, \; j = 1, \dots, N$

We adopt the GALERKIN approach. (3)

Minimize over $\dot{a}_k = \frac{d a_k(t)}{dt}$ s.t.

$$\int R[\varphi(x)]\,\varphi_k(x)\,dx = 0 =$$

$$\int \left[\sum_{n=1}^{N} \dot{a}_n \varphi_n + F\left(\sum_{n=1}^{N} a_n \varphi_n\right) \right] \varphi_k \, dx$$

$$\Rightarrow \boxed{\sum_n M_{nk}\,\dot{a}_n = -\int F\left(\sum_n a_n \varphi_n\right) \varphi_k \, dx}$$

for $k = 1, \dots, N$ LINEAR SYSTEM of eqs.

$$\boxed{M_{nk} = \int \varphi_n \varphi_k \, dx} \quad \text{(mass matrix)}$$

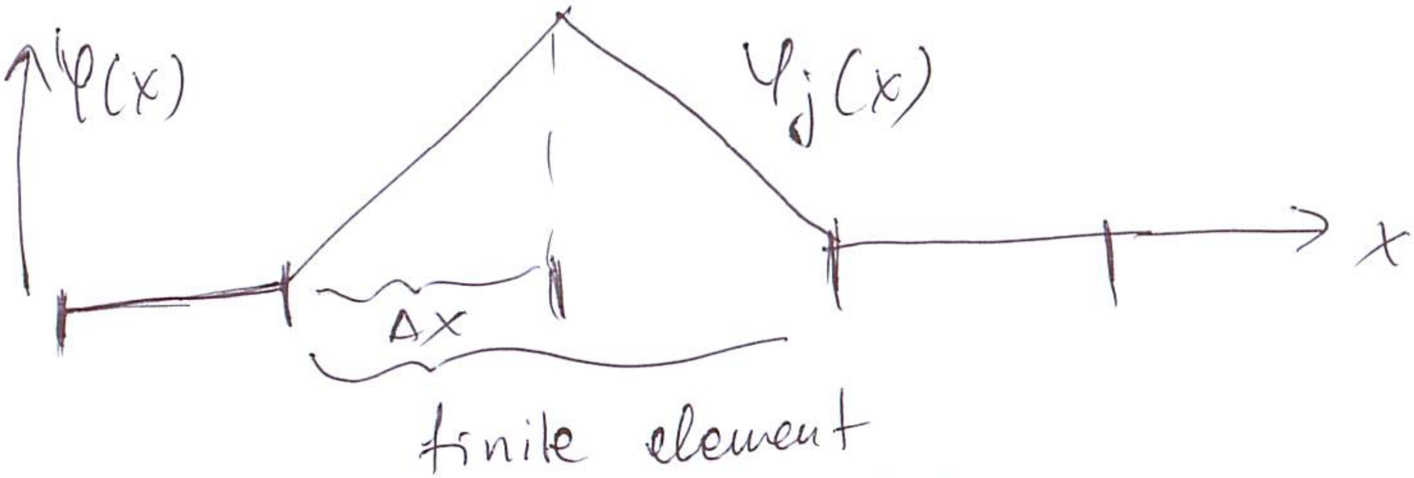
Note that if the basis functions are orthonormal then M is the identity matrix, $M = I$

We have now converted the PDE into a system of ODEs for $a_k(t)$
→ spatial discretization

Depending on the choice of basis functions, we can be doing spectral, finite-element, etc.

Finite - Element Method

The main difference with (pseudo) spectral is that now we choose a localized basis function set, e.g.



$$\int \varphi_j \varphi_{j+1}\, dx = \int_0^{\Delta x} \frac{x}{\Delta x} \frac{\Delta x - x}{\Delta x}\, dx = \frac{\Delta x}{6}$$

and we also need

$$\int \varphi_j^2\, dx = \frac{2\Delta x}{3}$$

$$-\underbrace{\int \frac{\partial \varphi_{j-1}}{\partial x}}\varphi_j \, dx = \frac{1}{2}$$

can be done using integration by parts if φ is not differentiable

So for $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$ this gives the Galerkin finite-element discretisation

$$\underbrace{\frac{\dot{a}_{j+1} + 4\dot{a}_j + \dot{a}_{j-1}}{6}}_{\text{sparse matrix}} + c\left(\frac{a_{j+1} - a_{j-1}}{2\Delta x}\right) = 0$$

Note that $a_j \equiv \psi_j$ for this basis

Denote $$M = \frac{1}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 4 \end{bmatrix} \Rightarrow$$

$$\dot{a} = \dot{\psi} = -M^{-1}(c\Delta a) = -c(M^{-1}\Delta)\psi$$

centered finite difference

$$\Rightarrow \boxed{\dot{\psi} = -c\,(M^{-1}\Delta)\,\psi}$$

Finite-difference Approximation of $\frac{\partial}{\partial x} \rightarrow$ called "compact finite difference" in the literature.

It turns out $M^{-1}\Delta$ is a <u>fourth-order</u> approximation of $\frac{\partial}{\partial x}$!

So this finite-element method is fourth order in space.

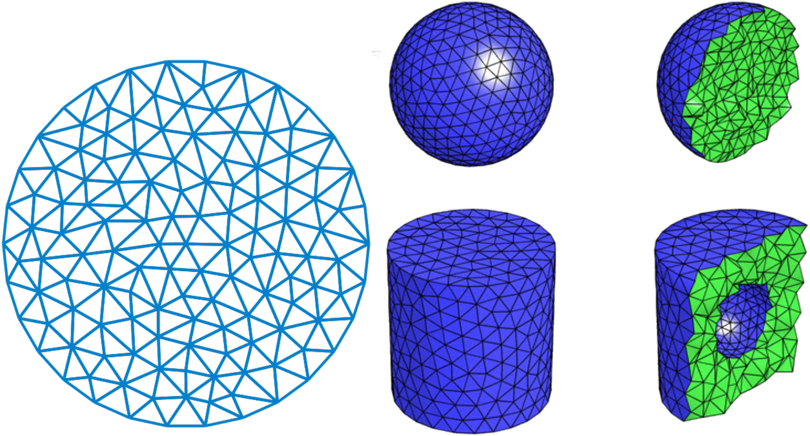
An alternative approach to minimizing the residual $R(\varphi) = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}$ is to replace the time derivative right-away with a difference:

$$\begin{cases} \dfrac{\varphi^{n+1} - \varphi^{n}}{\Delta t} + c\,\dfrac{\partial \varphi}{\partial x} = 0 \quad \ldots\ldots (*) \\ \text{where} \quad \varphi^{n} = \sum a_k^n \varphi_k(x) \end{cases}$$

Now minimize the residual in $(*)$ to get $\varphi^{n+1} \leftarrow \varphi^{n}$, i.e., $a^{n+1} \leftarrow a^{n}$

Irregular (Simplicial) Meshes

Any polygon can be triangulated into arbitrarily many **disjoint triangles**. Similarly **tetrahedral meshes** in 3D.



Basis functions on triangles

- For irregular grids the x and y directions are no longer separable.
- But the idea of using basis functions $\phi_{i,j}$, a **reference triangle**, and **piecewise polynomial interpolants** still applies.
- For a linear function we need 3 coefficients (x, y, const), for quadratic 6 $(x, y, x^2, y^2, xy, \text{const})$:

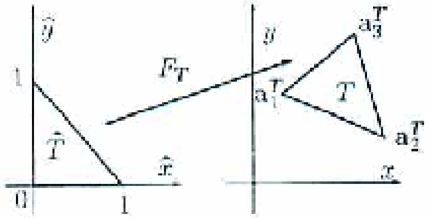


Fig. 8.8. Local interpolation nodes on $\hat{T}$ for $k = 0$ (left), $k = 1$ (center), $k = 2$ (right)

Piecewise constant / linear basis functions

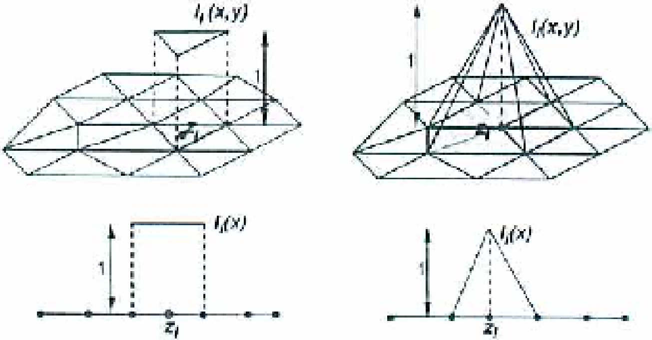
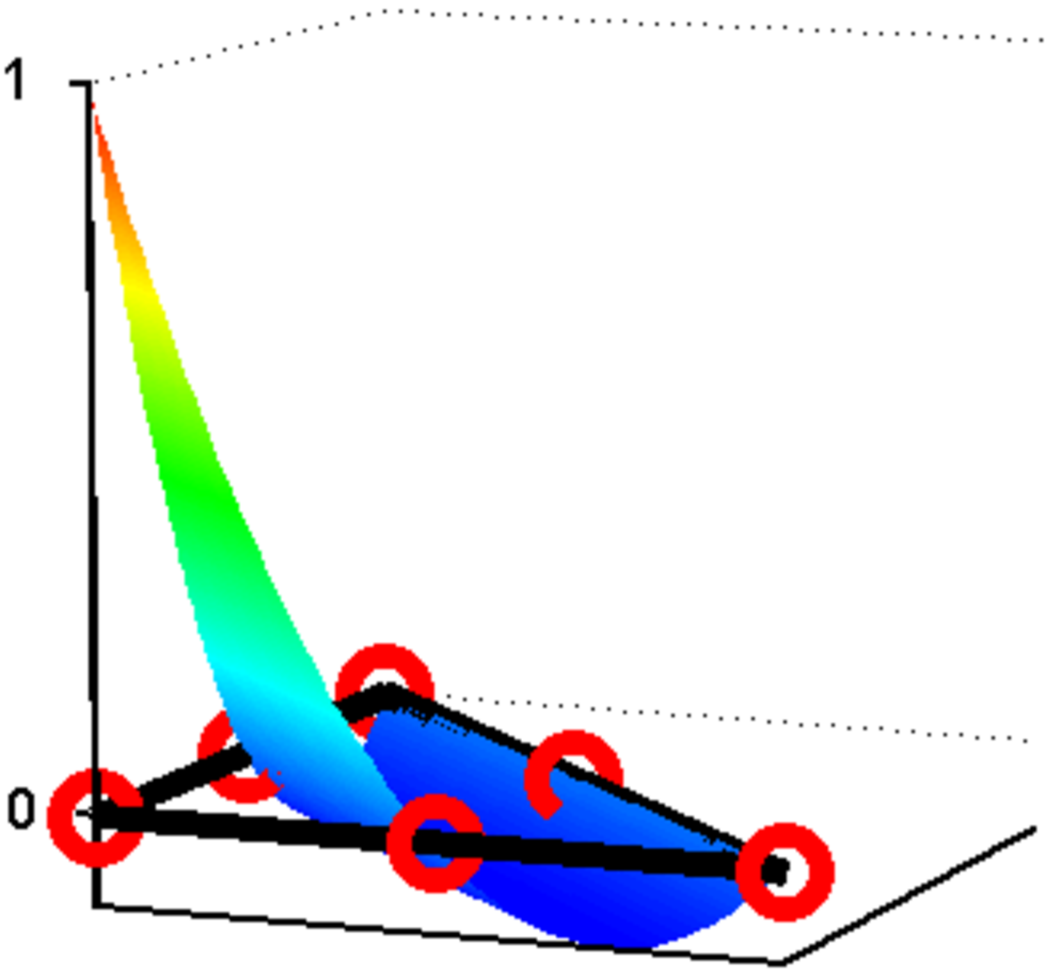


Fig. 8.7. Characteristic piecewise Lagrange polynomial, in two and one space dimensions. Left, $k = 0$; right, $k = 1$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