(1) CFD SPRING 2012 A. DONEV

ADVECTION – DIFFUSION EQUATIONS

based on book by Hundsdorfer / Verwer

1D :

$$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x}\Big[\underbrace{a(x,t)\,u(x,t)}_{\text{advective flux}}\Big] = \frac{\partial}{\partial x}\Big[\underbrace{d(x,t)\,u_x(x,t)}_{\text{diffusive flux}}\Big]$$

conservation law

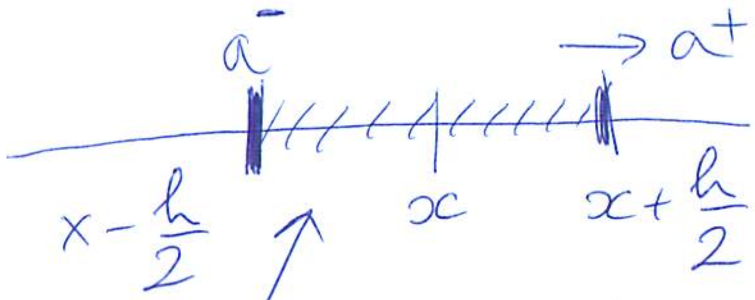
In this equation, $a(x,t)$ is a *given* advection velocity, and $d(x,t)$ is a *given* diffusion coefficient

$$\boxed{u_t + (au)_x = (du_x)_x + f(u)} \quad \dots (*)$$

Concentration of ions, pollutant, etc. (transported quantity)

forcing (reactions)

(2) The adv-diff equation is an expression of <u>conservation of mass</u>



finite volume cell

$$\bar{u}^x = \frac{1}{h}\int_{x-h/2}^{x+h/2} u(x,t)\,dx$$

$$h\frac{\partial}{\partial t}\bar{u}^x = -\left[a\left(x+\tfrac{h}{2}\right)u\left(x+\tfrac{h}{2}\right) - a\left(x-\tfrac{h}{2}\right)u\left(x-\tfrac{h}{2}\right)\right]$$

<u>exact!</u>

(not a discretization)

$$+\left[d\left(x+\tfrac{h}{2}\right)u_x\left(x+\tfrac{h}{2}\right) - d\left(x-\tfrac{h}{2}\right)u_x\left(x-\tfrac{h}{2}\right)\right]$$

lim $h \to 0$ gives (*)

Fick's law (constitutive relation) from <u>microscopic</u> physics

(3) For dimensions higher than one (typically dim = 2 or dim = 3)

nabla notation

$$\begin{cases} \nabla = (\partial_x, \partial_y, \partial_z)^T = \text{grad} \\ \Delta = \nabla \cdot \nabla = \text{div grad} = \partial_{xx} + \partial_{yy} + \partial_{zz} \end{cases}$$

I prefer ∇^2 instead of Δ

$$u_t + \nabla \cdot (\underline{a}\, u) = \nabla \cdot (D \nabla u)$$

$$\underline{a} \in \mathbb{R}^{dim}, \quad D \in \mathbb{R}^{dim \times dim} \succcurlyeq 0 \text{ (SPD)}$$

$$\nabla \cdot (\underline{a} u) = \partial_\alpha (a_\alpha u) = \quad (\alpha = 1, 2, .., dim)$$

$$(\partial_\alpha a_\alpha) u + a_\alpha \partial_\alpha u =$$

$$(\nabla \cdot \underline{a}) u + a \cdot \nabla u = a \cdot \nabla u$$

if $\boxed{\nabla \cdot \underline{a} = 0}$ solenoidal or incompressible

$$u_t + \underline{a} \cdot \nabla u = \nabla \cdot (D \nabla u)$$

advective form (non-conservative in general)

Another way in which advective form can arise:

$$\begin{cases} u_t + \nabla \cdot (\underline{a} u) = \nabla \cdot (D \nabla u) & \leftarrow \text{density of pollutant} \\ \rho_t + \nabla \cdot (\underline{a} \rho) = 0 & \leftarrow \text{true } \underline{\text{mass density}} \text{ (does } \underline{\text{NOT}} \text{ diffuse)} \end{cases}$$

Define $c = \dfrac{u}{\rho}$ as <u>concentration</u>

$$u = \rho c \Rightarrow u_t = \rho c_t + c \rho_t \Rightarrow$$

$$\rho c_t - c \nabla \cdot (\underline{a} \rho) + \nabla \cdot (\underline{a} u) = \rho (c_t + \underline{a} \cdot \nabla c)$$

$$\left\{ \text{Note } \nabla \cdot (\underline{a} \rho c) = c \nabla \cdot (\underline{a} \rho) + \rho \underline{a} \cdot \nabla c \right.$$

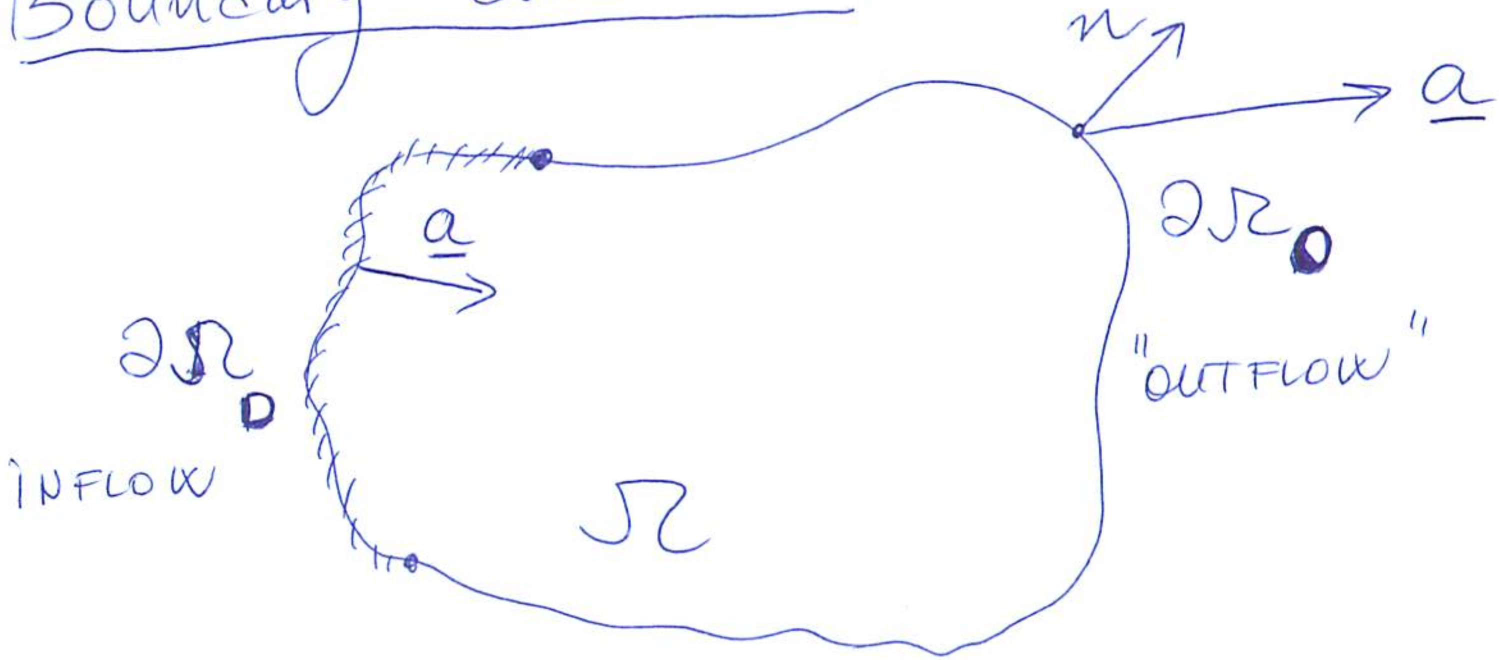
$$\rho(c_t + \underline{a}\cdot\nabla c) = \underbrace{\nabla\cdot(D\nabla u)}_{\text{Correct physics usually } \nabla\cdot(\rho D \nabla c)}$$

$$\boxed{D_t c = c_t + \underline{a}\cdot\nabla c} = \rho^{-1}\,\nabla\cdot(\rho D\nabla c)$$

advective or <u>material derivative</u>

Notice <u>only</u> conservative if $\rho = \text{const}$.

<u>Boundary conditions</u>:



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$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_O$$

$\underline{n} \cdot \underline{a} < 0$ on $\partial\Omega_D$

normal vector (outward)

$\underline{n} \cdot \underline{a} \geq 0$ on $\partial\Omega_O$

a) For <u>pure advection</u>,

<u>Dirichlet</u> BC $\boxed{u = g_D}$ on $\partial\Omega_D$

is <u>sufficient</u>

b) For diffusion or adv-diff, one needs BCs on $\partial\Omega_O$ as well, e.g.

<u>Neumann</u> BC : $\boxed{\underline{n} \cdot \nabla u = g_N}$ on $\partial\Omega_N$

(in) homogeneous

<u>Robin</u> or <u>zero flux</u> BC : $\boxed{\underline{n} \cdot (\underline{a} u - D \nabla u) = g_M}$ on $\partial\Omega_M$

We see that adding diffusion dramatically changes the character of the problem!

$u_t + \nabla\cdot(\underline{a}\, u) = 0$ is a pure <u>hyperbolic</u> eq.

$u_t = \nabla\cdot(D \nabla u)$ is a pure <u>parabolic</u> eq.

steady state $u_t = 0 = \nabla\cdot(D\nabla u)$ is a pure <u>elliptic</u> eq

In reality most equations have mixed character, but often problems are <u>advection-dominated</u> or <u>diffusion-dominated</u> and behave alike the limiting cases.