Block-Structured Adaptive Mesh Refinement

Lecture 2

- Incompressible Navier-Stokes Equations
  - Fractional Step Scheme
- 1-D AMR for "classical" PDE's
  - hyperbolic
  - elliptic
  - parabolic
- Accuracy considerations

Extension to More General Systems

How do we generalize the basic AMR ideas to more general systems?

Incompressible Navier-Stokes equations as a prototype

$$U_t + U \cdot \nabla U + \nabla p = \epsilon \Delta U$$

$$\nabla \cdot U = 0$$

- Advective transport
- Diffusive transport
- Evolution subject to a constraint

Vector field decomposition

Hodge decomposition: Any vector field V can be written as

$$V = U_d + \nabla\phi$$

where $\nabla \cdot U_d = 0$ and $U \cdot n = 0$ on the boundary

The two components, U_d and $\nabla\phi$ are orthogonal

$$\int U \cdot \nabla\phi \, dx = 0$$

With these properties we can define a projection $\mathbf{P}$

$$\mathbf{P} = I - \nabla(\Delta^{-1})\nabla\cdot$$

such that

$$U_d = \mathbf{P}V$$

with $\mathbf{P}^2 = \mathbf{P}$ and $||\mathbf{P}|| = 1$

Projection form of Navier-Stokes

Incompressible Navier-Stokes equations

$$U_t + U \cdot \nabla U + \nabla p = \epsilon \Delta U$$

$$\nabla \cdot U = 0$$

Applying the projection to the momentum equation recasts the system as an initial value problem

$$U_t + \mathbf{P}(U \cdot \nabla U - \epsilon \Delta U) = 0$$

Develop a fractional step scheme based on the projection form of equations

Design of the fractional step scheme takes into account issues that will arise in generalizing the methodology to

- More general Low Mach number models
- AMR

Discrete projection

Projection separates vector fields into orthogonal components

$$V = U_d + \nabla \phi$$

Orthogonality from integration by parts (with $U \cdot n = 0$ at boundaries)

$$\int U \cdot \nabla p \, dx = - \int \nabla \cdot U \; p \, dx = 0$$

Discretely mimic the summation by parts:

$$\sum U \cdot GP = - \sum (DU) \; p$$

In matrix form $D = -G^T$

Discrete projection

$$V = U_d + Gp$$

$$DV = DGp \qquad U_d = V - Gp$$

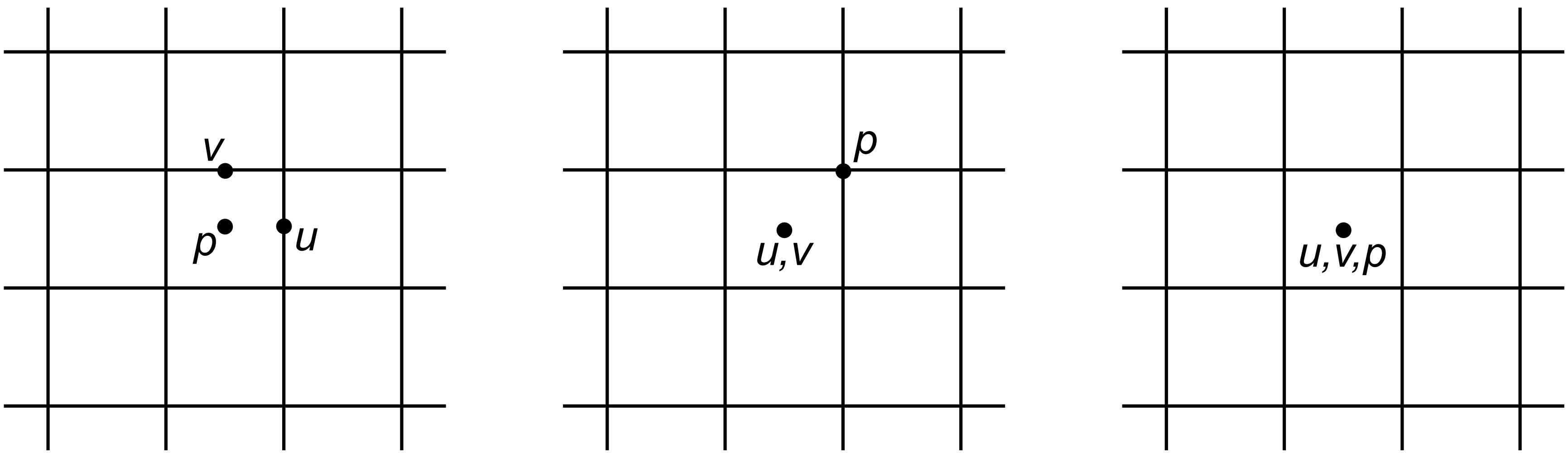
$$\mathbf{P} = I - G(DG)^{-1} D$$

Spatial discretization

Define discrete variables so that U, Gp defined at the same locations and DU, p defined at the same locations.

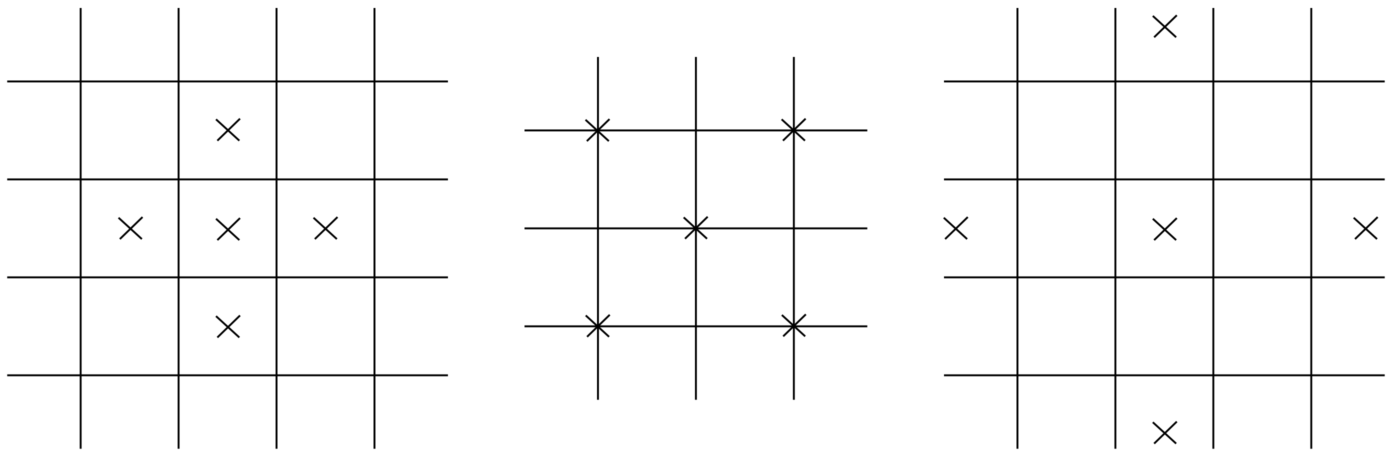
$$D : V_{space} \rightarrow p_{space} \quad G : p_{space} \rightarrow V_{space}$$

Candidate variable definitions:



Projection discretizations

What is the DG stencil corresponding to the different discretization choices



Non-compact stencils $\rightarrow$ decoupling in matrix

Decoupling is not a problem for incompressible Navier-Stokes with homogeneous boundary conditions but it causes difficulties for

- Nontrivial boundary conditions
- Low Mach number generalizations
- AMR

Fully staggered MAC discretization is problematic for AMR

- Proliferation of solvers
- Algorithm and discretization design issues

Approximate projection methods

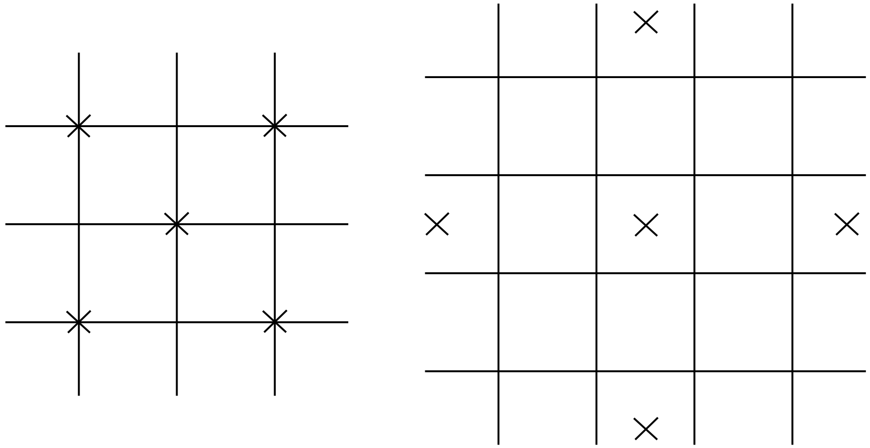
Based on AMR considerations, we will define velocities at cell-centers

Discrete projection

$$V = U_d + Gp$$

$$DV = DGp \qquad U_d = V - Gp$$

$$\mathbf{P} = I - G(DG)^{-1}D$$



Avoid decoupling by replacing inversion of DG in definition of $\mathbf{P}$ by a standard elliptic discretization.

Approximate projection methods

Analysis of projection options indicates staggered pressure has "best" approximate projection properties in terms of stability and accuracy.

$$
\begin{aligned}
DU_{i+\frac{1}{2},j+\frac{1}{2}} &= \frac{u_{i+1,j+1}+u_{i+1,j}-u_{i,j+1}-u_{i,j}}{2\Delta x} \\
&+ \frac{v_{i+1,j+1}+v_{i,j+1}-u_{i+1,j}-u_{i,j}}{2\Delta y}
\end{aligned}
$$

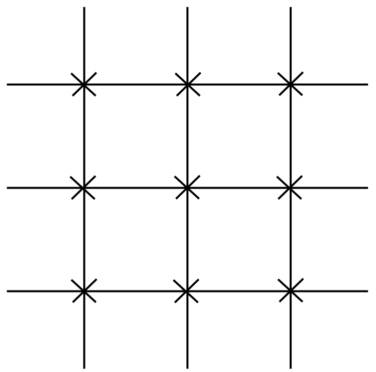
$$
Gp_{ij} = \begin{pmatrix} \dfrac{p_{i+1/2,j+1/2}+p_{i+1/2,j-1/2}-p_{i-1/2,j+1/2}-p_{i-1/2,j-1/2}}{2\Delta x} \\ \dfrac{p_{i+1/2,j+1/2}+p_{i-1/2,j+1/2}-p_{i+1/2,j-1/2}-p_{i-1/2,j-1/2}}{2\Delta y} \end{pmatrix}
$$

Projection is given by $\mathbf{P} = I - G(L)^{-1}D$

where L is given by bilinear finite element basis

$$(\nabla p, \nabla \chi) = (V, \nabla \chi)$$

Nine point discretization



2nd Order Fractional Step Scheme

First Step:

Construct an intermediate velocity field U^*:

$$\frac{U^* - U^n}{\Delta t} = -[U^{ADV} \cdot \nabla U]^{n+\frac{1}{2}} - \nabla p^{n-\frac{1}{2}} + \epsilon \Delta \frac{U^n + U^*}{2}$$

Second Step:

Project U^* onto constraint and update p. Form

$$V = \frac{U^*}{\Delta t} + Gp^{n-\frac{1}{2}}$$

Solve

$$Lp^{n+\frac{1}{2}} = DV$$

Set

$$U^{n+1} = \Delta t(V - Gp^{n+\frac{1}{2}})$$

Computation of Advective Derivatives

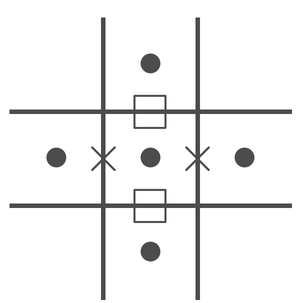
- Start with U^n at cell centers
- Predict normal velocities at cell edges using variation of second-order Godunov methodology $\Rightarrow u^{n+1/2}_{i+1/2,j}, v^{n+1/2}_{i,j+1/2}$
- MAC-project the edge-based normal velocities, i.e. solve

$$D^{MAC}(G^{MAC}\psi) = D^{MAC}U^{n+1/2}$$

and define normal advection velocities

$$u^{ADV}_{i+1/2,j} = u^{n+1/2}_{i+1/2,j} - G^x\psi, \quad v^{ADV}_{i,j+1/2} = v^{n+1/2}_{i,j+1/2} - G^y\psi$$

- Use these advection velocities to define $[U^{ADV}\cdot\nabla U]^{n+1/2}$.



$\times$: u
$\square$: v
$\bullet$: ψ

Second-order projection algorithm

Properties

- Second-order in space and time
- Robust discretization of advection terms using modern upwind methodology
- Approximate projection formulation

Algorithm components

- Explicit advection
- Semi-implicit diffusion
- Elliptic projections
  - 5-point cell-centered
  - 9-point node-centered

How do we generalize AMR to work for projection algorithm?

Look at discretization details in one dimension

- Revisit hyperbolic
- Elliptic
- Parabolic

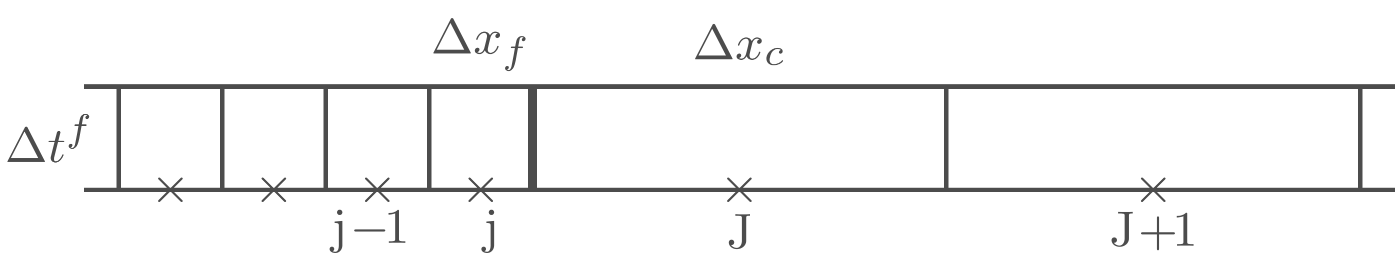
Spatial discretizations

Hyperbolic–1d

Consider $U_t + F_x = 0$ discretized with an explicit finite difference scheme:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{F_{i-1/2}^{n+\frac{1}{2}} - F_{i+1/2}^{n+\frac{1}{2}}}{\Delta x}$$

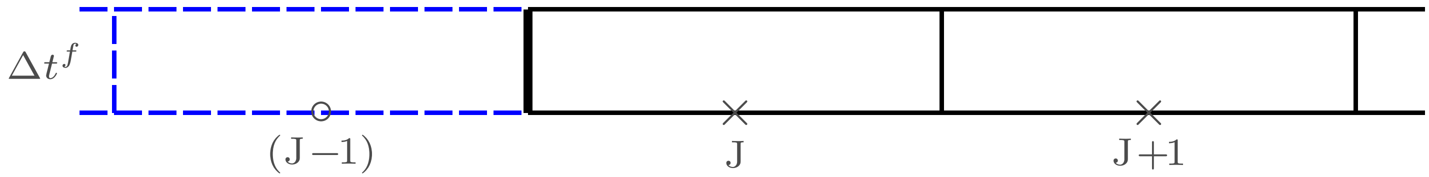
In order to advance the composite solution we must specify how to compute the fluxes:



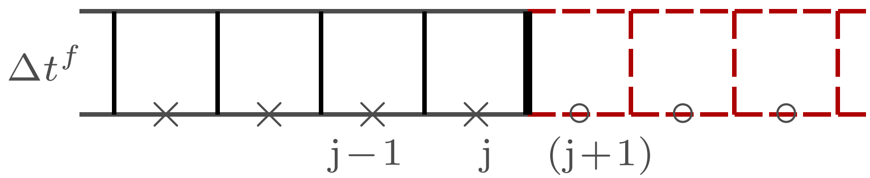
- Away from coarse/fine interface the coarse grid sees the average of fine grid values onto the coarse grid
- Fine grid uses interpolated coarse grid data
- The fine flux is used at the coarse/fine interface

Hyperbolic–composite

One can advance the coarse grid



then advance the fine grid



using "ghost cell data" at the fine level interpolated from the coarse grid data.

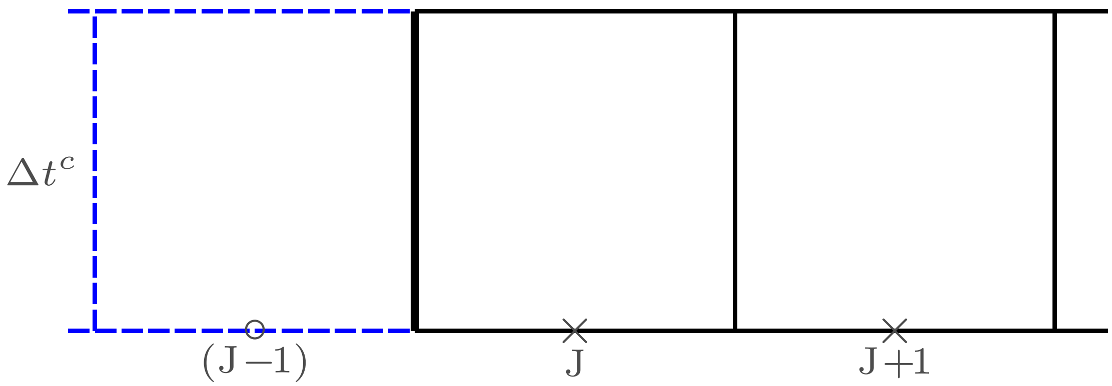
This results in a flux mismatch at the coarse/fine interface, which creates an error in U_J^{n+1}. The error can be corrected by refluxing, i.e. setting

$$\Delta x_c U_J^{n+1} := \Delta x_c U_J^{n+1} - \Delta t^f F^c_{J-1/2} + \Delta t^f F^f_{j+1/2}$$

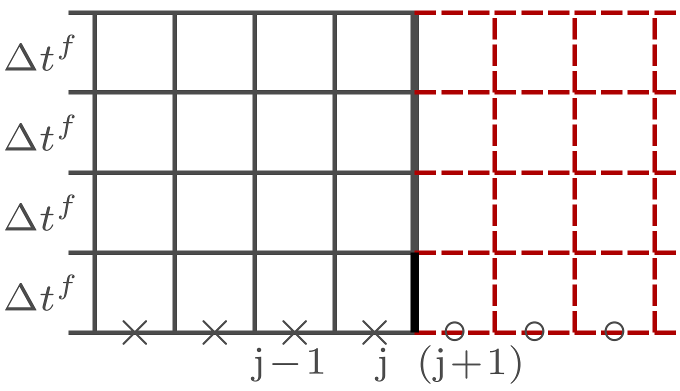
Before the next step one must average the fine grid solution onto the coarse grid.

Hyperbolic–subcycling

To subcycle in time we advance the coarse grid with Δt^c



and advance the fine grid multiple times with Δt^f.



The refluxing correction now must be summed over the fine grid time steps:

$$\Delta x_c U_J^{n+1} := \Delta x_c U_J^{n+1} - \Delta t^c F^c_{J-1/2} + \sum \Delta t^f F^f_{j+1/2}$$

AMR Discretization algorithms

Design Principles:

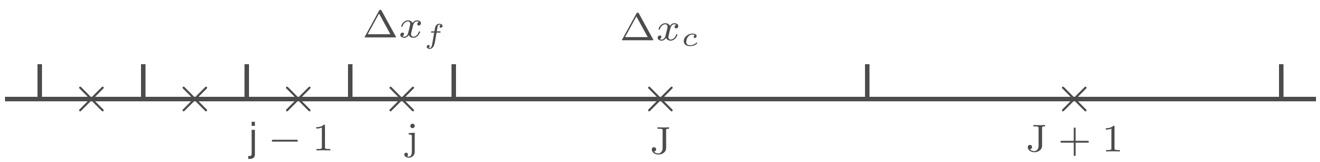
- Define what is meant by the *solution* on the grid hierarchy.
- Identify the errors that result from solving the equations on each level of the hierarchy “independently” (motivated by subcycling in time).
- Solve correction equation(s) to “fix” the solution.
- For subcycling, average the correction in time.

Coarse grid supplies Dirichlet data as boundary conditions for the fine grids.

Errors take the form of flux mismatches at the coarse/fine interface.

Elliptic

Consider $-\phi_{xx} = \rho$ on a locally refined grid:



We discretize with standard centered differences except at j and J. We then define a flux, ϕ_x^{c-f}, at the coarse / fine boundary in terms of ϕ^f and ϕ^c and discretize in flux form with

$$-\frac{1}{\Delta x_f}\left(\phi_x^{c-f} - \frac{(\phi_j - \phi_{j-1})}{\Delta x_f}\right) = \rho_j$$

at $i = j$ and

$$-\frac{1}{\Delta x_c}\left(\frac{(\phi_{J+1} - \phi_J)}{\Delta x_c} - \phi_x^{c-f}\right) = \rho_J$$

at $I = J$.

This defines a perfectly reasonable linear system but ...

Elliptic – composite

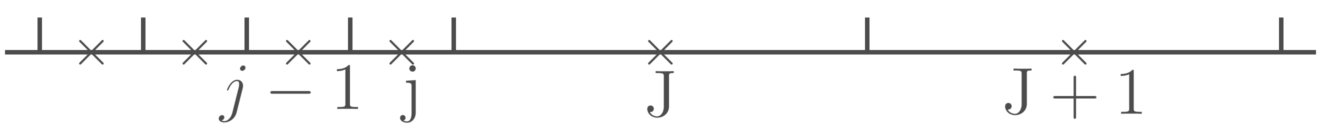
Suppose we solve

$$-\frac{1}{\Delta x_c}\left(\frac{(\overline{\phi}_{I+1}-\overline{\phi}_I)}{\Delta x_c}-\frac{(\overline{\phi}_I-\overline{\phi}_{I-1})}{\Delta x_c}\right)=\rho_I$$

at *all* coarse grid points I and then solve

$$-\frac{1}{\Delta x_f}\left(\frac{(\overline{\phi}_{i+1}-\overline{\phi}_i)}{\Delta x_f}-\frac{(\overline{\phi}_i-\overline{\phi}_{i-1})}{\Delta x_f}\right)=\rho_i$$

at all fine grid points $i \neq j$ and use the “correct” stencil at $i = j$, holding the coarse grid values fixed.



Elliptic – synchronization

The composite solution defined by $\overline{\phi}^c$ and $\overline{\phi}^f$ satisfies the composite equations everywhere except at J.

The error is manifest in the difference between ϕ_x^{c-f} and $\frac{(\overline{\phi}_J - \overline{\phi}_{J-1})}{\Delta x_c}$.

Let $e = \phi - \overline{\phi}$. Then $-\Delta^h e = 0$ except at $I = J$ where

$$-\Delta^h e = \frac{1}{\Delta x_c} \left(\frac{(\overline{\phi}_J - \overline{\phi}_{J-1})}{\Delta x_c} - \phi_x^{c-f} \right)$$

Solve the composite for e and correct

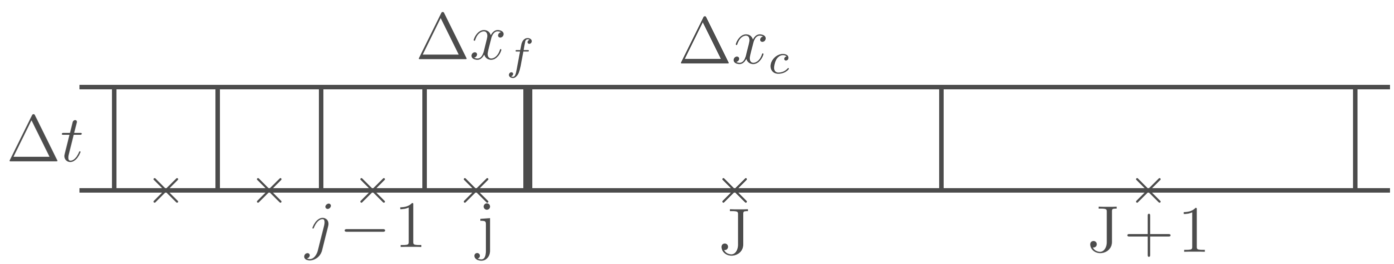
- $\phi^c = \overline{\phi}^c + e^c$
- $\phi^f = \overline{\phi}^f + e^f$

The resulting solution is the same as solving the composite operator

Parabolic – composite

Consider $u_t + f_x = \varepsilon u_{xx}$ and the semi-implicit time-advance algorithm:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{f_{i+1/2}^{n+\frac{1}{2}} - f_{i-1/2}^{n+\frac{1}{2}}}{\Delta x} = \frac{\varepsilon}{2}\Big((\Delta^h u^{n+1})_i + (\Delta^h u^n)_i\Big)$$



Again if one advances the coarse and fine levels separately, a mismatch in the flux at the coarse-fine interface results.

Let $\overline{u}^{c-f}$ be the initial solution from separate evolution

Parabolic – synchronization

The difference e^{n+1} between the exact composite solution u^{n+1} and the solution $\overline{u}^{n+1}$ found by advancing each level separately satisfies

$$(I - \frac{\varepsilon \Delta t}{2} \Delta^h)\, e^{n+1} = \frac{\Delta t}{\Delta x_c} (\delta f + \delta D)$$

$$\begin{aligned}
\Delta t\, \delta f &= \Delta t\, (-\overline{f}_{J-1/2} + f_{j+1/2}) \\
\Delta t\, \delta D &= \frac{\varepsilon \Delta t}{2} \Big((\overline{u}^{c,n}_{x,J-1/2} + \overline{u}^{c,n+1}_{x,J-1/2}) - (u_x^{c-f,n} + u_x^{c-f,n+1}) \Big)
\end{aligned}$$

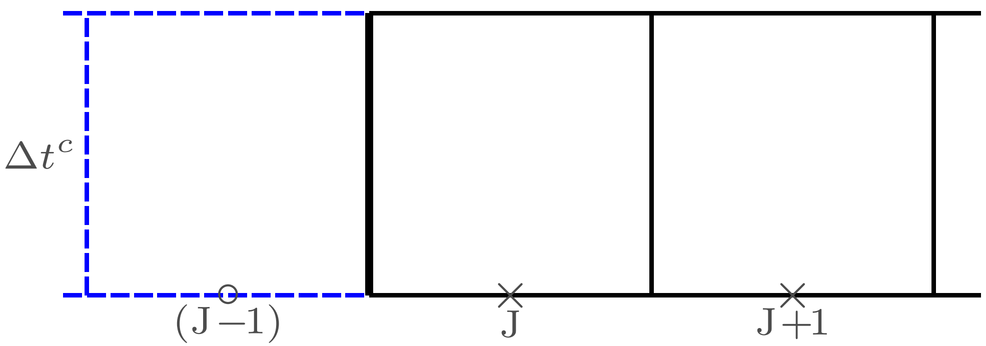
Source term is localized to to coarse cell at coarse / fine boundary

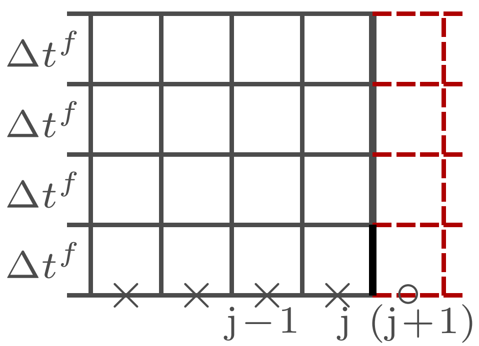
Updating $u^{n+1} = \overline{u}^{n+1} + e$ again recovers the exact composite solution

Parabolic – subcycling

Advance coarse grid

Advance fine grid r times





The refluxing correction now must be summed over the fine grid time steps:

$$(I - \frac{\varepsilon \Delta t^c}{2} \Delta^h)\; e^{n+1} = \frac{\Delta t^c}{\Delta x_c} (\delta f + \delta D)$$

$$\begin{aligned}
\Delta t^c\; \delta f &= -\Delta t^c\; \overline{f}_{J-1/2} + \sum \Delta t^f\; f_{j+1/2} \\
\Delta t^c\; \delta D &= \frac{\varepsilon \Delta t^c}{2} (\overline{u}^{c,n}_{x,J-1/2} + \overline{u}^{c,n+1}_{x,J-1/2}) \\
&- \sum \frac{\varepsilon \Delta t^f}{2} (u_x^{c-f,n} + u_x^{c-f,n+1})
\end{aligned}$$

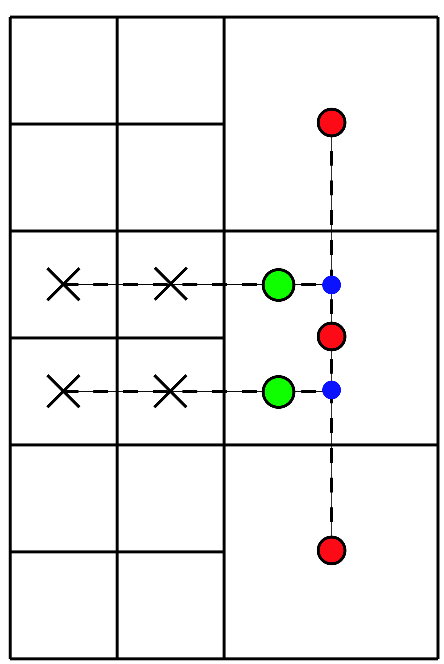
Spatial accuracy – cell-centered

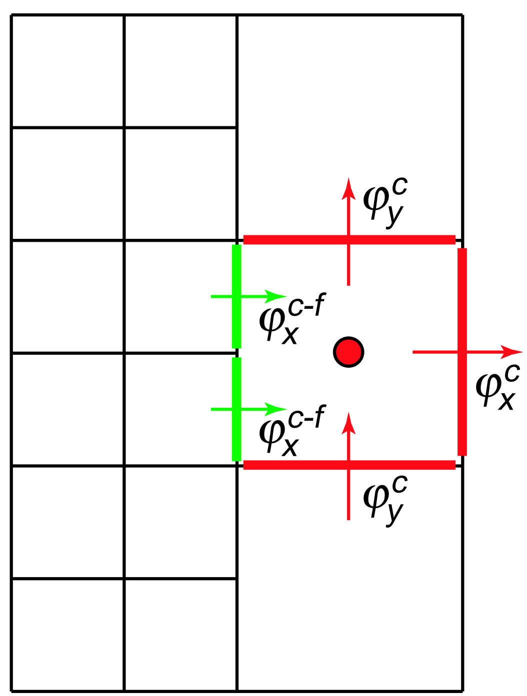
Modified equation gives

$$\psi^{comp} = \psi^{exact} + \Delta^{-1}\tau^{comp}$$

where τ is a *local* function of the solution derivatives.

Simple interpolation formulae are not sufficiently accurate for second-order operators





Convergence results

Local Truncation Error

D	Norm	Δx	$\|\|L(U_e) - \rho\|\|_h$	$\|\|L(U_e) - \rho\|\|_{2h}$	R	P
2	L_∞	1/32	1.57048e-02	2.80285e-02	1.78	0.84
2	L_∞	1/64	8.08953e-03	1.57048e-02	1.94	0.96
3	L_∞	1/16	2.72830e-02	5.60392e-02	2.05	1.04
3	L_∞	1/32	1.35965e-02	2.72830e-02	2.00	1.00
3	L_1	1/32	8.35122e-05	3.93200e-04	4.70	2.23

Solution Error

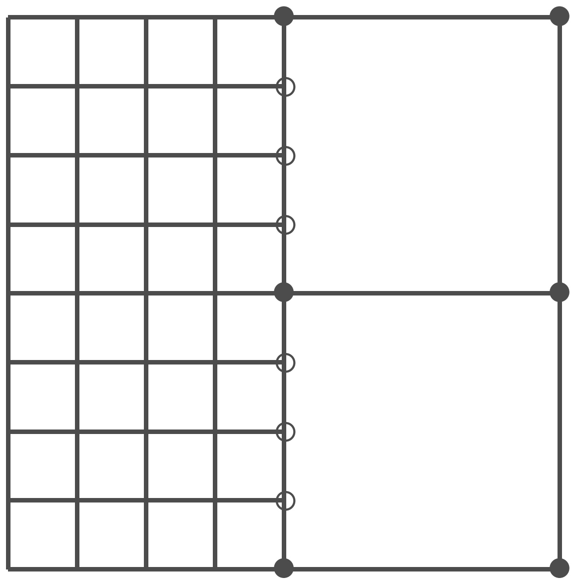
D	Norm	Δx	$\|\|U_h - U_e\|\|$	$\|\|U_{2h} - U_e\|\|$	R	P
2	L_∞	1/32	5.13610e-06	1.94903e-05	3.79	1.92
2	L_∞	1/64	1.28449e-06	5.13610e-06	3.99	2.00
3	L_∞	1/16	3.53146e-05	1.37142e-04	3.88	1.96
3	L_∞	1/32	8.88339e-06	3.53146e-05	3.97	1.99

$$\psi^{computed} = \psi^{exact} + L^{-1}\overline{\tau}$$

Solution operator smooths the error

Spatial accuracy – nodal

Node-based solvers:



- Symmetric self-adjoint matrix
- Accuracy properties given by approximation theory

Recap

Solving coarse grid then solving fine grid with interpolated Dirichlet boundary conditions leads to a flux mismatch at boundary

Synchronization corrects mismatch in fluxes at coarse / fine boundaries.

Correction equations match the structure of the process they are correcting.

- For explicit discretizations of **hyperbolic** PDE's the correction is an explicit flux correction localized at the coarse/fine interface.
- For an **elliptic** equation (e.g., the projection) the source is localized on the coarse/fine interface but an elliptic equation is solved to distribute the correction through the domain. Discrete analog of a layer potential problem.
- For Crank-Nicolson discretization of **parabolic** PDE's, the correction source is localized on the coarse/fine interface but the correction equation diffuses the correction throughout the domain.

Efficiency considerations

For the elliptic solves, we can substitute the following for a full composite solve with no loss of accuracy

- Solve $\Delta\psi^c = g^c$ on coarse grid
- Solve $\Delta\psi^f = g^f$ on fine grid using interpolated Dirichlet boundary conditions
- Evaluate composite residual on the coarse cells adjacent to the fine grids
- Solve for correction to coarse and fine solutions on the composite hierarchy

Because of the smoothing properties of the elliptic operator, we can, in some cases, substitute either a two-level solve or a coarse level solve for the full composite operator to compute the *correction* to the solution.

- Source is localized at coarse cells at coares / fine boundary
- Solution is a discrete harmonic function in interior of fine grid
- This correction is exact in 1-D