Computational Fluid Dynamics, Fall 2018
Homework 4: Space-Time Methods for Advection-Diffusion

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Consider numerically solving the advection-diffusion equation

$$u_t + a u_x = (d(x) u_x)_x \,,$$

on the periodic domain $0 \leq x < 1$, for $a = 1$ and initial condition

$$u(x,0) = [\sin(\pi x)]^{100} \,,$$

on a uniform grid, as we did back in Homework 1 for $d(x) = d = \text{const.}$

1 Linear PDE

[Note: This was an optional part for HW1]
Solve this equation using the Lax-Wendroff or Fromm's (recommended!) method for the advection, and make the diffusion coefficient non-constant,

$$d(x) = \epsilon \, (2 + \cos(2\pi x)).$$

In this case, it is a bit harder to compute an exact solution (please do not try, this is *not* the point of this homework!). Use a small value for the diffusion coefficient in the range $\epsilon = 0.01 - 0.05$.

1. Write down a spatio-temporal discretization. Explain how you handled diffusion in your spatio-temporal discretization and what you expect the order of accuracy of the method to be (do your best to make it second-order, of course).
2. Validate your code in some way (e.g., by solving problem with $d(x) = d = \text{const.}$ and/or using a manufactured analytical solution).
3. Refine the resolution in *both space and time* (i.e., in space-time, not space or time separately) to empirically estimate the spatio-temporal order of convergence.
4. Investigate the stability of your scheme – is it limited in stability by both advection and diffusion or only advection?
   Note: You may find that Fromm's method behaves differently from Lax-Wendroff.
5. [Optional] Put in boundary conditions as in the second part of HW2 and see if you can still get second-order accuracy.

2 [Optional] Nonlinear PDE

Now try your hand at solving the nonlinear equation

$$u_t + a u_x = (d(u) u_x)_x \,,$$

by repeating the above tasks (as many as you can), using, for example,

$$d(x) = 0.05 \, \exp(u) \,.$$