Computational Fluid Dynamics, Fall 2018
Homework 5
Pseudo-spectral solver for the two-dimensional Navier-Stokes equations

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The Taylor vortex

$$v_x = v_0 - 2e^{-8\pi^2\mu t/L^2} \cos\left(\frac{2\pi(x-v_0 t)}{L}\right) \sin\left(\frac{2\pi(y-v_0 t)}{L}\right), \tag{1}$$

$$v_y = v_0 + 2e^{-8\pi^2\mu t/L^2} \sin\left(\frac{2\pi(x-v_0 t)}{L}\right) \cos\left(\frac{2\pi(y-v_0 t)}{L}\right), \tag{2}$$

$$\pi = -e^{-16\pi^2\mu t/L^2} \left[\cos\left(\frac{4\pi(x-v_0 t)}{L}\right) + \cos\left(\frac{4\pi(y-v_0 t)}{L}\right)\right]. \tag{3}$$

is a well-known exact solution of the unforced Navier-Stokes equation

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}\pi + \mu\boldsymbol{\nabla}^2\boldsymbol{v}, \tag{4}$$

in two dimensions for a periodic domain of size $L \times L$. You can set $v_0 = 1$, $L = 1$ and $\mu = 0.05$ for this homework. Note that to capture the non-zero mean of the velocity in the vorticity-stream function formulation you need to keep track of it separately and add it "by hand",

$$\boldsymbol{v} = \boldsymbol{\nabla}^{\perp}\psi + \boldsymbol{v}_0.$$

Note that you can easily make the velocity less smooth by increasing the wavenumber and replacing, for example, $\cos\left(\frac{2\pi(x-v_0 t)}{L}\right)$ with $\cos\left(\frac{\kappa\pi(x-v_0 t)}{L}\right)$ where κ is a larger integer. I strongly recommend that you write your code to have the wavenumber κ be an input parameter so you can easily play around with different smoothness of the velocity field in part 2 of this homework.

1 Velocity Solver

Write a pseudo-spectral solver for the two-dimensional Navier-Stokes equations in the vorticity-stream formulation, as we discussed in class. Choose a temporal integrator and discuss your choice. Solve the NS equations from time $t = 0$ (using the Taylor vortex as initial condition) to time $t = 0.25$ and compare the numerical solution to the exact solution. Discuss (both numerically and analytically) the order of accuracy of the scheme you developed, as well as the computational cost. What would be different (better, worse) if you used a finite-volume spatial discretization?

Try with and without anti-aliasing and comment on your observations.

2 Advection-Diffusion Solver

Solve also a forced scalar advection-diffusion equation for the concentration $c(\boldsymbol{r}, t)$ a passive tracer,

$$\partial_t c + \boldsymbol{v}\cdot\boldsymbol{\nabla}c = D\boldsymbol{\nabla}^2 c + f(r,t). \tag{5}$$

Here the velocity $\boldsymbol{v}(x, y, t)$ may come from the numerical solution in the first part of this homework or be specified analytically. In many practical applications the concentration can couple back into the velocity equation but here we do not consider this case.

If you set $D = \mu$ and $f = -\partial_x \pi$ then $c = v_x$ from the Taylor vortex is a solution of (5). Use this to test your advection-diffusion solver.

Now set $f = 0$ and $D = 0$ (pure advection) and start with the initial condition

$$c(x, y; t = 0) = \left[\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\right]^{100}.$$

First, set $\boldsymbol{v}(x, y, t) = \boldsymbol{v}_0 =$ const and advect the peak to time $t = 1$, when it should come back to its starting position unchanged. Examine numerically and theoretically what the stability limit is for the time step size. Then try the velocity coming from the first part of this homework with $v_0 = 0$ and observe what happens to the peak as it is advected around by the divergence-free flow for different grid resolutions and wavenumbers κ; observe that the velocity now decays exponentially so after some time the peak should reach a steady final state. Comment on your observations. Try some anti-aliasing strategy and see if it helps.