Computational Fluid Dynamics, Fall 2018
Homework 3: Implicit Temporal Integrators

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Oct 2nd, 2018
Due Sunday **Oct. 21st 2018**

1 Implicit Timestepping for the Advection Equation in 1D

Consider numerically solving the advection equation with constant coefficients

$$u_t + au_x = 0,$$

on the periodic domain $0 \leq x < 1$, for $a = 1$ and initial condition

$$u(x,0) = [\sin(\pi x)]^{100},$$

on a uniform grid with $m = 1000$ points (this is the same problem as in Homework 1), using a method-of-lines (MOL) approach.

Spatially discretize this equation using the *third-order upwind biased scheme* (as in HW1), and integrate it forward in time to a time $T = 1$ using a time step size corresponding to an advective CFL number of $\nu = a\Delta t/\Delta x = 1$, 10, 100 and the following implicit temporal integrators:

1. Backward Euler method.
2. Implicit trapezodial method (Crank-Nicolson).

Compare the three numerical solutions $\boldsymbol{w}(t = 1)$ to the exact solution $u(x, 1)$ and comment on your observations. Explain how you solved the linear system arising in the implicit temporal discretization. Integrate the spatially-discrete system in time "exactly" based on your solution to HW1 (you can improve your solution based on discussion in class, of course) and show the result for comparison.

2 Implicit Timestepping for the Diffusion Equation

2.1 One Dimension

Consider solving the diffusion equation in one dimension

$$u_t = u_{xx},$$

on the domain $0 < x < 1$ with Dirichlet BCs, $u(0, t) = 0$, $u(1, t) = 1$, and a discontinuous initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } x \leq 1/2 \\ 1 & \text{otherwise.} \end{cases}$$

Choose the method of spatial discretization and use a grid spacing of $h = 1/100$. Write code to integrate the resulting semi-discrete system forward in time using the backward Euler (BE) and the implicit trapezoidal method (Crank-Nicolson, CN) for a specified time step size.

1. Using a time step size corresponding to a diffusive CFL number of $\mu = \Delta t/\Delta x^2 = 1$, 10, 100, compare the numerical solutions $\boldsymbol{w}(t = 0.01)$ for the two methods (BE vs CN) and comment on your observations. Explain how you solved the linear system arising in the implicit temporal discretization. Try to construct the exact solution of the PDE and show that also for comparison.
2. Compare the solutions obtained by the two temporal integrators at a much longer time $t = 10$ for a time step size corresponding to a diffusive CFL number of $\mu = 1000$ and comment on your observations. Compare the numerical solution to the steady state solution $u(x, \infty)$ of the PDE and discuss your observations of the differences between CN and BE.

2.2 [Optional] Two dimensions

Consider solving the diffusion equation in two dimensions

$$u_t = u_{xx} + u_{yy},$$

on the square $\Omega : (x, y)\,,\ -1 < (x, y) < 1$, and a discontinuous initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Consider periodic boundary conditions.

Choose the method of spatial discretization and some reasonable grid size (say 64^2 grid points). Write code to integrate the resulting semi-discrete system forward in time using the backward Euler (BE) and the implicit trapezoidal method (Crank-Nicolson, CN) for a specified time step size. Using a time step size corresponding to a diffusive CFL number of $\mu = 1$, 10, 100, compare the numerical solutions $\boldsymbol{w}(t = 1)$ for the two methods (BE vs CN) and comment on your observations. Explain how you solved the linear system arising in the implicit temporal discretization and why you chose the method you chose [*Hint: FFT is your friend for periodic BCs*].