Computational Fluid Dynamics, Fall 2018
Homework 2: Boundary Conditions for Advection-Diffusion Equations

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1 Finite-Volume Discretization of Heat Equation

Consider constructing a spatial semi-discretization for the diffusion equation with constant coefficients

$$u_t = u_{xx},$$

on the domain $0 < x < 1$ with Dirichlet BCs at $x = 0$ and Neumann BCs at $x = 1$. Use the manufactured analytical solution

$$u(x,t) = \exp^{-\pi^2 t/4} \cos\left(\frac{\pi x}{2}\right)$$

to obtain the specific forms of the initial and boundary conditions.

Write a finite-volume (flux based) second-ordered centered difference scheme and solve this equation up to time $T = 1/4$ (explain how you solved the ODEs and why, and be warned that you may run into some numerical issues) with different grid spacings, and:

1. Find the numerical order of convergence in the L_1, L_2 and L_∞ norms.
2. Find the local truncation error at the left and right boundaries, and if possible, use that to prove second-order accuracy in some norm.
3. **Optional**: Prove stability in some norm.

2 Boundary Layers for Advection-Diffusion Equation

Consider constructing a spatial semi-discretization for the advection-diffusion equation with constant coefficients

$$u_t + u_x = d\, u_{xx},$$

on the domain $0 < x < 1$, for initial condition $u(x,0) = \sin^p(\pi x)$ where $p = 2$ is an exponent, and boundary conditions

$$\begin{aligned} u(0,t) &= \sin^p(-\pi t) \\ u_x(1,t) &= 0. \end{aligned}$$

Observe that if $d = 0$ the exact solution here is $u(0,t) = \sin^p(\pi(x-t))$ which is the problem we studied in HW1 with periodic BCs.

Develop a finite-volume method (choose the advective/diffusive stencils, the boundary condition treatment, number of grid points, etc., and *explain your choices*) to solve the equation up to time $T = 1$.

1. Show the spatially-discrete solution at this time for $\epsilon = 0.1$, $\epsilon = 0.01$ and $\epsilon = 0.001$ and comment on your observations and experiences. Discuss what happens as $\epsilon \to 0$ with the PDE and its true analytical solution, and compare to your numerical method and the numerical solution. Is your method consistent for $\epsilon = 0$?
2. Repeat with a Dirichlet condition on the right, $u(1,t) = 0$ and comment on the differences repeating all of the steps you did for a Neumann BC.
3. Numerically (empirically) determine the spatial order of convergence for your discretization.
   Does it agree with theoretical expectations (explain what your expectation is)? What is the order of convergence for $\epsilon = 0$?
4. [Optional] Change the exponent to $p = 100$ and comment on any new observations you make. How much diffusion do you need (i.e., how small can d be) to get a sensible-looking solution now?