Computational Fluid Dynamics, Fall 2018
Homework 1: Advection-Diffusion Equations

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This and most of the other homeworks will be in 1D and therefore very simple and quick to do computationally in MATLAB. You could easily solve them with a very fine grid (say 1024 points) (or very small time step size for later homeworks), but you will learn nothing from that: Every convergent method will solve a smooth problem very accurately if you make the space-time grid fine enough. However, the goal here is to learn about the **issues** that will also appear in **2D and 3D**, where it would be very challenging indeed to do a simulation with a 1024^3 grid! So please think carefully about how you choose the grid size. The goal is always to make the grid size and time step size as large as possible while controlling the error, so we can solve large problems over a long time in 3D. So **explore** how much you can **reduce the resolution until problems appear**, and try to explain/understand when problems appear and why. The goal here is to think about how the theory in class connects to this specific problem, not to solve the (trivial!) problem itself.

Also important: Please do **not** use any specific form of input functions I give in the solution. For example, the initial condition below is $[\sin(\pi x)]^{100}$. Do **not** use this specific form in your solution. For example, do not give an exact analytical solution to the PDE for this specific form. These homeworks are exercises in **numerical analysis** and not PDE analysis. Do everything numerically and write your code in a way that one can easily change the specific functions appearing in the problem specification.

Please make an effort to write good code. Again, think about how easily this code could be modified to solve a similar but different problem (e.g., if the advection velocity is space-dependent as well); a modular well-designed code will be reusable, will not have repeated

1 Advection-Diffusion Equation in 1D

Consider numerically solving the advection-diffusion equation

$$u_t + a u_x = \left(d(x) u_x\right)_x,$$

on the periodic domain $0 \le x < 1$, for $a = 1$ and initial condition (remember not to use the specific analytical form of this in your solution, rather, treat it as some general smooth periodic function given to you by a *problem-specific input*)

$$u(x,0) = [\sin(\pi x)]^{100},$$

on a uniform grid.

Here we focus on spatial discretization, and do not discretize time but rather study the consistency and stability of the semi-discrete approximation

$$\boldsymbol{w}'(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{g}(t), \tag{1}$$

where $\boldsymbol{A}$ depends on the choice of the spatial discretization.

Find some way to solve (1) with sufficient temporal accuracy. Ideally, we want to solve it "exactly", i.e., to numerical roundoff, so try that *instead* of using Matlab's ODE solvers as the lecture notes suggest. Explain what you did and indicate which norm(s) you used.

Hint: Use the Discrete Fourier Transform (DFT) to diagonalize the system of ODEs (1) and then solve it numerically using the FFT algorithm.

1.1 Pure advection

Consider pure advection, $d = 0$ and:

1. Compare the numerical solution $\boldsymbol{w}(t = 1)$ to the exact solution $u(x, 1)$ for the first-order upwind scheme, the second-order centered scheme, and the third-order upwind biased scheme, for several smartly-chosen grid resolutions. Comment on your observations, in particular their relation to artificial dissipation and dispersion as discussed in class.
2. Compute the *relative* global error $\|\epsilon(t)\| = \|\boldsymbol{w}(t) - \boldsymbol{u}_h(t)\| \,/\, \|\boldsymbol{u}_h(t)\|$ at time $t = 1$ for different grid resolutions (think about how you choose them!), and estimate the spatial order of accuracy empirically.
3. Plot the evolution of the *relative* global error $\|\epsilon(t)\| = \|\boldsymbol{w}(t) - \boldsymbol{u}_h(t)\| \,/\, \|\boldsymbol{u}_h(t)\|$ over the time interval $0 \leq t \leq 10$ for the first-order upwind scheme and the second-order centered scheme, and compare to the theoretical estimate from class.

1.2 Advection-Diffusion

For this part choose one or several "good" spatial discretizations and *write down* the scheme you used (I should not need to look in the code). For each scheme, explain what its advantages and disadvantages are, and what its theoretical (spatial) order of accuracy is.

1.2.1 Constant Diffusion

Let's add a small amount of constant diffusion, $d = 0.001$.

1. Compare the numerical solution $\boldsymbol{w}(t = 1)$ to the exact solution $u(x, 1)$ (explain how you computed the "exact" solution to roundoff tolerance) for several resolutions and comment on your observations.
2. Compute the *relative* global error $\|\epsilon(t)\| = \|\boldsymbol{w}(t) - \boldsymbol{u}_h(t)\| \,/\, \|\boldsymbol{u}_h(t)\|$ at time $t = 1$ for different grid resolutions, and estimate the spatial order of accuracy empirically.
3. [Optional] Plot the evolution of the absolute and relative over the time interval $0 \leq t \leq 10$, and compare to the theoretical estimate from class.