Multiresolution Analysis

DS-GA 1013 / MATH-GA 2824 Optimization-based Data Analysis

http://www.cims.nyu.edu/~cfgranda/pages/OBDA_fall17/index.html

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Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding

Definition

Let $\mathcal{V}$ be an inner-product space

A frame of $\mathcal{V}$ is a set of vectors $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ such that for every $\vec{x} \in \mathcal{V}$

$$c_L \, ||\vec{x}||^2_{\langle \cdot,\cdot \rangle} \leq \sum_{v \in \mathcal{F}} |\langle \vec{x}, \vec{v} \rangle|^2 \leq c_U \, ||\vec{x}||^2_{\langle \cdot,\cdot \rangle}$$

for fixed positive constants $c_U \geq c_L \geq 0$

The frame is a *tight frame* if $c_L = c_U$

Frames span the whole space

Any frame $\mathcal{F} := \{\vec{v}_1, \vec{v}_2, \ldots\}$ of $\mathcal{V}$ spans $\mathcal{V}$

Proof:

Assume $\vec{y} \notin \text{span}\left(\vec{v}_1, \vec{v}_2, \ldots\right)$

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Assume $\vec{y} \notin \text{span}\,(\vec{v}_1, \vec{v}_2, \ldots)$

Then $\mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)^\perp}\,\vec{y}$ is nonzero and

$$\sum_{\vec{v} \in \mathcal{F}} \left| \left\langle \mathcal{P}_{\text{span}(\vec{v}_1, \vec{v}_2, \ldots)^\perp}\,\vec{y}, \vec{v} \right\rangle \right|^2 =$$

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$$\sum_{\vec{v} \in \mathcal{F}} \left| \left\langle \mathcal{P}_{\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots)^\perp} \vec{y}, \vec{v} \right\rangle \right|^2 = 0$$

Orthogonal bases are tight frames

Any orthonormal basis $\mathcal{B} := \left\{ \vec{b}_1, \vec{b}_2, \ldots \right\}$ is a tight frame

Proof:

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For any vector $\vec{x} \in \mathcal{V}$

$$||\vec{x}||^2_{\langle \cdot, \cdot \rangle}$$

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Proof:

For any vector $\vec{x} \in \mathcal{V}$

$$||\vec{x}||^2_{\langle \cdot, \cdot \rangle} = \left|\left| \sum_{\vec{b} \in \mathcal{B}} \left\langle \vec{x}, \vec{b} \right\rangle \vec{b} \right|\right|^2_{\langle \cdot, \cdot \rangle}$$

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Proof:

For any vector $\vec{x} \in \mathcal{V}$

$$
\begin{aligned}
\|\vec{x}\|_{\langle\cdot,\cdot\rangle}^2 &= \left\|\sum_{\vec{b}\in\mathcal{B}} \left\langle \vec{x}, \vec{b} \right\rangle \vec{b}\right\|_{\langle\cdot,\cdot\rangle}^2 \\
&= \sum_{\vec{b}\in\mathcal{B}} \left|\left\langle \vec{x}, \vec{b} \right\rangle\right|^2 \left\|\vec{b}\right\|_{\langle\cdot,\cdot\rangle}^2
\end{aligned}
$$

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$$\begin{aligned} \|\vec{x}\|_{\langle \cdot, \cdot \rangle}^2 &= \left\| \sum_{\vec{b} \in \mathcal{B}} \left\langle \vec{x}, \vec{b} \right\rangle \vec{b} \right\|_{\langle \cdot, \cdot \rangle}^2 \\ &= \sum_{\vec{b} \in \mathcal{B}} \left| \left\langle \vec{x}, \vec{b} \right\rangle \right|^2 \left\| \vec{b} \right\|_{\langle \cdot, \cdot \rangle}^2 \\ &= \sum_{\vec{b} \in \mathcal{B}} \left| \left\langle \vec{x}, \vec{b} \right\rangle \right|^2 \end{aligned}$$

Analysis operator

The analysis operator Φ of a frame maps a vector to its coefficients

$$\Phi(\vec{x})[k] = \langle \vec{x}, \vec{v}_k \rangle$$

For any finite frame $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ of $\mathbb{C}^n$ the analysis operator is

$$F := \begin{bmatrix} \vec{v}_1^* \\ \vec{v}_2^* \\ \cdots \\ \vec{v}_m^* \end{bmatrix}$$

Frames in finite-dimensional spaces

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are a frame of $\mathbb{C}^n$ if and only F is full rank

In that case,

$$c_U = \sigma_1^2$$
$$c_L = \sigma_n^2$$

Proof:

$$\sigma_n^2 \leq ||F\vec{x}||_2^2 = \sum_{j=1}^{m} \langle \vec{x}, \vec{v}_j \rangle^2 \leq \sigma_1^2$$

Pseudoinverse

If an $n \times m$ tall matrix A, $m \geq n$, is full rank, then its pseudoinverse

$$A^\dagger := (A^* A)^{-1} A^*$$

is well defined, is a left inverse of A

$$A^\dagger A = I$$

and equals

$$A^\dagger = V S^{-1} U^*$$

where $A = USV^*$ is the SVD of A

Proof

$$A^\dagger := (A^* A)^{-1} A^*$$

Proof

$$
\begin{aligned}
A^\dagger &:= (A^* A)^{-1} A^* \\
&= (VSU^* USV^* A)^{-1} VSU^*
\end{aligned}
$$

Proof

$$
\begin{aligned}
A^{\dagger} &:= (A^{*}A)^{-1}\,A^{*} \\
&= (VSU^{*}USV^{*}A)^{-1}\,VSU^{*} \\
&= \left(VS^{2}V^{*}\right)^{-1}\,VSU^{*}
\end{aligned}
$$

Proof

$$
\begin{aligned}
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Proof

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&= VS^{-2}V^*VSU^* \\
&= VS^{-1}U
\end{aligned}
$$

Proof

$$
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&= (VSU^* USV^* A)^{-1} VSU^* \\
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\end{aligned}
$$

$$A^{\dagger} A$$

Proof

$$\begin{aligned} A^\dagger &:= (A^* A)^{-1} A^* \\ &= (VSU^* USV^* A)^{-1} VSU^* \\ &= \left(VS^2 V^*\right)^{-1} VSU^* \\ &= VS^{-2} V^* VSU^* \\ &= VS^{-1} U \end{aligned}$$

$$A^\dagger A = VS^{-1} UV^* USV^*$$

Proof

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$$A^\dagger A = VS^{-1} UV^* USV^* = I$$

Frames

Short-time Fourier transform (STFT)

Wavelets

Thresholding

Motivation

Spectrum of speech, music, etc. varies over time

Idea: Compute frequency representation of *time segments* of the signal

Short-time Fourier transform

The short-time Fourier transform (STFT) of a function $f \in \mathcal{L}_2[-1/2, 1/2]$ is

$$\text{STFT}\left\{f\right\}(k, \tau) := \int_{-1/2}^{1/2} f(t)\,\overline{w(t-\tau)}e^{-i2\pi kt}\,\text{d}t$$

where $w \in \mathcal{L}_2[-1/2, 1/2]$ is a window function

Frame vectors: $v_{k,\tau}(t) := w(t-\tau)\,e^{i2\pi kt}$

Discrete short-time Fourier transform

The STFT of a vector $\vec{x} \in \mathbb{C}^n$ is

$$\text{STFT}\left\{f\right\}(k, l) := \left\langle \vec{x} \circ \vec{w}_{[l]}, \vec{h}_k \right\rangle$$

where $w \in \mathbb{C}^n$ is a window vector

Frame vectors: $v_{k,l}(t) := \vec{w}_{[l]} \circ \vec{h}_k$

STFT

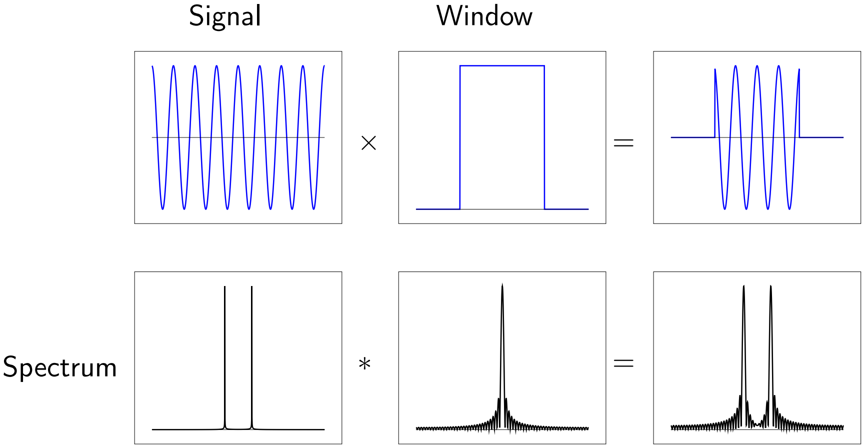
Length of window and shifts are chosen so that shifted windows overlap

In that case the STFT is a frame

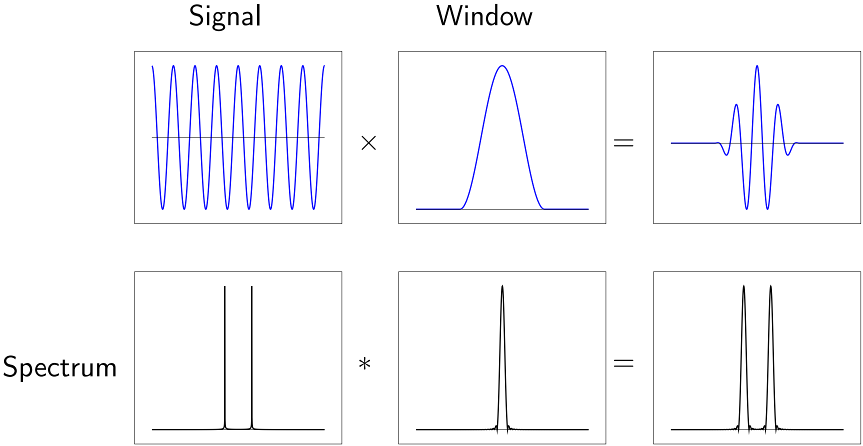
We can invert it using fast algorithms based on the FFT

Window should not produce spurious high-frequency artifacts

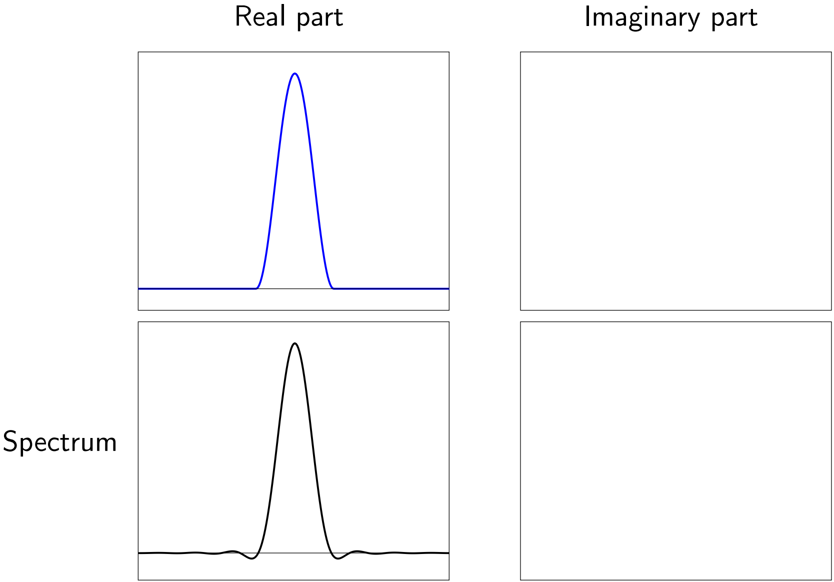
Rectangular window



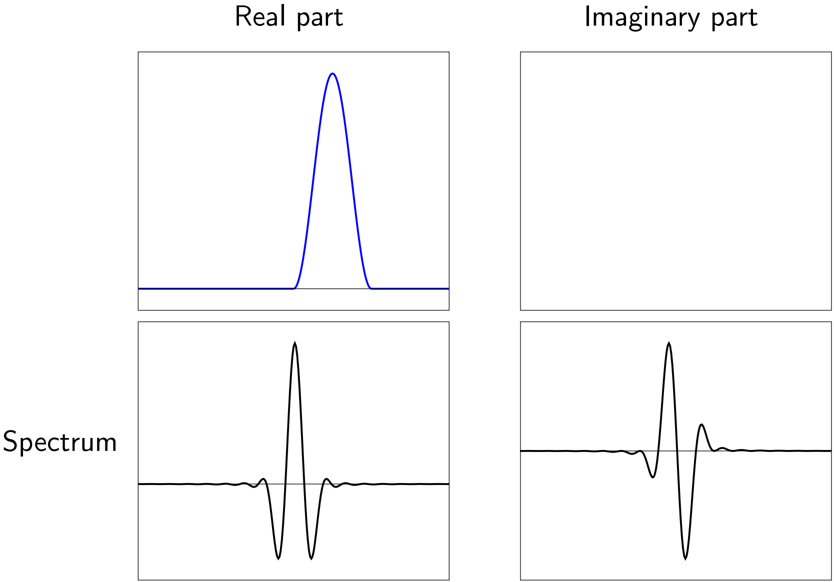
Hann window



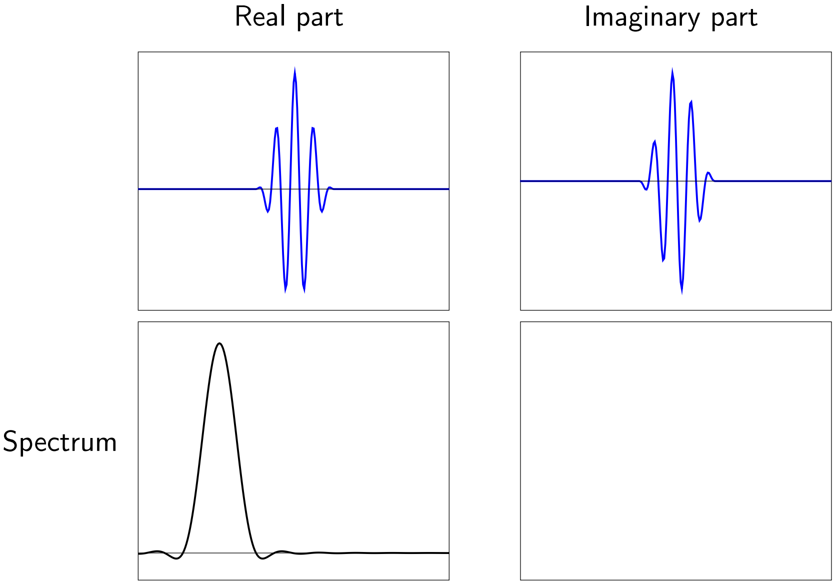
Frame vector $l = 0$, $k = 0$



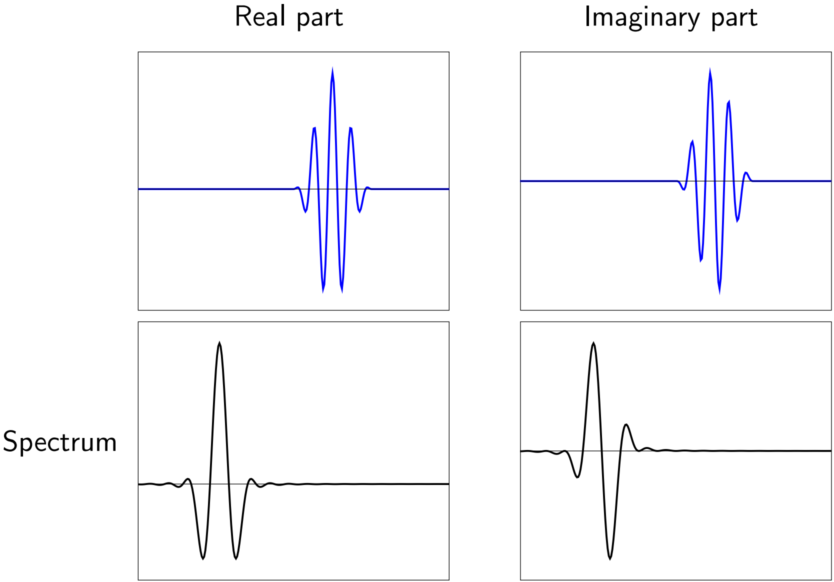
Frame vector $l = 1/32$, $k = 0$



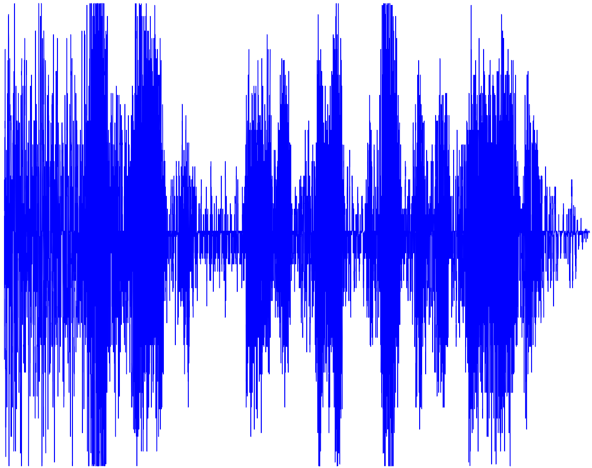
Frame vector $l = 0$, $k = 64$



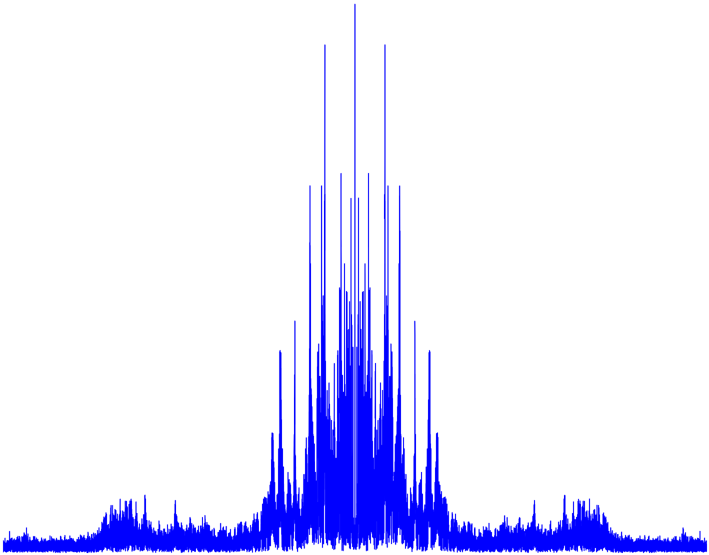
Frame vector $l = 1/32$, $k = 64$



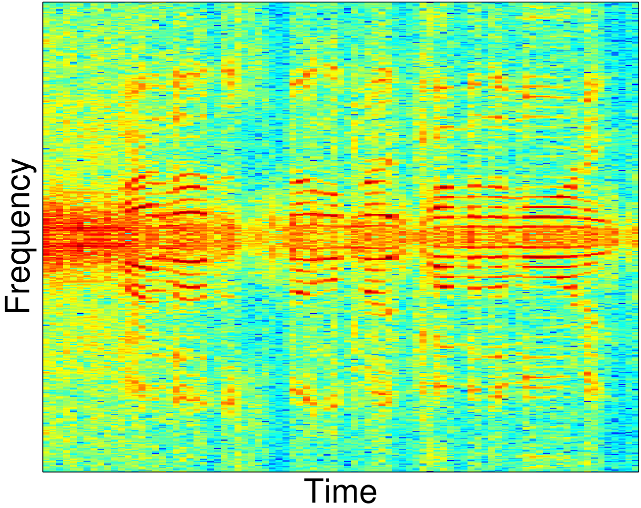
Speech signal



Spectrum



Spectrogram (log magnitude of STFT coefficients)



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Wavelet transform

Motivation: Extracting features at different scales

Idea: Frame vectors are scaled, shifted copies of a fixed function

An additional function captures low-pass component at largest scale

Wavelet transform

The wavelet transform of a function $f \in \mathcal{L}_2[-1/2, 1/2]$ depends on a choice of scaling function (or *father wavelet*) ϕ and wavelet function (or *mother wavelet*) ψ

The scaling coefficients are

$$W_\phi \{f\} (\tau) := \frac{1}{\sqrt{s}} \int f(t) \overline{\phi(t-\tau)} \, dt$$

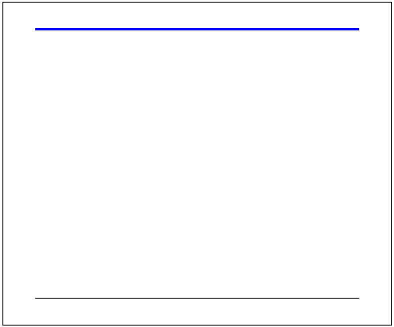
The wavelet coefficients are

$$W_\psi \{f\} (s, \tau) := \frac{1}{\sqrt{s}} \int_0^1 f(t) \overline{\psi\left(\frac{t-\tau}{s}\right)} \, dt$$

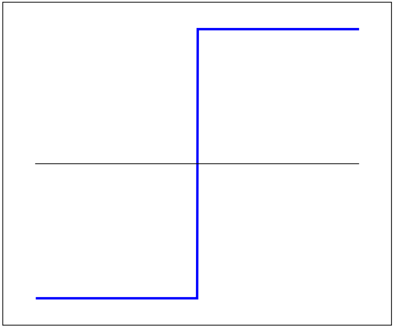
Wavelets can be designed to be bases or frames

Haar wavelet

Scaling function



Mother wavelet



Wavelets are band-pass filters, scaling functions are low-pass filters

Discrete wavelet transform

The discrete wavelet transform depends on a choice of scaling vector $\vec{\phi}$ and wavelet $\vec{\psi}$

The scaling coefficients are

$$\mathrm{W}_{\vec{\phi}}\left\{f\right\}(l) := \left\langle \vec{x}, \vec{\phi}_{[l]} \right\rangle$$

The wavelet coefficients are

$$\mathrm{W}_{\vec{\psi}}\left\{f\right\}(s, l) := \left\langle \vec{x}, \vec{\psi}_{[s,l]} \right\rangle,$$

where

$$\vec{\psi}_{[s,l]}[j] := \vec{\psi}\left[\frac{j-l}{s}\right]$$

Wavelets can be designed to be bases or frames

Orthonormal wavelet basis

Scale	Basis functions
2^0

Orthonormal wavelet basis

Scale | Basis functions

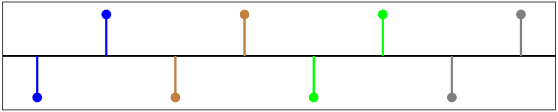
2^0

Orthonormal wavelet basis

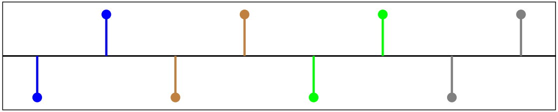
Scale

Basis functions

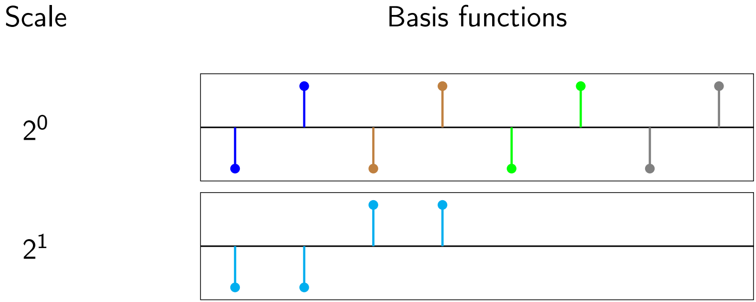
2^0



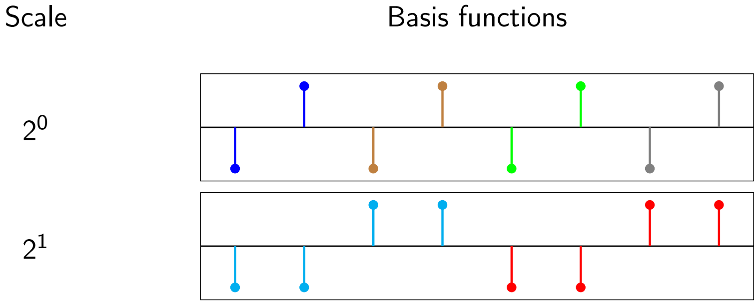
Orthonormal wavelet basis

Scale	Basis functions
2^0	
2^1	

Orthonormal wavelet basis



Orthonormal wavelet basis



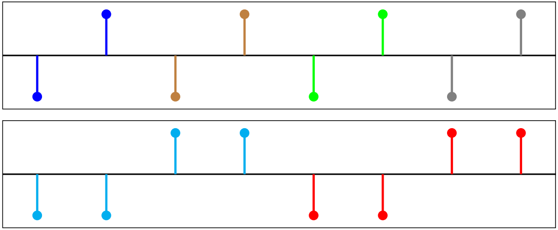
Orthonormal wavelet basis

Scale	Basis functions

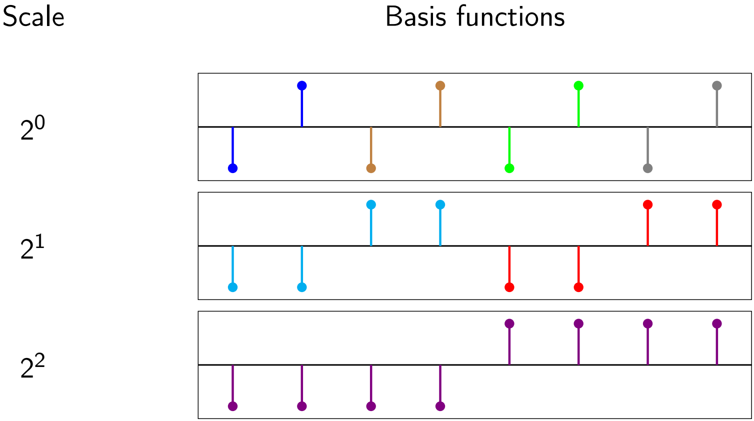
2^0

2^1

2^2

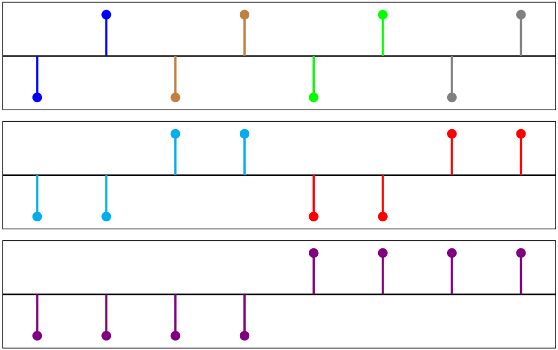


Orthonormal wavelet basis



Orthonormal wavelet basis

Scale | Basis functions



2^0

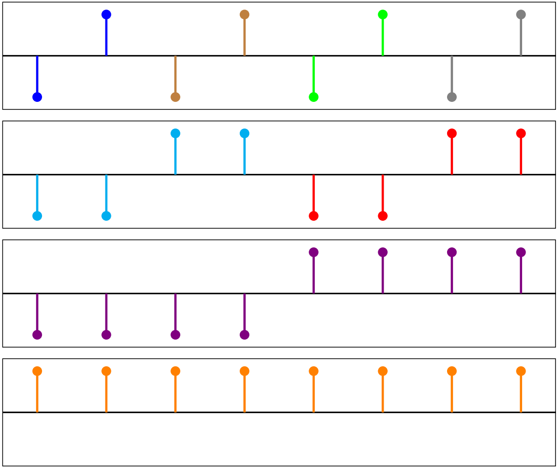
2^1

2^2

2^3

Orthonormal wavelet basis

Scale	Basis functions
2^0	
2^1	
2^2	
2^3 (scaling vector)	



Multiresolution decomposition

Sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_K$ representing different scales

Fix a scaling vector $\vec{\phi}$ and a wavelet $\vec{\psi}$

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$\mathcal{V}_K$ is the span of $\vec{\phi}$

Multiresolution decomposition

Sequence of subspaces $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_K$ representing different scales

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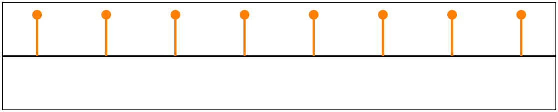
$\mathcal{V}_K$



Multiresolution decomposition

$\mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1}$

$\mathcal{W}_k$ is the span of $\vec{\psi}$ dilated by 2^k and shifted by multiples of 2^{k+1}

Multiresolution decomposition

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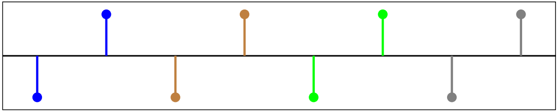
$\mathcal{W}_k$ is the span of $\vec{\psi}$ dilated by 2^k and shifted by multiples of 2^{k+1}

$\mathcal{W}_0$



Multiresolution decomposition

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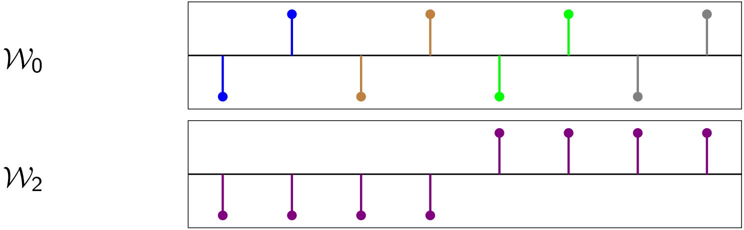


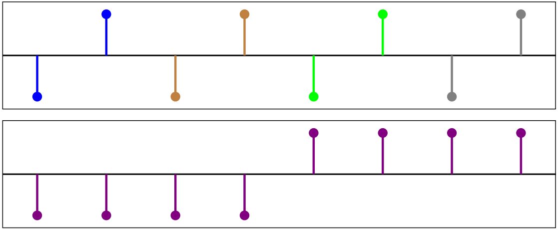
Multiresolution decomposition

$\mathcal{V}_k := \mathcal{W}_k \oplus \mathcal{V}_{k+1}$

$\mathcal{W}_k$ is the span of $\vec{\psi}$ dilated by 2^k and shifted by multiples of 2^{k+1}

$\mathcal{W}_0$

$\mathcal{W}_2$



$\mathcal{P}_{\mathcal{V}_k}\,\vec{x}$ is an approximation of $\vec{x}$ at scale 2^k

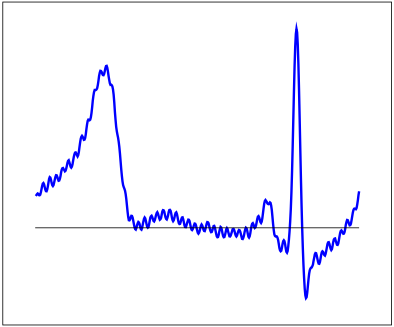
Multiresolution decomposition

Properties

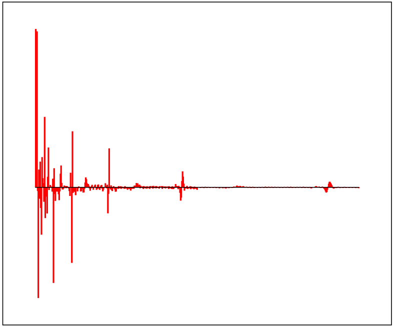
- $\mathcal{V}_0 = \mathbb{C}^n$ (approximation at scale 2^0 is perfect)
- $\mathcal{V}_k$ is invariant to translations of scale 2^k
- Dilating vectors in $\mathcal{V}_j$ by 2 yields vectors in $\mathcal{V}_{j+1}$

Electrocardiogram

Signal

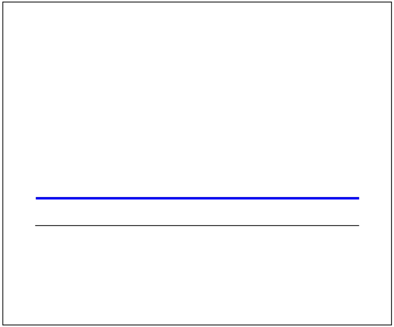


Haar transform

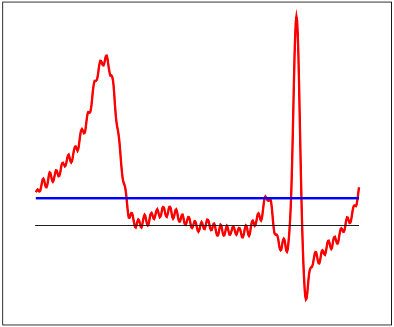


Scale 2^9

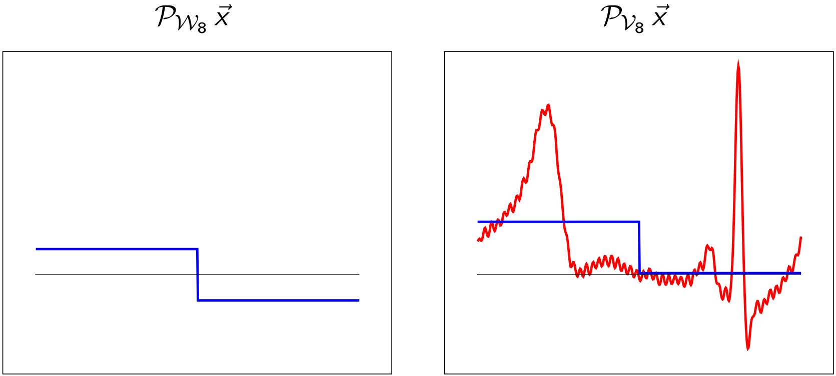
$\mathcal{P}_{\mathcal{W}_9}\,\vec{x}$



$\mathcal{P}_{\mathcal{V}_9}\,\vec{x}$

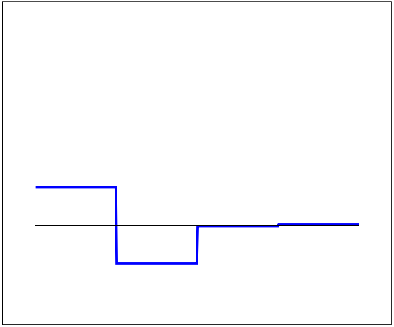


Scale 2^8

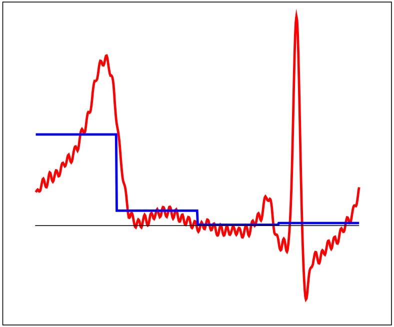


Scale 2^7

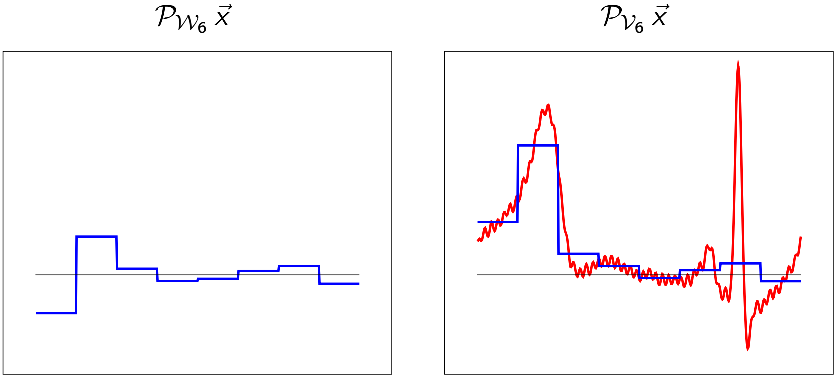
$\mathcal{P}_{\mathcal{W}_7}\,\vec{x}$



$\mathcal{P}_{\mathcal{V}_7}\,\vec{x}$

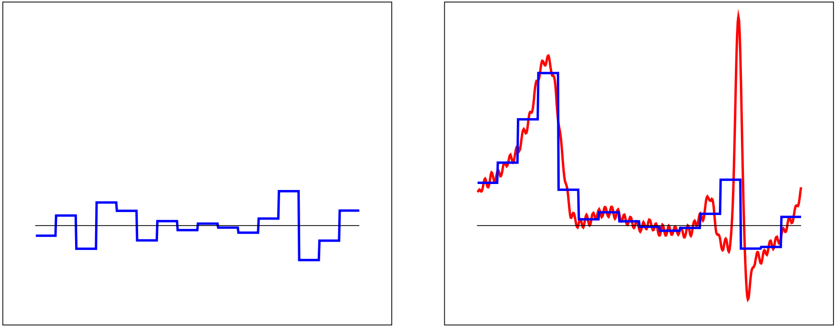


Scale 2^6



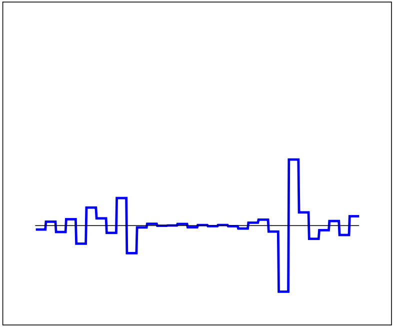
Scale 2^5

$\mathcal{P}_{\mathcal{W}_5}\, \vec{x}$ $\qquad$ $\mathcal{P}_{\mathcal{V}_5}\, \vec{x}$

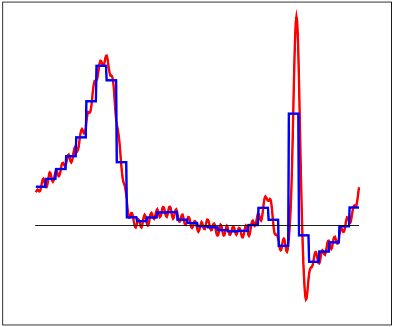


Scale 2^4

$\mathcal{P}_{\mathcal{W}_4}\,\vec{x}$

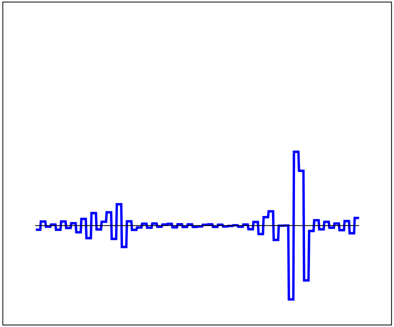


$\mathcal{P}_{\mathcal{V}_4}\,\vec{x}$

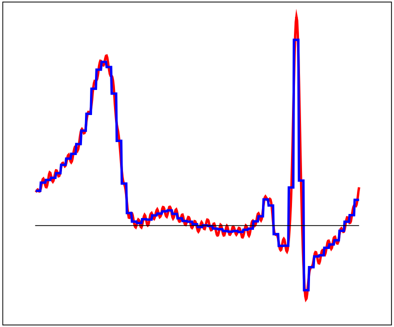


Scale 2^3

$\mathcal{P}_{\mathcal{W}_3}\,\vec{x}$

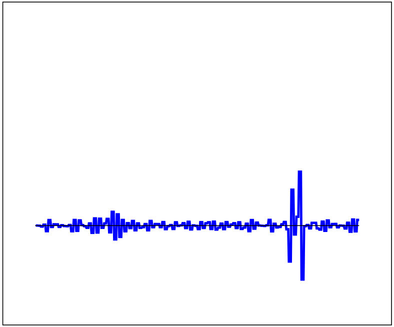


$\mathcal{P}_{\mathcal{V}_3}\,\vec{x}$

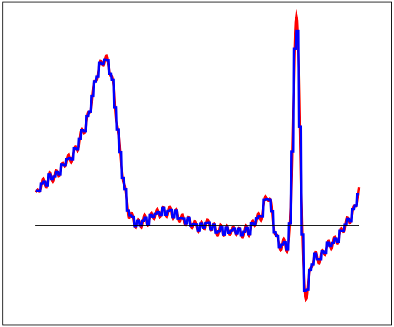


Scale 2^2

$\mathcal{P}_{\mathcal{W}_2}\,\vec{x}$

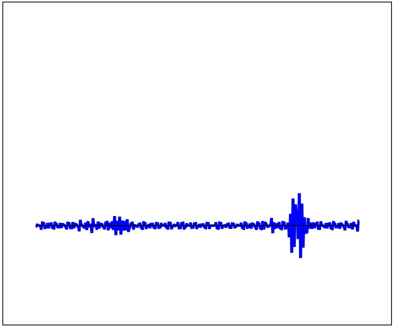


$\mathcal{P}_{\mathcal{V}_2}\,\vec{x}$

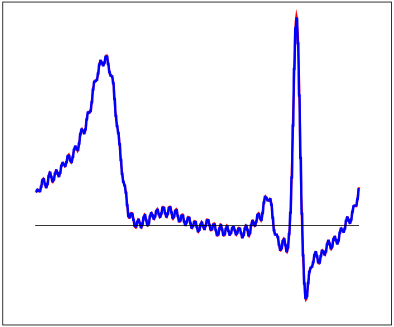


Scale 2^1

$\mathcal{P}_{\mathcal{W}_1}\,\vec{x}$

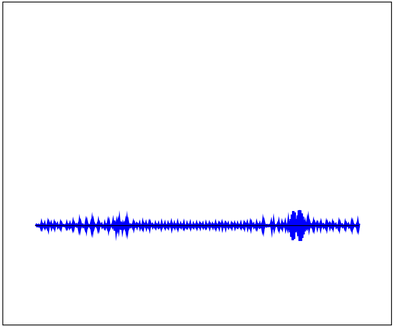


$\mathcal{P}_{\mathcal{V}_1}\,\vec{x}$

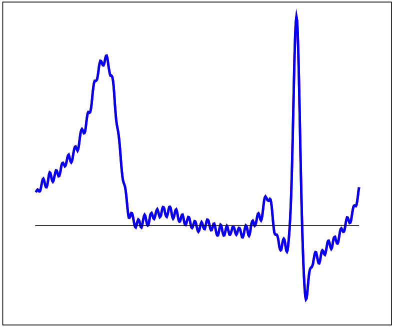


Scale 2^0

$\mathcal{P}_{\mathcal{W}_0}\,\vec{x}$



$\mathcal{P}_{\mathcal{V}_0}\,\vec{x}$



2D Wavelets

Extension to 2D by using outer products of 1D atoms

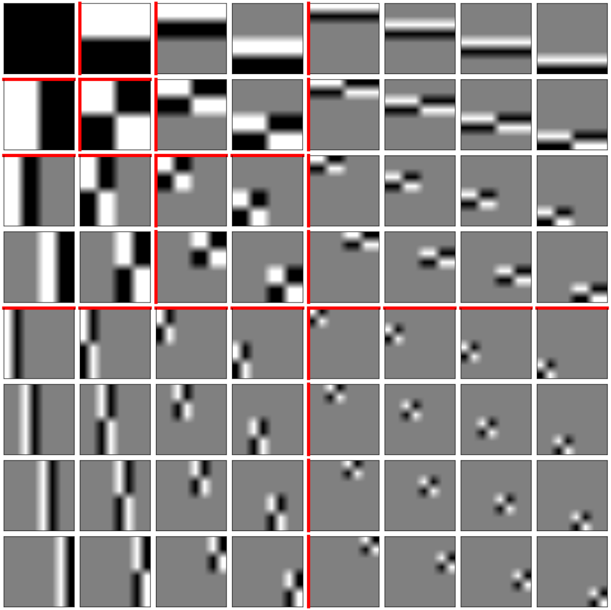
$$\xi^{\text{2D}}_{s_1,s_2,k_1,k_2} := \xi^{\text{1D}}_{s_1,k_1} \left(\xi^{\text{1D}}_{s_2,k_2} \right)^T$$

The JPEG 2000 compression standard is based on 2D wavelets

Many extensions:

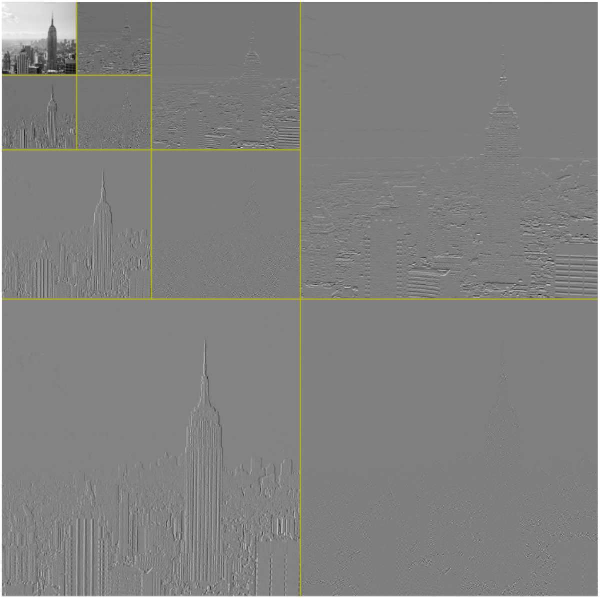
Steerable pyramid, ridgelets, curvelets, bandlets, ...

2D Haar transform



2D wavelet transform

2D wavelet transform



Denoising

Aim: Extracting information (signal) from data in the presence of uninformative perturbations (noise)

Additive noise model

$$\text{data} = \text{signal} + \text{noise}$$
$$\vec{y} = \vec{x} + \vec{z}$$

Prior knowledge about structure of signal vs structure of noise is required

Assumption

- Signal is a sparse superposition of basis/frame vectors
- Noise is not

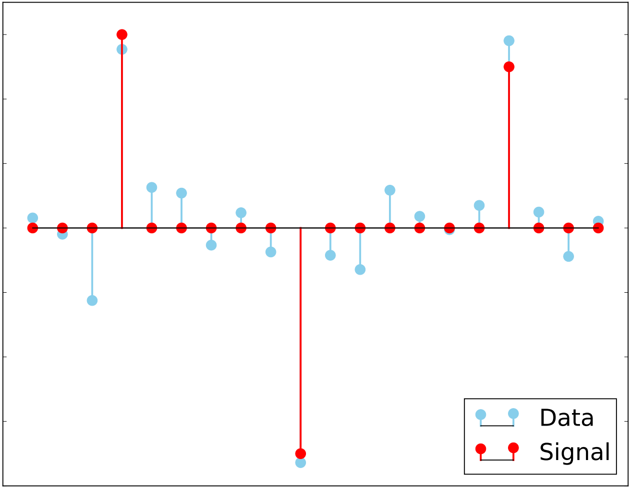
Assumption

- Signal is a sparse superposition of basis/frame vectors
- Noise is not

  Example:

  Gaussian noise $\vec{z}$ with covariance matrix $\sigma^2 I$, distribution of $F\vec{z}$?

Example

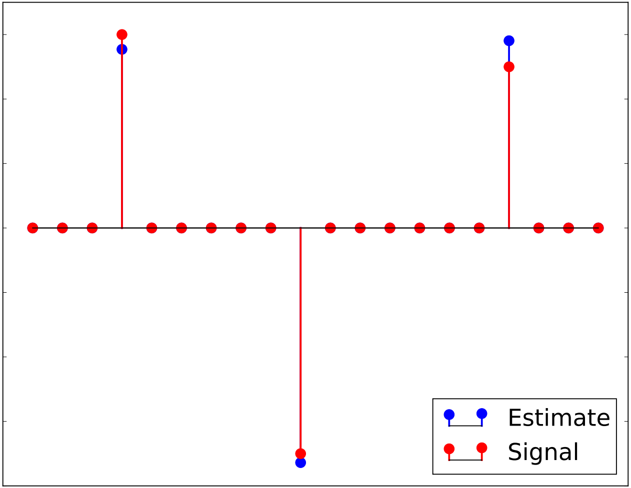


Thresholding

Hard-thresholding operator

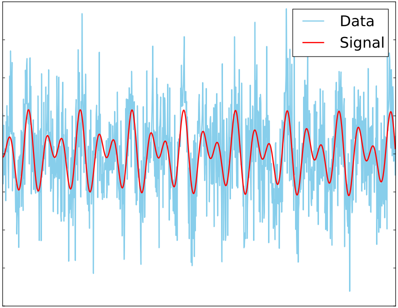
$$\mathcal{H}_{\eta}\left(\vec{v}\right)[j] := \begin{cases} \vec{v}\,[j] & \text{if } |\vec{v}\,[j]| > \eta \\ 0 & \text{otherwise} \end{cases}$$

Denoising via hard thresholding

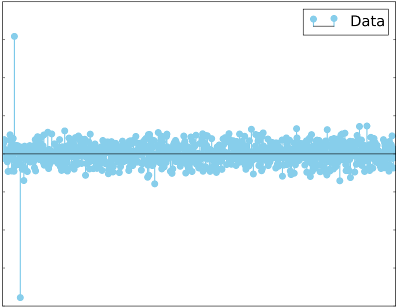


Multisinusoidal signal

$\vec{y}$



$F\vec{y}$



Denoising via hard thresholding

Data: $\vec{y} = \vec{x} + \vec{z}$

Assumption: $F\vec{x}$ is sparse, $F\vec{z}$ is not

1. Apply the hard-thresholding operator $\mathcal{H}_\eta$ to $F\vec{y}$

2. If F is a basis, then

$$\vec{x}_{\text{est}} := F^{-1}\mathcal{H}_\eta\left(F\vec{y}\right)$$

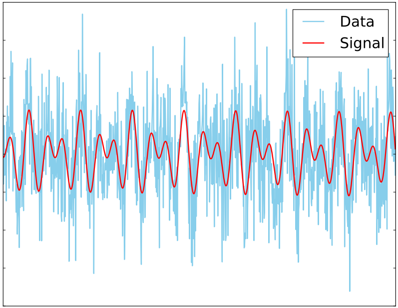
   If F is a frame,

$$\vec{x}_{\text{est}} := F^{\dagger}\mathcal{H}_\eta\left(F\vec{y}\right),$$

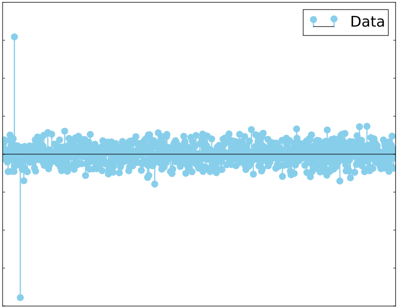
   where $F^{\dagger}$ is the pseudoinverse of F (other left inverses of F also work)

Denoising via hard thresholding in Fourier basis

$\vec{y}$

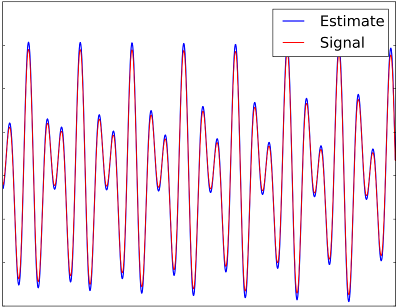


$F\vec{y}$



Denoising via hard thresholding in Fourier basis

$$F^{-1}\mathcal{H}_\eta(F\vec{y})$$



$$\mathcal{H}_\eta(F\vec{y})$$

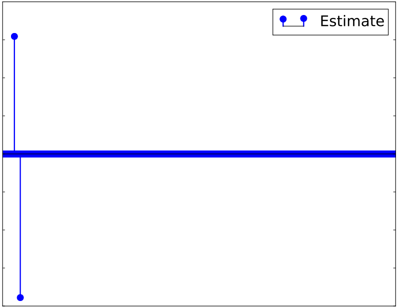


Image denoising

$\vec{x}$

$F\vec{x}$

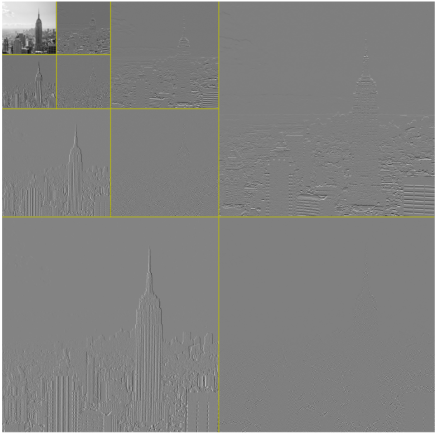
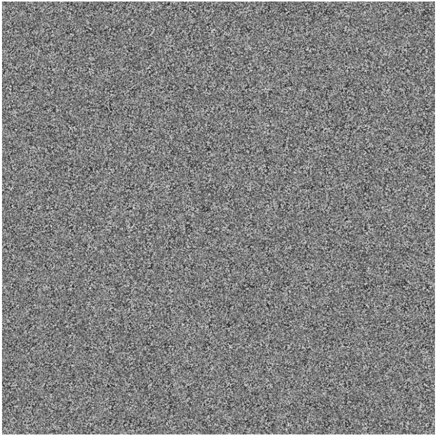
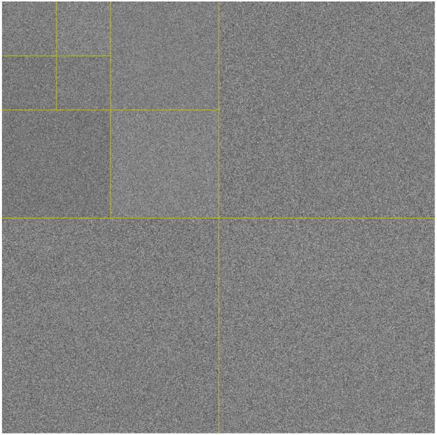


Image denoising

$\vec{z}$



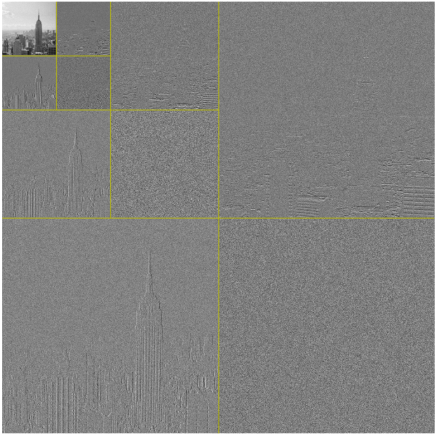
$F\vec{z}$



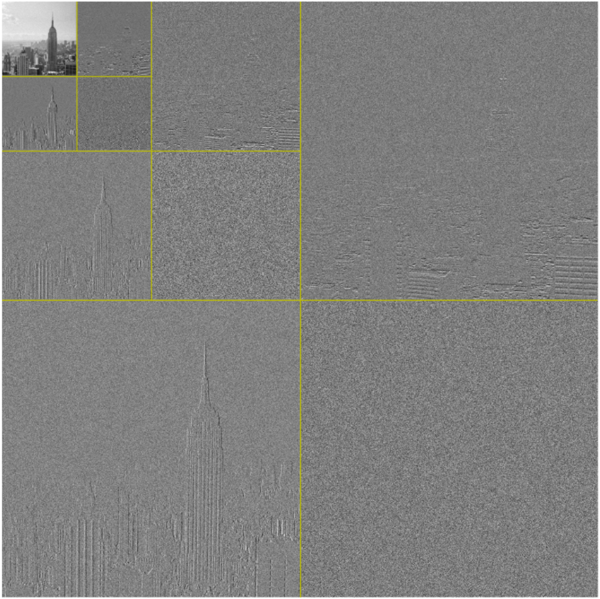
Data (SNR=2.5)

$\vec{y}$

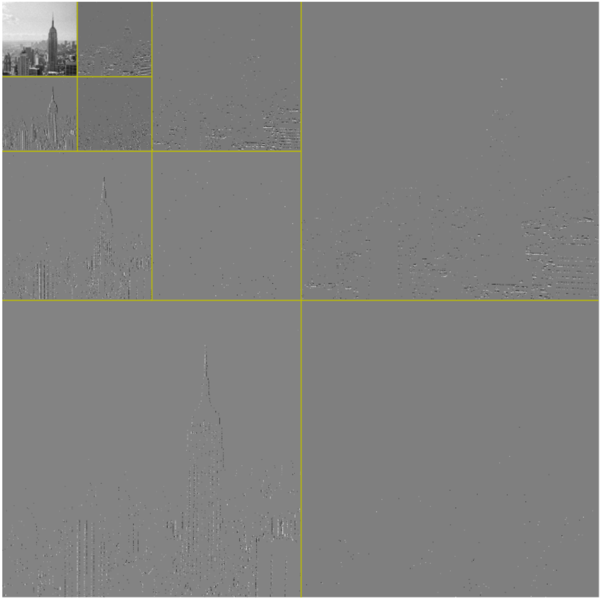
$F\vec{y}$



$F\vec{y}$



$\mathcal{H}_\eta(F\vec{y})$



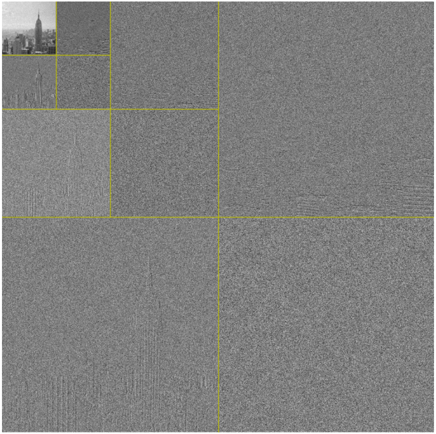
$F^{-1}\mathcal{H}_\eta\left(F\vec{y}\right)$

$\vec{y}$

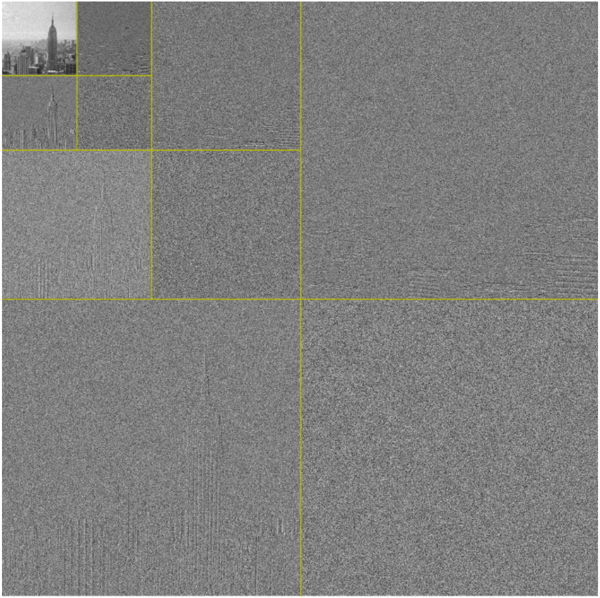
Data (SNR=1)

$\vec{y}$

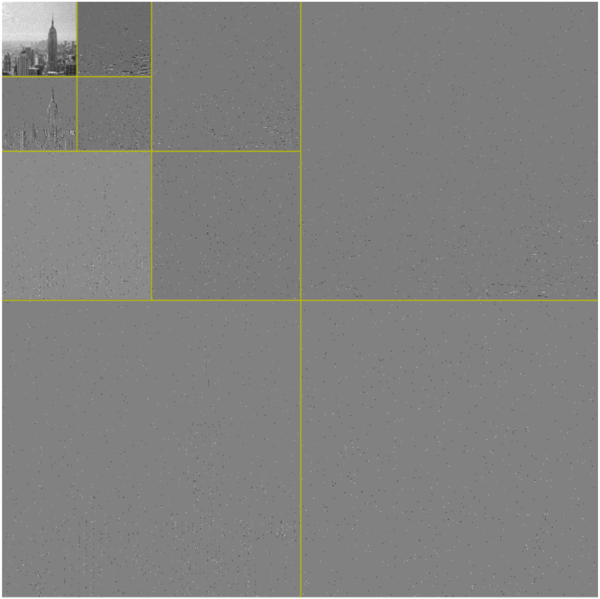
$F\vec{y}$



$F\vec{y}$



$\mathcal{H}_\eta(F\vec{y})$



$F^{-1}\mathcal{H}_\eta\left(F\vec{y}\right)$

$\vec{y}$

Image denoising

$\vec{y}$

$F^{-1}\mathcal{H}_\eta(F\vec{y})$

$\vec{x}$

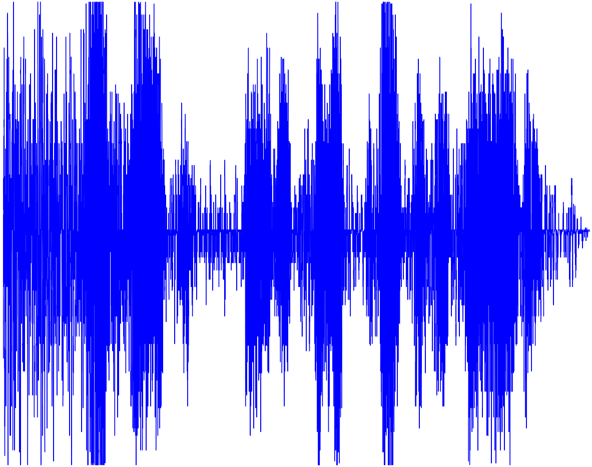
Denoising via thresholding

$\vec{y}$

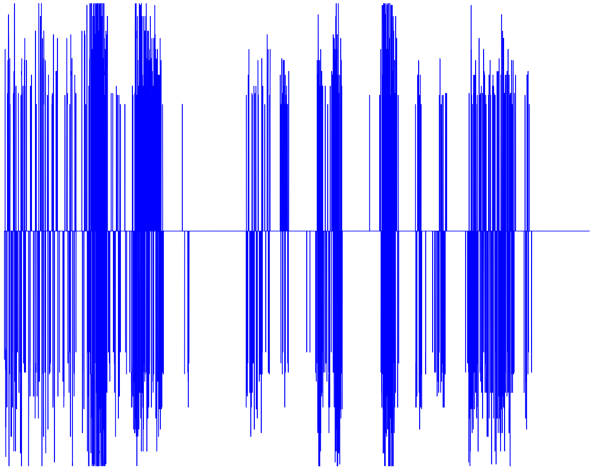
$F^{-1}\mathcal{H}_{\eta}(F\vec{y})$

$\vec{x}$

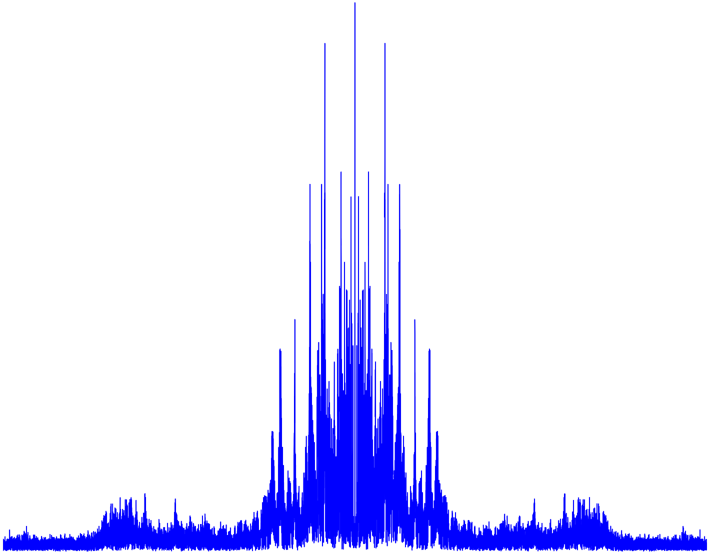
Speech denoising



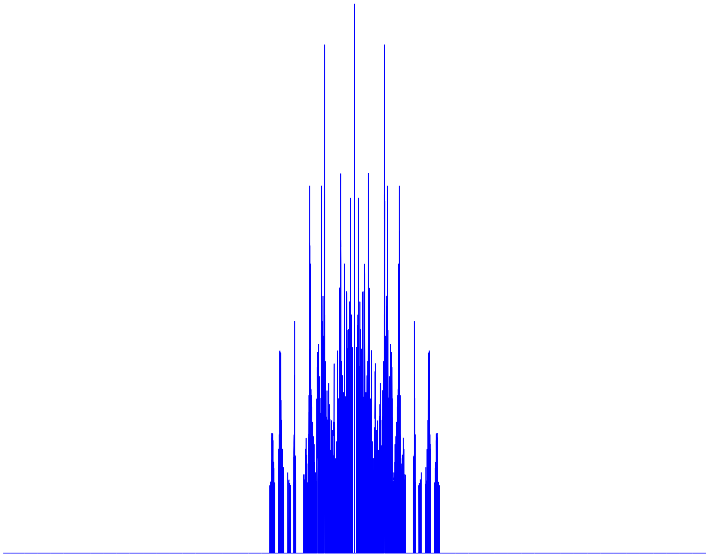
Time thresholding



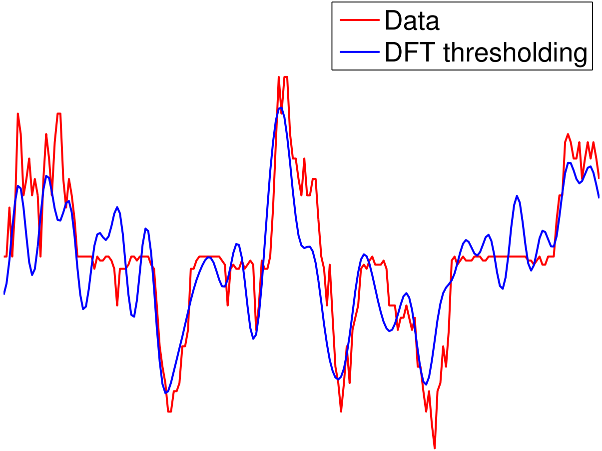
Spectrum



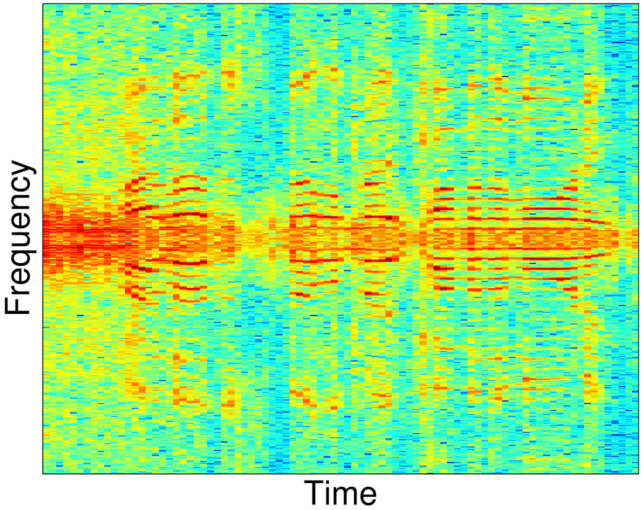
Frequency thresholding



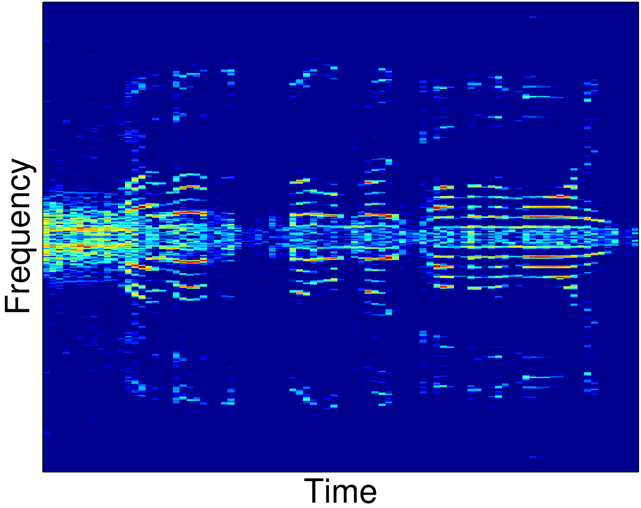
Frequency thresholding



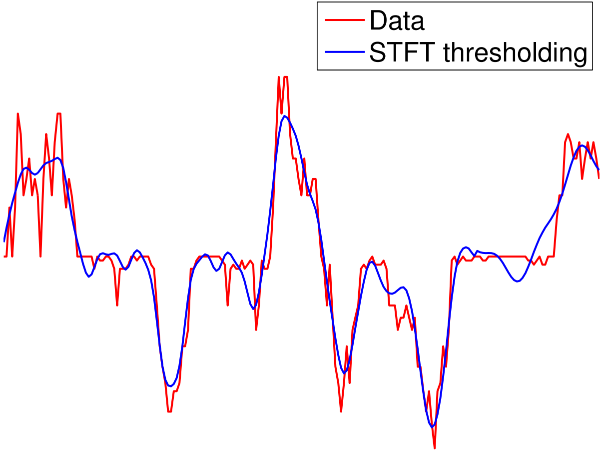
Spectrogram (STFT)



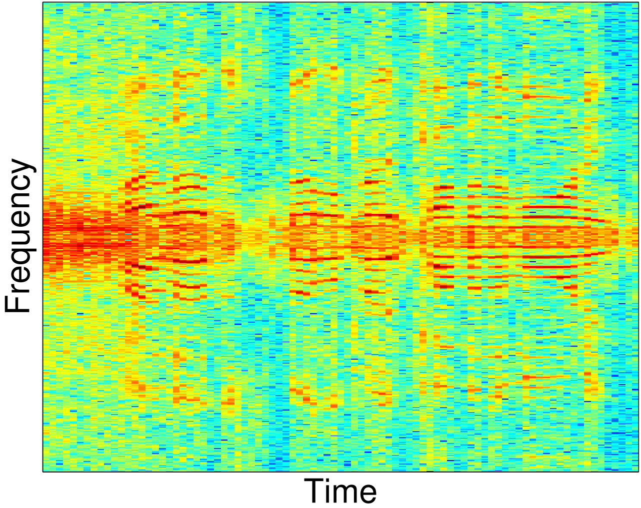
STFT thresholding



STFT thresholding



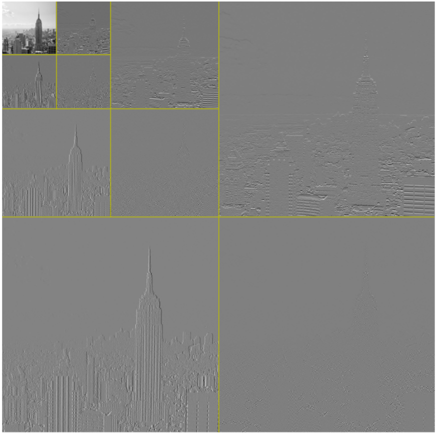
Coefficients are structured



Coefficients are structured

$\vec{x}$

$F\vec{x}$



Block thresholding

Assumption: Coefficients are *group sparse*, nonzero coefficients *cluster* together

Partition coefficients into blocks $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k$
and threshold whole blocks

$$\mathcal{B}_\eta \left(\vec{v}\right)[j] := \begin{cases} \vec{v}\,[j] & \text{if } j \in \mathcal{I}_j \quad \text{such that } \left|\left|\vec{v}_{\mathcal{I}_j}\right|\right|_2 > \eta, , \\ 0 & \text{otherwise,} \end{cases}$$

Denoising via block thresholding

1. Apply the hard-thresholding operator $\mathcal{B}_\eta$ to $F\vec{y}$
2. If F is a basis, then

$$\vec{x}_{\text{est}} := F^{-1}\mathcal{B}_\eta\left(F\vec{y}\right)$$

   If F is a frame,

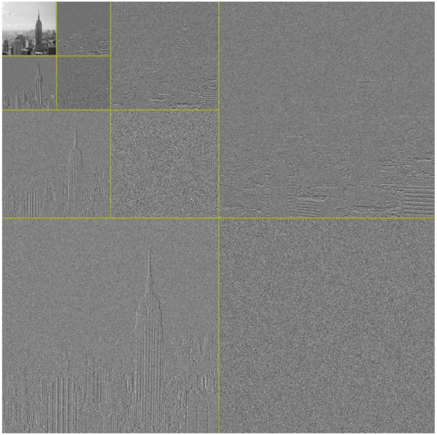
$$\vec{x}_{\text{est}} := F^{\dagger}\mathcal{B}_\eta\left(F\vec{y}\right),$$

   where $F^{\dagger}$ is the pseudoinverse of F (other left inverses of F also work)

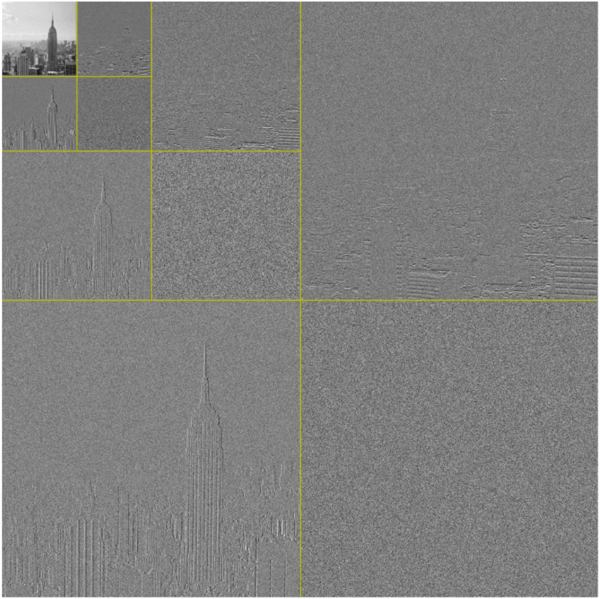
Image denoising (SNR=2.5)

$\vec{y}$

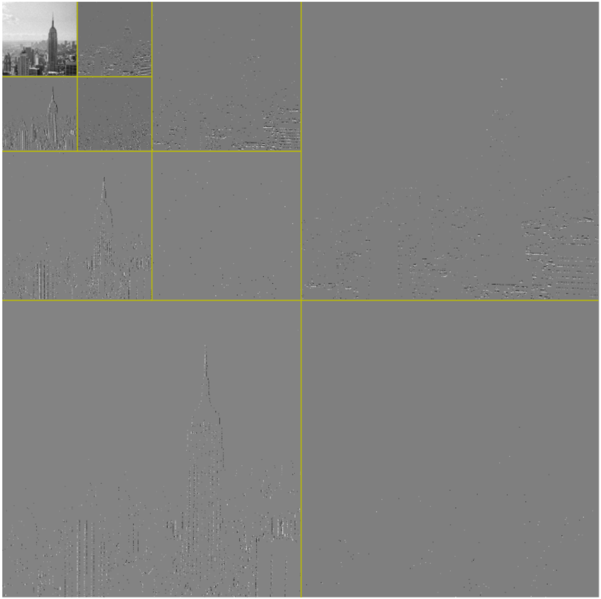
$F\vec{y}$



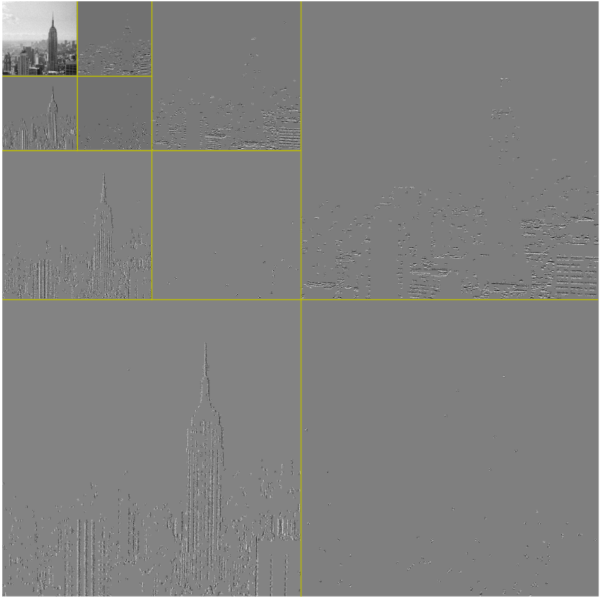
$F\vec{y}$



$\mathcal{H}_\eta(F\vec{y})$



$\mathcal{B}_\eta\,(F\vec{y})$



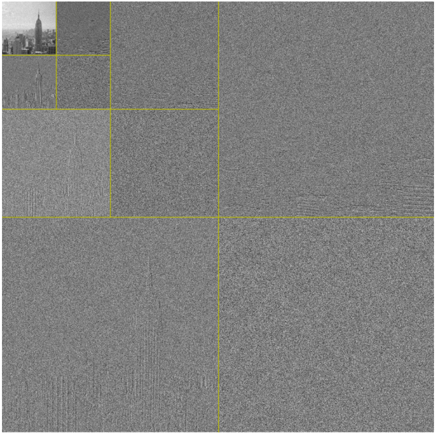
$F^{-1}\mathcal{H}_\eta\left(F\vec{y}\right)$

$F^{-1}\mathcal{B}_{\eta}\left(F\vec{y}\right)$

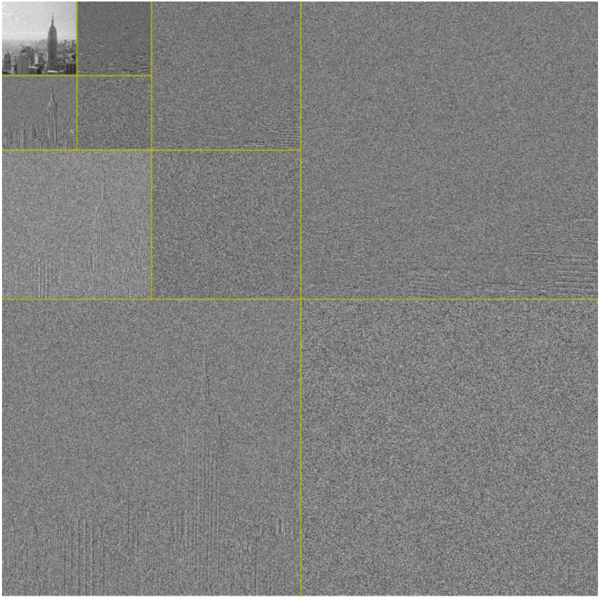
Image denoising (SNR=1)

$\vec{y}$

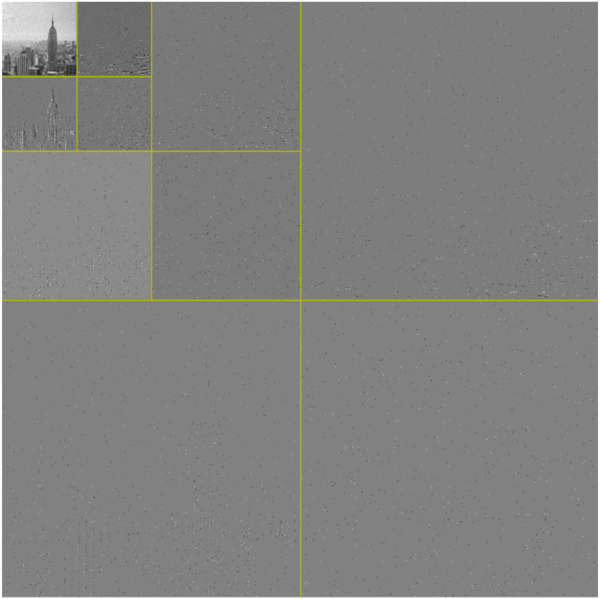
$F\vec{y}$



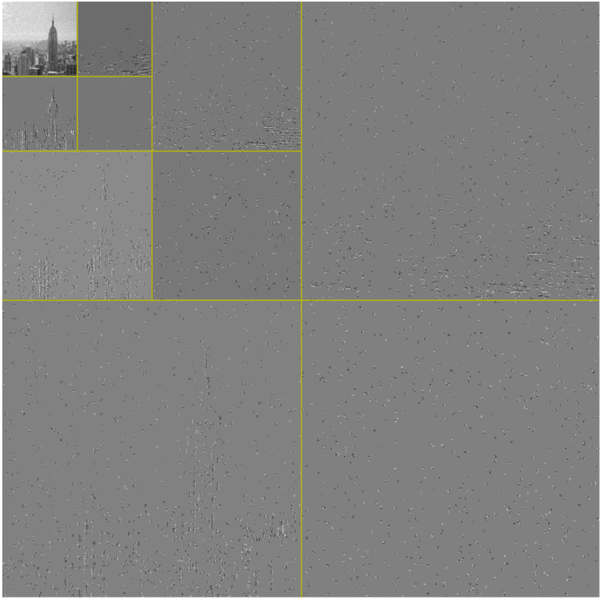
$F\vec{y}$



$\mathcal{H}_\eta(F\vec{y})$



$\mathcal{B}_\eta\,(F\vec{y})$



$F^{-1}\mathcal{H}_{\eta}(F\vec{y})$

$F^{-1}\mathcal{B}_\eta\left(F\vec{y}\right)$

Denoising via thresholding

$\vec{y}$

$F^{-1}\mathcal{H}_{\eta}\left(F\vec{y}\right)$

$\vec{x}$

Denoising via thresholding

$\vec{y}$

$F^{-1}\mathcal{B}_{\eta}\left(F\vec{y}\right)$

$\vec{x}$

Denoising via thresholding

$\vec{y}$

$F^{-1}\mathcal{H}_\eta(F\vec{y})$

$\vec{x}$

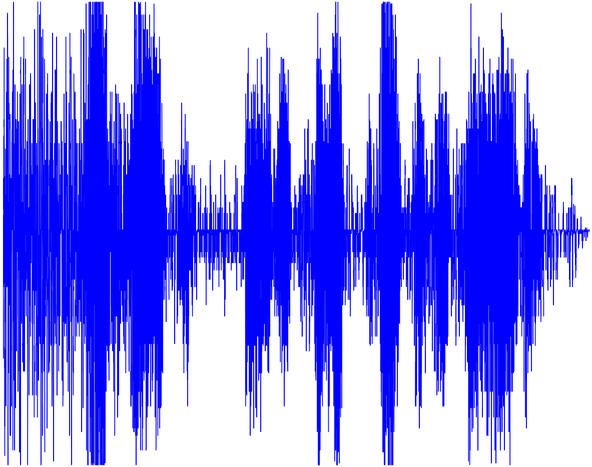
Denoising via thresholding

$\vec{y}$

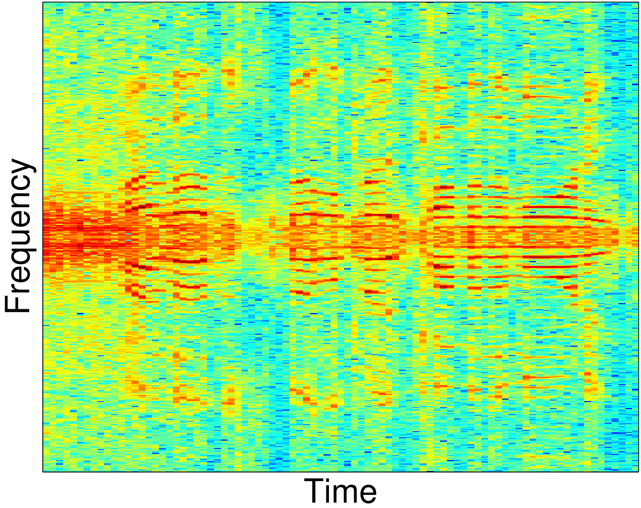
$F^{-1}\mathcal{B}_\eta(F\vec{y})$

$\vec{x}$

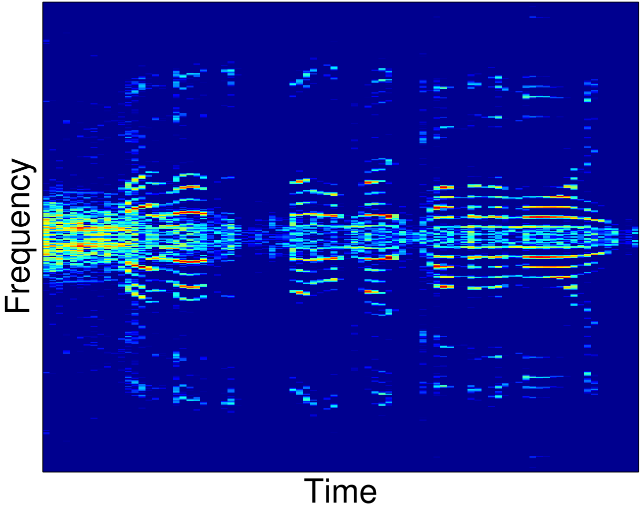
Speech denoising



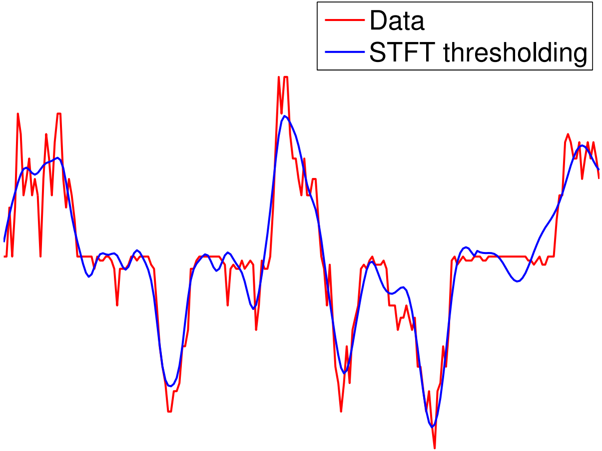
Spectrogram (STFT)



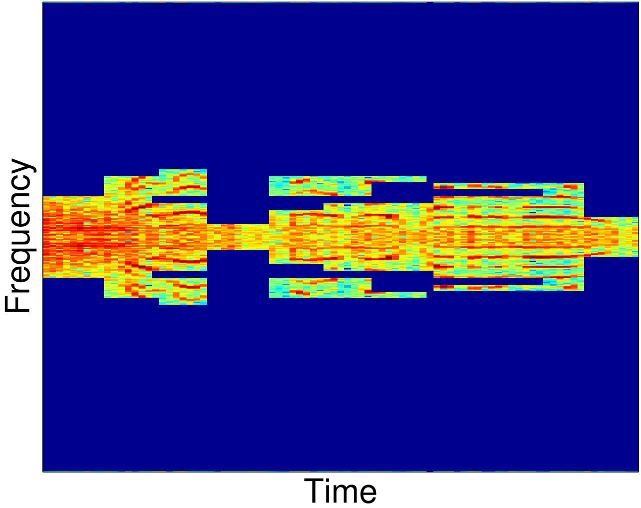
STFT thresholding



STFT thresholding



STFT block thresholding



STFT block thresholding