Probability, homework 8, due November 11.

Exercise 1. Let (X_1, X_2) be a Gaussian vector with mean (m_1, m_2) and non-degenerate covariance matrix $(C_{ij})_{1\leq i,j\leq 2}$. Prove that

$$\mathbb{E}[X_1 \mid X_2] = m_1 + \frac{C_{12}}{C_{22}}(X_2 - m_2).$$

Exercise 2. Let X be a random variable such that $\mathbb{P}(X > t) = \exp(-t)$ for any $t \geq 0$. Let $Y = \min(X, s)$, where $s > 0$ is fixed. Prove that, almost surely,

$$\mathbb{E}[X \mid Y] = Y\mathbb{1}_{Y<s} + (1+s)\mathbb{1}_{Y=s}.$$

Exercise 3. Let μ and ν be two probability measures such that $\mu \ll \nu$ and $\nu \ll \mu$ (usually abbreviated $\mu \sim \nu$). Let $X = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

(i) Prove that $\nu(X = 0) = 0$.
(ii) Prove that $\frac{1}{X} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ almost surely (for μ or ν).

Exercise 4. Let $(X_n)_{n\geq 0}$ be defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Assume this sequence converges in probability (under $\mathbb{P}$) to X. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{A})$ assumed to be absolutely continuous w.r.t. $\mathbb{P}$. Prove that $X_n \to X$ in probability under $\mathbb{Q}$.

Exercise 5. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-field of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-field of $\mathcal{G}$. Show that $\mathbb{E}\left(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}\right) = \mathbb{E}\left(Y \mid \mathcal{H}\right)$ (almost surely).

Exercise 6. Let $X_1, \ldots, X_n$ be i.i.d. integrable random variables, and $S = \sum_{i=1}^n X_i$. Calculate $\mathbb{E}[S \mid X_1]$ and $\mathbb{E}[X_1 \mid S]$.

Exercise 7. For fixed $a, b > 0$, let (X, Y) be a $\mathbb{N} \times \mathbb{R}_+$-valued random variable such that

$$\mathbb{P}(X = n, Y \leq t) = b \int_0^t \frac{(ay)^n}{n!} e^{-(a+b)y} \mathrm{d}y.$$

For $h : \mathbb{R}_+ \to \mathbb{R}$ continuous and bounded, calculate $\mathbb{E}[h(Y) \mid X]$. Calculate $\mathbb{E}[\frac{Y}{X+1}]$. Calculate $\mathbb{P}(X = n \mid Y)$. Calculate $\mathbb{E}[X \mid Y]$.

Exercise 8. Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}, \mathcal{H}$ sub σ-fields of $\mathcal{F}$ such that $\sigma(\mathcal{G}, \mathcal{H}) = \mathcal{F}$. Find counterexamples to the following assertions:

(i) If $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$ then X and Y are independent.
(ii) If $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}] = 0$ then $X = 0$.
(iii) If X and Y are independent then so are $\mathbb{E}[X \mid \mathcal{G}]$ and $\mathbb{E}[Y \mid \mathcal{G}]$.

Exercise 9. Let $(X_n)_{n\geq 1}$ be a sequence of nonnegative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and $(\mathcal{F}_n)_{n\geq 0}$ a sequence of sub σ-fields of $\mathcal{F}$. Assume that $\mathbb{E}(X_n \mid \mathcal{F}_n)$ converges to 0 in probability.

(i) Show that X_n converges to 0 in probability.
(ii) Show that the reciprocal is wrong.

Exercise 10. On the same probability space, let X, Y be positive random variables such that $\mathbb{E}[X \mid Y] = Y$ and $\mathbb{E}[Y \mid X] = X$ (almost surely). Prove that $X = Y$ almost surely.

Exercise 11. For any $d \geq 1$, we admit that there is only one probability measure μ on $\mathcal{S}_d$, (the $(d-1)$-th dimensional sphere embedded in $\mathbb{R}^d$) that is uniform, in the following sense: for any isometry $A \in \mathrm{O}(d)$ (the orthogonal group in $\mathbb{R}^d$), and any continuous function $f : \mathcal{S}_d \to \mathbb{R}$,

$$\int_{\mathcal{S}_d} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{S}_d} f(Ax)\mathrm{d}\mu(x).$$

Let $X = (X_1, \ldots, X_d)$ be a vector of independent centered and reduced Gaussian random variables.

a) Prove that the random variable $U = X/\|X\|_{L^2}$ is uniformly distributed on the sphere.

b) Prove that, as $d \to \infty$, the main part of the globe is concentrated close to the Equator, i.e. for any $\varepsilon > 0$,

$$\int_{x \in \mathcal{S}_d, |x_1| < \epsilon} \mathrm{d}\mu(x) \to 1.$$