Probability, homework 7, solutions.

Long problem. The goal of this problem is to prove the iterated logarithm law, first for Gaussian random variables. In other words, for $X_1, X_2 \dots$ i.i.d. standard Gaussian random variables, denoting $S_n = X_1 + \cdots + X_n$, we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right) = 1 \tag{0.1}$$

(1) Prove that

$$\mathbb{P}(X_1 > \lambda) \underset{\lambda\to\infty}{\sim} \frac{1}{\lambda\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}.$$

In the following questions we denote $f(n) = \sqrt{2n\log\log n}$, $\lambda > 1$, $c, \alpha > 0$, $A_k = \{S_{\lfloor\lambda^k\rfloor} \geq cf(\lambda^k)\}$, $C_k = \{S_{\lfloor\lambda^{k+1}\rfloor} - S_{\lfloor\lambda^k\rfloor} \geq cf(\lambda^{k+1} - \lambda^k)\}$ and $D_k = \{\sup_{n\in[\![\lambda^k,\lambda^{k+1}]\!]} \frac{S_n - S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)} \geq \alpha\}$.

(2) Prove that for any $c > 1$ we have $\sum_{k\geq 1} \mathbb{P}(A_k) < \infty$ and

$$\limsup_{k\to\infty} \frac{S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)} \leq 1 \text{ a.s.}$$

(3) Prove that for any $c < 1$ we have $\sum_{k\geq 1} \mathbb{P}(C_k) = \infty$ and

$$\mathbb{P}(C_k \text{ i.o.}) = 1.$$

(4) Let $\varepsilon > 0$ and choose $c = 1 - \varepsilon/10$. Prove that almost surely the following inequality holds for infinitely many k:

$$\frac{S_{\lfloor\lambda^{k+1}\rfloor}}{f(\lambda^{k+1})} \geq c\frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} - (1+\varepsilon)\frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) By choosing a large enough λ in the previous inequality, prove that almost surely

$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \geq 1.$$

(6) Prove that for any $n \in [\![\lambda^k, \lambda^{k+1}]\!]$ and $S_n > 0$ we have

$$\frac{S_n}{f(n)} \leq \frac{S_{\lfloor\lambda^k\rfloor}}{f(\lfloor\lambda^k\rfloor)} + \frac{S_n - S_{\lfloor\lambda^k\rfloor}}{f(\lfloor\lambda^k\rfloor)}.$$

(7) Prove that

$$\mathbb{P}(D_k) \underset{k\to\infty}{\sim} 2\mathbb{P}\left(X_1 \geq \frac{\alpha f(\lambda^k)}{\sqrt{\lambda^{k+1} - \lambda^k}}\right) \underset{k\to\infty}{\sim} \frac{\text{cst}}{\sqrt{\log k}} \left(\frac{1}{k}\right)^{\frac{\alpha^2}{\lambda-1}}.$$

(8) Prove that for $\alpha^2 > \lambda - 1$, almost surely

$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq \limsup_{n\to\infty} \frac{S_{\lfloor\lambda^k\rfloor}}{f(\lambda^k)} + \alpha.$$

(9) By choosing appropriate λ and α, prove that almost surely

$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq 1.$$

(10) State a result similar to (0.1) for i.i.d. uniformly bounded random variables. Which steps in the above proof need to be modified to prove this universality result? How?

Solution.

(1) By integration by parts, we have

$$\mathbb{P}(X_1 > \lambda) = \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty \left(-\frac{1}{x}\right) \cdot (-x)\, e^{-\frac{x^2}{2}}\, \mathrm{d}x = \frac{1}{\lambda\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} - \frac{1}{\sqrt{2\pi}} \int_\lambda^{+\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}}\, \mathrm{d}x.$$

Moreover, $\int_\lambda^{+\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}} \mathrm{d}x \leq \lambda^{-2} \int_\lambda^{+\infty} e^{-\frac{x^2}{2}} \mathrm{d}x = \mathrm{o}\left(\mathbb{P}(X_1 > \lambda)\right)$, which concludes the proof.
(2) From the provious estimate, we have

$$\mathbb{P}(A_k) = \mathbb{P}(S_{\lfloor \lambda^k \rfloor} \geq cf(\lambda^k)) = \mathbb{P}(X_1 \geq cf(\lambda^k)/\sqrt{\lfloor \lambda^k \rfloor}) = \mathbb{P}(X_1 \geq c\sqrt{2\log\log(\lambda^k)} \cdot \sqrt{\lambda^k/\lfloor \lambda^k \rfloor})$$
$$\underset{k\to\infty}{\sim} \frac{1}{2c\sqrt{\pi \log k}} \exp(-c^2 \log_2(\lambda^k) \frac{\lambda^k}{\lfloor \lambda^k \rfloor}) = \frac{1}{2c\sqrt{\pi \log k}} \exp(-c^2(\log k + \log_2 \lambda)(1+\mathrm{O}(\lambda^{-k})))$$
$$\sim \frac{\exp(-c^2 \log_2 \lambda)}{2c\sqrt{\pi \log k}} \cdot k^{-c^2}.$$

Therefore for any $c > 1$ we have $\sum_{k\geq 1} \mathbb{P}(A_k) < \infty$ and $\mathbb{P}(A_k \text{ i.o.}) = 0$ by the Borel-Cantelli lemma, which is equivalent to the claim.
(3) A similar calculation gives

$$\mathbb{P}(C_k) = \mathbb{P}(X_1 \geq cf(\lambda^{k+1} - \lambda^k)/\sqrt{\lfloor \lambda^{k+1} \rfloor - \lfloor \lambda^k \rfloor})$$
$$= \mathbb{P}(X_1 \geq c\sqrt{2\log\log(\lambda^k(\lambda-1))} \cdot \sqrt{(\lambda^{k+1} - \lambda^k)/(\lfloor \lambda^{k+1} \rfloor - \lfloor \lambda^k \rfloor)})$$
$$\underset{k\to\infty}{\sim} \frac{1}{2c\sqrt{\pi \log k}} \exp(-c^2 \log_2(\lambda^k(\lambda-1)) \frac{\lambda^{k+1} - \lambda^k}{\lfloor \lambda^{k+1} \rfloor - \lfloor \lambda^k \rfloor})$$
$$\sim \frac{\exp(-c^2 \log_2 \lambda)}{2c\sqrt{\pi \log k}} \cdot k^{-c^2}.$$

Therefore for any $c < 1$ we have $\sum_{k\geq 1} \mathbb{P}(A_k) = \infty$ and $\mathbb{P}(C_k \text{ i.o.}) = 1$ by the Borel-Cantelli lemma for independent events.
(4) The event C_k holding infinitely often can be rephrased as

$$\frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq c\frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} + \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \frac{f(\lambda^k)}{f(\lambda^{k+1})} \text{ i.o.}$$

By (2), for any $\varepsilon > 0$, almost surely $\frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \geq -1 - \varepsilon$ for large enough k, so that the following holds infinitely often, almost surely:

$$\frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq c\frac{f(\lambda^{k+1} - \lambda^k)}{f(\lambda^{k+1})} - (1+\varepsilon)\frac{f(\lambda^k)}{f(\lambda^{k+1})}.$$

(5) As $k \to \infty$, the above right hand side is equivalent to $c\sqrt{1-\lambda^{-1}} - \frac{1+\varepsilon}{\sqrt{\lambda}}$. Hence for any $\theta > 0$, by choosing $c = 1 - \theta/2$, $\varepsilon = \theta/10$ and $\lambda^{-1} = (\theta/100)^{-1}$, we obtain that i.o. $\frac{S_{\lfloor \lambda^{k+1} \rfloor}}{f(\lambda^{k+1})} \geq 1 - \theta$, which answers the question as θ is arbitrarily small.
(6) For any $n \in [\![\lambda^k, \lambda^{k+1}]\!]$ we have

$$\frac{S_n}{f(n)} = \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \frac{f(\lambda^k)}{f(n)} + \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(n)} \leq \frac{|S_{\lfloor \lambda^k \rfloor}|}{f(\lambda^k)} + \frac{|S_n - S_{\lfloor \lambda^k \rfloor}|}{f(\lambda^k)}.$$

(7) This is the most difficult question. More generally, one needs to evaluate the tail distribution of $R_n = \max(0, S_1, \ldots, S_n)$ where $S_k = X_1 + \cdots + X_k$ and the X_k's are i.i.d. with finite large enough moments.

There are at least three possible approaches:

- As in Varadhan's book, use Lévy's inequality, Lemma 3.8.

- Use the reflection principle when the X_i's are ± 1 and then compare the general case to this Bernoulli case (e.g. via Lindeberg's method). (Note that both methods mentioned so far are related, the proof of Lévy's inequality relying on ideas from the reflection principle)

- Use the surprising Sptizer's formula: for any t and λ sucht at the series below are summable, we have

$$\sum_{n\geq 0} t^n\, \mathbb{E}(e^{\lambda R_n}) = \exp\left(\sum_{n\geq 1} \frac{\mathbb{E}(e^{\lambda S_n^+})}{n} t^n\right).$$

(For a beautiful proof, read Wendel's article *Spitzer's formula: a short proof*). Let us conclude sketching how to conclude from the above formula. For standard Gaussian X_i's and $\lambda > 0$, we have $\mathbb{E}(e^{\lambda S_n^+}) \asymp e^{\frac{\lambda^2}{2} n}$ so that we obtain

$$\sum_{n\geq 0} t^n\, \mathbb{E}(e^{\lambda R_n}) \asymp \exp\Big(-\log(1 - te^{\lambda^2/2})\Big) = \frac{1}{1 - te^{\lambda^2/2}},$$

and one easily concludes that $\mathbb{E}(e^{\lambda R_n}) \asymp e^{\frac{\lambda^2}{2} n}$. This implies that

$$\mathbb{P}(R_n > x) \leq e^{-\lambda x}\, \mathbb{E}(e^{\lambda R_n}) \lesssim e^{\frac{\lambda^2}{2} n - \lambda x}.$$

Optimization over λ gives

$$\mathbb{P}(R_n > x) \lesssim e^{-\frac{x^2}{2n}}.$$

In our setting, with $n = \lfloor \lambda^{k+1} \rfloor - \lfloor \lambda^k \rfloor$ this gives

$$\mathbb{P}(D_k) \lesssim \mathbb{P}(R_n > \alpha f(\lambda^k)) \lesssim e^{-\frac{\alpha^2 f(\lambda^k)^2}{2n}} \lesssim k^{-\alpha^2/(\lambda - 1)}.$$

(8) If $\alpha^2 > \lambda - 1$, from the Borel-Cantelli lemma a.s. for lage enough k the event D_k does not happen. Naturally the same conclusion applies to

$$\tilde{D}_k = \{\inf_{n \in [\![\lambda^k, \lambda^{k+1}]\!]} \frac{S_n - S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} \leq -\alpha\}.$$

With (6) this implies that a.s.

$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq \limsup_{n\to\infty} \frac{S_{\lfloor \lambda^k \rfloor}}{f(\lambda^k)} + \alpha.$$

(9) For $\varepsilon > 0$, pick $\lambda = 1 + \varepsilon^4$ and $\alpha = \varepsilon$. Then (2) and (8) together give

$$\limsup_{n\to\infty} \frac{S_n}{f(n)} \leq 1 + \varepsilon.$$

As ε is arbitrarilly small, this concludes the proof.

(10) Most steps are robust, only the tails estimates on S_n and R_n require more arguments. For S_n, calculate the Laplace transofrm and use Marjkov's inequality to obtain some upper tail estimates. Use a quantitative central limit theorem obtained via characteristic functions for lower tail estimates.

For R_n, we just need the upper tail, and the approaches mentiones in (7) all work.