

Probability, homework 6, due October 21.

Exercise 1. Let $(X_i)_{i \geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n \rightarrow \infty} \frac{X_1 + \dots + X_n}{X_1^2 + \dots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 2. Let f be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \rightarrow \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \dots + x_n}{n}\right) dx_1 \dots dx_n.$$

Exercise 3. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0, 1]$. The n -th Bernstein polynomial is

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

a) Let $S_n(x) = B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x : $B^{(n,x)} = \sum_{i=1}^n X_i$ where the X_i 's are independent and $\mathbb{P}(X_i = 1) = x$, $\mathbb{P}(X_i = 0) = 1 - x$. Prove that $B_n(x) = \mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n - f\|_{L^\infty([0,1])} \rightarrow 0$ as $n \rightarrow \infty$.

Exercise 4. Calculate $\lim_{n \rightarrow \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!}$.

Exercise 5. Let $(X_n)_{n \geq 0}$ be a sequence of i.i.d random variables, with uniform distribution on $[0, 1]$. Let $Y_n = (X_n)^n$.

- (1) Calculate the distribution of Y_n .
- (2) Show that $(Y_n)_{n \geq 0}$ converges to 0 in probability.
- (3) Show that $(Y_n)_{n \geq 0}$ converges in L^1 .
- (4) Show that almost surely $(Y_n)_{n \geq 0}$ does not converge.

Long problem. The goal is to prove the Erdős-Kac theorem: if $w(m)$ denotes the number of distinct prime factors of m and k is a random variable uniformly distributed on $\llbracket 1, n \rrbracket$, then the following convergence in distribution holds:

$$\frac{w(k) - \log \log n}{\sqrt{\log \log n}} \xrightarrow[n \rightarrow \infty]{} \mathcal{N}(0, 1).$$

- (i) Prove that if $(X_n)_{n \geq 1}$ converges in distribution to $\mathcal{N}(0, 1)$ and $\sup_{n \geq 1} \mathbb{E}[X_n^{2k}] < \infty$ for any $k \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0, 1)^k]$ for any $k \in \mathbb{N}$.
- (ii) Prove that for any $x \in \mathbb{R}$ and $d \geq 1$ we have

$$\left| e^{ix} - \sum_{\ell=0}^d \frac{(ix)^\ell}{\ell!} \right| \leq \frac{|x|^{d+1}}{(d+1)!}.$$

- (iii) Assume that $\lim_{n \rightarrow \infty} \mathbb{E}[X_n^k] = \mathbb{E}[\mathcal{N}(0, 1)^k]$ for any $k \in \mathbb{N}$. Prove that X_n converges in distribution to X .
- (iv) Let $w_y(m)$ be the number of prime factors of m which are smaller than y . Let $(B_p)_{p \text{ prime}}$ be independent random variables such that $\mathbb{P}(B_p = 1) = 1 - \mathbb{P}(B_p = 0) = \frac{1}{p}$, $W_y = \sum_{p \leq y} B_p$, $\mu_y = \sum_{p \leq y} \frac{1}{p}$, $\sigma_y^2 = \sum_{p \leq y} \left(\frac{1}{p} - \frac{1}{p^2}\right)$. Prove that if $y = n^{o(1)}$, then for any $d \in \mathbb{N}$ we have

$$\mathbb{E} \left[\left(\frac{w_y(k) - \mu_y}{\sigma_y} \right)^d \right] - \mathbb{E} \left[\left(\frac{W_y - \mu_y}{\sigma_y} \right)^d \right] \rightarrow 0$$

as $n \rightarrow \infty$.

- (v) Conclude.