# INCIDENCE ESTIMATE TABLE

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This table records four estimates from 2D incidence geometry: the Szemerédi–Trotter bound, Cauchy–Schwarz(Lines) (uses that two lines intersect in 1 point), Cauchy–Schwarz(Points) (uses that two points span 1 line), and the high case of the high-low bound. The 6 variables are:

- $|\mathcal{P}|$, the number of points.
- $|\mathcal{L}|$, the number of lines.
- $r_P$, the number of lines through each point.
- $r_L$, the number of points on each line.
- $\mathcal{I}(P,L) = r_P|\mathcal{P}| = r_L|\mathcal{L}|$, the number of incidences.
- $|\pi_\theta(P)| = \frac{|\mathcal{P}|}{r_L}$, the size of a typical projection, if $\mathcal{L}$ comes in a projection family.

Any 3 of these variables determine the others. In the table below, each column represents an incidence estimate, and each row records a different way to write that estimate, using one left-hand-side variable (marked 1) and two right-hand-side variables (marked ✓).

| $\mathcal{P}$ | $\mathcal{L}$ | $r_P$ | $r_L$ | $\mathcal{I}$ | $\lvert\pi_\theta(P)\rvert$ | ST | CS(Lines) | CS(Points) | High case |
|---|---|---|---|---|---|---|---|---|---|
| 1 | ✓ | ✓ | | | | $\lvert\mathcal{P}\rvert \lesssim \frac{\lvert\mathcal{L}\rvert^2}{r_P^3}$ | $\lvert\mathcal{P}\rvert \lesssim \frac{\lvert\mathcal{L}\rvert^2}{r_P^2}$ | | $\lvert\mathcal{P}\rvert \lesssim \delta^{-1}\frac{\lvert\mathcal{L}\rvert}{r_P^2}$ |
| ✓ | 1 | ✓ | | | | $\lvert\mathcal{L}\rvert \gtrsim \lvert\mathcal{P}\rvert^{1/2} r_P^{3/2}$ | $\lvert\mathcal{L}\rvert \gtrsim \lvert\mathcal{P}\rvert^{1/2} r_P$ | $\lvert\mathcal{L}\rvert \gtrsim r_P^2$ | $\lvert\mathcal{L}\rvert \gtrsim \delta\lvert\mathcal{P}\rvert\, r_P^2$ |
| ✓ | ✓ | 1 | | | | $r_P \lesssim \lvert\mathcal{L}\rvert^{2/3}\lvert\mathcal{P}\rvert^{-1/3}$ | $r_P \lesssim \lvert\mathcal{L}\rvert\lvert\mathcal{P}\rvert^{-1/2}$ | $r_P \lesssim \lvert\mathcal{L}\rvert^{1/2}$ | $r_P \lesssim \lvert\mathcal{L}\rvert^{1/2}\lvert\mathcal{P}\rvert^{-1/2}\delta^{-1/2}$ |
| 1 | ✓ | | ✓ | | | $\lvert\mathcal{P}\rvert \gtrsim \lvert\mathcal{L}\rvert^{1/2} r_L^{3/2}$ | $\lvert\mathcal{P}\rvert \gtrsim r_L^2$ | $\lvert\mathcal{P}\rvert \gtrsim \lvert\mathcal{L}\rvert^{1/2} r_L$ | $\lvert\mathcal{P}\rvert \gtrsim \delta\lvert\mathcal{L}\rvert r_L^2$ |
| ✓ | 1 | | ✓ | | | $\lvert\mathcal{L}\rvert \lesssim \frac{\lvert\mathcal{P}\rvert^2}{r_L^3}$ | | $\lvert\mathcal{L}\rvert \lesssim \frac{\lvert\mathcal{P}\rvert^2}{r_L^2}$ | $\lvert\mathcal{L}\rvert \lesssim \delta^{-1}\frac{\lvert\mathcal{L}\rvert}{r_L^2}$ |
| ✓ | ✓ | | 1 | | | $r_L \lesssim \lvert\mathcal{P}\rvert^{2/3}\lvert\mathcal{L}\rvert^{-1/3}$ | $r_L \lesssim \lvert\mathcal{P}\rvert^{1/2}$ | $r_L \lesssim \lvert\mathcal{P}\rvert\lvert\mathcal{L}\rvert^{-1/2}$ | $r_L \lesssim \lvert\mathcal{P}\rvert^{1/2}\lvert\mathcal{L}\rvert^{-1/2}\delta^{-1/2}$ |
| 1 | | ✓ | ✓ | | | $\lvert\mathcal{P}\rvert \gtrsim r_P r_L^2$ | $\lvert\mathcal{P}\rvert \gtrsim r_L^2$ | $\lvert\mathcal{P}\rvert \gtrsim r_P r_L$ | $r_P r_L \lesssim \delta^{-1}$ |
| ✓ | | 1 | ✓ | | | $r_P \lesssim \frac{\lvert\mathcal{P}\rvert}{r_L^2}$ | | $r_P \lesssim \frac{\lvert\mathcal{P}\rvert}{r_L}$ | $r_P \lesssim \frac{\delta^{-1}}{r_L}$ |
| ✓ | | ✓ | 1 | | | $r_L \lesssim \left(\frac{\lvert\mathcal{P}\rvert}{r_P}\right)^{1/2}$ | $r_L \lesssim \lvert\mathcal{P}\rvert^{1/2}$ | $r_L \lesssim \frac{\lvert\mathcal{P}\rvert}{r_P}$ | $r_L \lesssim \frac{\delta^{-1}}{r_P}$ |
| | 1 | ✓ | ✓ | | | $\lvert\mathcal{L}\rvert \gtrsim r_L r_P^2$ | $\lvert\mathcal{L}\rvert \gtrsim r_P r_L$ | $\lvert\mathcal{L}\rvert \gtrsim r_P^2$ | $r_P r_L \lesssim \delta^{-1}$ |
| | ✓ | 1 | ✓ | | | $r_P \lesssim \left(\frac{\lvert\mathcal{L}\rvert}{r_L}\right)^{1/2}$ | $r_P \lesssim \frac{\lvert\mathcal{L}\rvert}{r_L}$ | $r_P \lesssim \lvert\mathcal{L}\rvert^{1/2}$ | $r_P \lesssim \frac{\delta^{-1}}{r_L}$ |
| | ✓ | ✓ | 1 | | | $r_L \lesssim \frac{\lvert\mathcal{L}\rvert}{r_P^2}$ | $r_L \lesssim \frac{\lvert\mathcal{L}\rvert}{r_P}$ | | $r_L \lesssim \frac{\delta^{-1}}{r_P}$ |
| ✓ | ✓ | | | 1 | | $\mathcal{I} \lesssim \lvert\mathcal{P}\rvert^{2/3}\lvert\mathcal{L}\rvert^{2/3}$ | $\mathcal{I} \lesssim \lvert\mathcal{L}\rvert\lvert\mathcal{P}\rvert^{1/2}$ | $\mathcal{I} \lesssim \lvert\mathcal{P}\rvert\lvert\mathcal{L}\rvert^{1/2}$ | $\mathcal{I} \lesssim \delta^{-1/2}\lvert\mathcal{P}\rvert^{1/2}\lvert\mathcal{L}\rvert^{1/2}$ |
| 1 | | ✓ | | | ✓ | $\lvert\mathcal{P}\rvert \lesssim \frac{\lvert\pi_\theta(P)\rvert^2}{r_P}$ | $\lvert\mathcal{P}\rvert \lesssim \lvert\pi_\theta(P)\rvert^2$ | | $\lvert\mathcal{P}\rvert \lesssim \delta^{-1}\frac{\lvert\pi_\theta(P)\rvert}{r_P}$ |
| ✓ | | 1 | | | ✓ | $r_P \lesssim \frac{\lvert\pi_\theta(P)\rvert^2}{\lvert\mathcal{P}\rvert}$ | | $r_P \lesssim \lvert\pi_\theta(P)\rvert$ | $r_P \lesssim \delta^{-1}\frac{\lvert\pi_\theta(P)\rvert}{\lvert\mathcal{P}\rvert}$ |
| ✓ | | ✓ | | | 1 | $\lvert\pi_\theta(P)\rvert \gtrsim \lvert\mathcal{P}\rvert^{1/2} r_P^{1/2}$ | $\lvert\pi_\theta(P)\rvert \gtrsim \lvert\mathcal{P}\rvert^{1/2}$ | $\lvert\pi_\theta(P)\rvert \gtrsim r_P$ | $\lvert\pi_\theta(P)\rvert \gtrsim \delta\lvert\mathcal{P}\rvert r_P$ |

TABLE 1. Incidence bounds in different forms