THEORY OF PROBABILITY: HOMEWORK 7 SOLUTIONS

VLADIMIR KOBZAR

The following problems are from Ross, Chapter 3. Ross, Chapter 3, Problems 70 and Theoretical Exercises 1 and 8.

Problem (13). cf. Ross, Example 3.2g.

\[ P(E_1) = \frac{\binom{4}{1} \binom{48}{12}}{\binom{52}{13}} \]
\[ P(E_2|E_1) = \frac{\binom{3}{1} \binom{36}{12}}{\binom{39}{13}} \]
\[ P(E_2|E_1 \cap E_2) = \frac{\binom{2}{1} \binom{24}{12}}{\binom{26}{13}} \]
\[ P(E_3|E_1 \cap E_2 \cap E_3) = 1 \]
\[ P(E_1 \cap E_2 \cap E_3 \cap E_4) = \frac{\binom{4}{1} \binom{48}{12}}{\binom{52}{13}} \cdot \frac{\binom{3}{1} \binom{36}{12}}{\binom{39}{13}} \cdot \frac{\binom{2}{1} \binom{24}{12}}{\binom{26}{13}} \approx .1055 \]

Problem (30). Let 1 be the event that the box containing 1 black and 1 white marble is chosen, 2 be the event that the box containing 2 black and 1 white marble is chosen, W be the event that a white ball is chosen from a randomly selected box and B be the event that a black ball is chosen from a randomly selected box. By the Bayes’s Formula

\[ P(1) = 1/2; P(B) = P(B|1)P(1) + P(B|2)P(2) \]
\[ = 1/2 \cdot 1/2 + 2/3 \cdot 1/2 = 7/12; \]
\[ P(W) = 1 - P(B) = 5/12; P(1|W) = \frac{P(W|1)P(1)}{P(W)} = \frac{1/2 \cdot 1/2}{5/12} = 3/5; \]

Problem (49). (a) Let \( D \) = (a person tested has the disease) and \( E \) = (test is positive).

Date: July 22, 2016.
By Bayes’s, if the physician is 70 percent certain that a male has prostate cancer,

\[
P(D|E) = \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)}
\]

\[
= \frac{(.268)(.7)}{(.268)(.7) + (.135)(.3)} \approx .8224
\]

If the physician is 30 percent certain that a male has prostate cancer,

\[
P(D|E) = \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)}
\]

\[
= \frac{(.268)(.3)}{(.268)(.3) + (.135)(.7)} \approx .4597
\]

(b) If the physician is 70 percent certain that a male has prostate cancer,

\[
P(D|E^c) = \frac{P(E^c|D)P(D)}{P(E^c|D)P(D) + P(E^c|D^c)P(D^c)}
\]

\[
= \frac{(.732)(.7)}{(.732)(.7) + (.865)(.3)} \approx .6638
\]

If the physician is 30 percent certain that a male has prostate cancer,

\[
P(D|E^c) = \frac{P(E^c|D)P(D)}{P(E^c|D)P(D) + P(E^c|D^c)P(D^c)}
\]

\[
= \frac{(.732)(.3)}{(.732)(.3) + (.865)(.7)} \approx .2662
\]

**Problem** (56). Let \( E \) be the event that the \( n \)th coupon is new. \( F_i \) be the event that the \( n \)th coupon is a type \( i \) coupon (other coupons can be of any type). Note that \( E|F_i \) occurs if and only if all \( 1, \ldots, n-1 \) are of type other than \( i \). Therefore \( P(E|F_i) = (1 - p_i)^{n-1} \) Since \( F_1, \ldots, F_m \) partition the space all possible collections of \( n \) coupons, by the law of total probability, we have

\[
P(E) = \sum_{i=1}^{m} P(E|F_i)P(F_i) = \sum_{i=1}^{m} (1 - p_i)^{n-1} p_i
\]

**Problem** (70). Let \( Q \) be the event that the queen has hemophilia, \( H_i \) be the event that the \( i \)th prince has hemophilia \( (1 \leq i \leq 4) \). Note we need to assume a medical fact that if the queen is not a carrier than a prince will not have hemophilia.

Let \( Q' \) be the event that the queen is a carrier conditioned on her having one son without hemophilia, \( Q_i \) is the same for \( i \) sons.
\[ P(Q_1) = P(Q|H_1^c) = \frac{P(H_1^c|Q)P(Q)}{P(H_1^c|Q)P(Q) + P(H_1^c|Q^c)P(Q^c)} \]
\[ = \frac{(0.5)(0.5)}{(0.5)(0.5) + 1(0.5)} = 1/3 \]

\[ P(Q_2) = P(Q_1|H_2^c) = \frac{P(H_2^c|Q_1)P(Q_1)}{P(H_2^c|Q_1)P(Q_1) + P(H_2^c|Q_1^c)P(Q_1^c)} \]
\[ = \frac{(0.5)(1/3)}{(0.5)(1/3) + 1(2/3)} = 1/5 \]

\[ P(Q_3) = P(Q_2|H_3^c) = \frac{P(H_3^c|Q_2)P(Q_2)}{P(H_3^c|Q_2)P(Q_2) + P(H_3^c|Q_2^c)P(Q_2^c)} \]
\[ = \frac{(0.5)(1/5)}{(0.5)(1/5) + 1(4/5)} = 1/9 \]

\[ P(H_4) = 1/2 \cdot 1/9 \]

**Exercise** (Theoretical 1). The left-hand side of the inequality can be expressed

\[ P(AB|A) = \frac{P(AB)}{P(A)} \]

Similarly, for the right-hand side

\[ P(AB|A \cup B) = \frac{P(AB(A \cup B))}{P(A \cup B)} = \frac{P(AB)}{P(A \cup B)} \]

Therefore, to prove the desired inequality, it suffices to prove \( P(A \cup B) \geq P(A) \), and that result follows from the monotonicity of \( P \).

**Exercise** (Theoretical 8). (a) Multiply both sides of each inequality by \( P(C) \) or \( P(C^c) \), and add them up.

\[ P(A|C)P(C) > P(B|C)P(C); \]
\[ P(A|C^c)P(C^c) > P(B|C^c)P(C^c); \]
\[ P(A|C)P(C) + P(A|C^c)P(C^c) = P(A) \]
\[ > P(B) = P(B|C)P(C) + P(B|C^c)P(C^c) \]

(b) Per hint, let \( A \) and \( B \) be the events that the first and the second die respectively land on 6. And \( C \) be the event that the sum of the
value of two dies is equal to 10. Then $C = \{(6, 4), (5, 5), (4, 6)\}$, and we see that

\[
\begin{align*}
P(A|C) &= 1/3 = 11/33 > P(A|C^c) = 5/33; \\
P(B|C) &= 11/33 > P(B|C^c) = 5/33; \\
P(A \cap B|C) &= 0 < (A \cap B|C^c) = 1/33;
\end{align*}
\]

References