Filtering with Lagrangian Floaters in Stochastic Navier Stokes Flow

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Stochastic Navier Stokes Equations (SNS):

- Incompressible periodic velocity field $u_t \in H^1(\mathbb{T}^2)$:

$$
du_t = (\nu \Delta u_t + (u \cdot \nabla)u + \nabla p) dt + dW_t;
$$

$$
\nabla \cdot u_t = 0.
$$

- $W_t$ stands for some divergence free Gaussian process.

- $r$ floaters are dispatched into $u_t$. Their locations $x_t^i$ follow the following equation:

$$
x_t^i = x_0^i + \int_0^t u_s(x_s^i) ds, \quad i = 1, \ldots, r.
$$
Oceanography

- Data that Oceanographers receive:
  \[ Y^i_n = x^i_{t_n} + \xi^i_n; \]
- Oceanographers want to recover \( u_t \) based on \( Y^i_n \).
This model is a Hidden Markov Chain:

- $Z_n = (u_{tn}, x_{tn}^1, \ldots, x_{tn}^r)$ is a Markov Chain;
- $Y_n^i$ is a perturbed partial observation of $Z_n$.
- Want to recover $\Pi_n = P(Z_n \in \cdot | Y_0, \ldots, Y_n)$.

\[
\cdots \rightarrow Z_{n-1} \rightarrow Z_n \rightarrow Z_{n+1} \rightarrow \cdots
\]

\[
\cdots \quad Y_{n-1} \quad Y_n \quad Y_{n+1} \quad \cdots
\]
The evolution of $\Pi_n = \mathbf{P}(Z_n \in \cdot | Y_0, \ldots, Y_n)$:

- Denote joint distribution of $Z_0$ as $\mu$, let $\Pi_0^\mu(\cdot) = \mu(Z_0 \in \cdot | Y_0)$;
- Compute $\Pi_n^\mu = \mathbf{P}^\mu(Z_n \in \cdot | Y_0, \ldots, Y_n)$ using the recursive Bayes’ formula:

$$
\Pi_{n+1}^\mu(A) = \frac{\int_A \Pi_n^\mu(dz_n)p(z_n, dz_{n+1})g(z_{n+1}, Y_{n+1})}{\int \Pi_n^\mu(dz_n)p(z_n, dz_{n+1})g(z_{n+1}, Y_{n+1})}.
$$

We assume the transition kernel from $Z_n$ to $Z_{n+1}$ is $p(z_n, dz_{n+1})$; and the transition kernel from $Z_n$ to $Y_n$ has density $g(Z_n, Y_n)$. This process is called nonlinear filtering or optimal filtering.
Nonlinear Filtering

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The selection of the prior distribution $\mu$ appears to be a problem,
- We need to know the distribution of $u_0$, the velocity field at time 0, which is implausible;
- We can plug in reasonable approximation;
- How does the selection of $\mu$ impact the result?
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Filter Stability

- Desired: nonlinear filtering is self correcting:

  \[ |P^\mu(Z_n \in \cdot | Y_0, \ldots, Y_n) - P^\nu(Z_n \in \cdot | Y_0, \ldots, Y_n)| \xrightarrow{n \to \infty} 0 \text{ a.s.} \]

- Intuition: when there is no observation:

  \[ \| P^\mu(Z_n \in \cdot) - P^\nu(Z_n \in \cdot) \|_{TV} \xrightarrow{n \to \infty} 0. \]

- Intuition: true if the unconditioned chain has ergodic behavior.
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Conditional Ergodicity with Asymptotic Coupling:

Asymptotic coupling:

- The space of $Z_n$ has a metric $d$;
- Assume for any two initial condition $z_0 = (u_0, x_0^1, \ldots, x_0^r)$, $	ilde{z}_0 = (\tilde{u}_0, \tilde{x}_0^1, \ldots, \tilde{x}_0^r)$, there is a coupling $Q$ such

$$Q\left(\sum_n d(Z_n, \tilde{Z}_n)^2 < \infty \right) > a > 0.$$ 

Theorem (van Handel and Tong, 2012)

*If a Hinden Markov Chain has asymptotic coupling, assume also the observation density $g > 0$, then for any $d$-Lip function $f$,

$$|\Pi_n^\mu(f) - \Pi_n^\pi(f)| \to 0, \quad \pi\text{-a.s.}$$*
Coupling

Coupling is useful to show memoryless of initial condition. e.g.,

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Asymptotic Coupling

- For infinite dimensional systems, e.g. SNS, exact coupling is impossible;
- A weaker version, asymptotic coupling, can be achieved:
Heuristic behind Asymptotic Coupling

- At each time $t$, with prob $1 - q(z_t)$ the distance decreases to its half;
- The prob that you can repeat this coupling forever is about

$$[1 - q(z_0)][1 - q(\frac{1}{2}z_0)] \cdots \sim O(e^{-q'(0)})$$

- It will be strictly positive if we can verify the differentiability of the Kolmogorov transition density: $\nabla_z P(t, z)$. 
Known Facts

- The case where the observations are done at fixed points has asymptotic coupling. (M. Hairer, J.C. Mattingly and M. Scheutzow. 2009)
- We have verified the same mechanic works for single floater case;
- Multiple floater remain unknown.
Interesting Consequence of Asymptotic Coupling

- Multiple floaters will separate no matter how close they are in the beginning;
- Fracture behavior of the flow map.
Thank You for Your Attention!