MATH-GA 2150.001: Homework 2

1. Let \( k \) be an algebraically closed field. For each of the following plane curves over \( k \) write down three open affine charts and determine the intersection with the three coordinate lines (\( X = 0, Y = 0 \) or \( Z = 0 \)).

   (a) \( Y^2Z = X^3 + aXZ^2 + bZ^3 \);
   (b) \( X^2Y^2 + X^2Z^2 + Y^2Z^2 = 2XYZ(X + Y + Z) \);
   (c) \( XZ^3 = (X^2 + Z^2)Y^2 \).

2. (a) Let \( k \) be a field. Let \( P = (0 : 0 : \ldots : 0 : 1) \in \mathbb{P}_k^n \). Show that the set of lines \( \mathcal{L}_P \) in \( \mathbb{P}_k^n \) passing by \( P \) could be identified with a projective space \( \mathbb{P}_{n-1}^k \).

   (b) Let \( X \subset \mathbb{P}_k^n \) be a quadric: \( X \) is a projective variety defined by a homogeneous form \( q(x_0, \ldots, x_n) \) of degree 2. Assume that \( X \) passes by \( P \) and at least one of the derivatives \( \frac{\partial q}{\partial x_i}(P) \) is not zero (\( X \) is smooth at \( P \)).

   Let \( T_P \) be a hyperplane given by the equation \( \sum_{i=0}^n \frac{\partial q}{\partial x_i}(P)x_i = 0 \) (the tangent hyperplane to \( X \) at \( P \)).

   i. Show that the set of lines in \( \mathcal{L}_P \), that are not contained in \( T_P \), is a nonempty open \( U_P \subset \mathbb{P}_{n-1}^k \).

   ii. Show that a line \( L \in U_P \) intersects \( X \) in exactly two distinct points: \( P \) and a second point, that we call \( P_L \).

   iii. Deduce that the projection \( U \to X, L \mapsto P_L \) is bijective on its image.

3. Let \( k \) be an algebraically closed field.

   (a) Show that the set of lines in \( \mathbb{P}_k^2 \) form a projective space.

   (b) Let \( d \geq 2 \) be an integer. Consider the set of maps \( f : \mathbb{P}_k^1 \to \mathbb{P}_k^2 \) of degree \( d \). Recall that such a map is given by \( (x : y) \mapsto (f_0(x, y) : f_1(x, y) : f_2(x, y)) \) where \( f_0, f_1, f_2 \in k[x, y] \) are homogeneous polynomials of degree \( d \) without a common factor.

   i. Show that the vector of coefficients of \( f_0, f_1 \) and \( f_2 \) gives a point in a projective space \( \mathbb{P}_k^N \), write explicitly \( N \) in terms of \( d \).

   ii. Show that the ideal \( I = (f_0, f_1, f_2) \) of \( k[x, y] \) contains some power of the maximal ideal \( (x, y) \).

   iii. For \( m \geq 0 \) denote \( k[x, y]_m \) the set of homogeneous polynomials of degree \( m \) in \( k[x, y] \). Show that \( k[x, y]_m \) is a \( k \)-vector space and determine its dimension.

   iv. Consider a map

   \[
   S_m : (k[x, y]_m)^3 \to k[x, y]_{m+2}, (g_0, g_1, g_2) \mapsto \sum_{i=0}^2 f_ig_i.
   \]
Show that $S_m$ is a linear map and that if $S_m$ is not surjective then all $(m + d + 1)$-minors of some matrix, whose entries are linear combinations of the coefficients of $f_0$, $f_1$ and $f_2$, vanish.

v. Show that for some $m$ the map $S_m$ is surjective.

vi. Deduce that the set maps $f : \mathbb{P}^1_k \to \mathbb{P}^2_k$ of degree $d$ corresponds to a Zariski open in the projective space $\mathbb{P}^N_k$ corresponding to the coefficients of $f$. 