Problems

March 25, 2014

Exercise. Find the $2 \times 2$ matrix that gives the orthogonal projection onto the line $y = 2x$ in the plane.

Exercise. Prove that for a matrix $A$ with eigenvalues $\lambda_i$, that $\text{Tr}(A^k) = \sum_i \lambda_i^k$. Does this still hold if $A$ is not diagonalizable?

Exercise. Let $f(t) = \det (A + tB)$. What is $f'(0)$? (Hint: first do the case $B = I$, then the case $B$ is invertible)

Problem. (#4 Jan 04) Prove that the non-zero eigenvalues of $AB$ and $BA$ are the same, where $A$ and $B$ are rectangular matrices of size $n \times m$ and $m \times n$ respectively. Express the corresponding eigenvalues of $AB$ in terms of those of $BA$.

Problem. (#5 Sept 05) (Note the wording on this is hard to read!!) Let $V$ be a real finite dimensional vector space with inner product. A set of $k$ vectors $(y_1, \ldots, y_k) \in V$ defines the linear operator $A : V \to V$ by $Ax = \sum_{j=1}^k \langle y_j, x \rangle y_j$. Define the $k \times k$ matrix, $M$, with entries $M_{ij} = \langle y_i, y_j \rangle$. The norm of $A$ is the operator norm coming from the vector norm $\|x\|^2 = \langle x, x \rangle$. The norm of $M$ is the matrix norm coming from the vector norm of $\mathbb{R}^k$ given by $\|z\|^2 = \sum_{j=1}^k z_j^2$. Show that $\|A\| = \|M\|$.

Problem. (#2 Jan 05) Let $C$ be an $n \times n$ matrix. Prove that $\text{Tr}(C) = 0$ if and only if $C$ is similar to a matrix $D$ with all diagonal elements equal to 0. (Hint: Make one zero at a time on the diagonal). Show that this condition is both necessary and sufficient for $C$ to be equal to $AB - BA$ for some $n \times n$ matrices $A, B$.

Problem. (#1 Sept 02) Let $A$ be the $n \times n$ matrix:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$


b) Give a relation between its nonzero eigenvalues.

c) Compute $A^2$ and deduce another relation and finally the values of its eigenvalues.

d) Give the minimal polynomial of $A$. 