RAYLEIGH QUOTIENT AND THE MIN-MAX THEOREM

**Definition 1.** A Matrix is called Hermitian if $A^* = A$. Notice that if $A = A^*$ then $A^* A = A A^*$ so by the diagonalization theorem for normal matrices, $A$ is unitarily diagonalizable, i.e., it has an orthonormal basis of eigenvectors.

**Lemma 2.** Let $A$ be Hermitian and suppose its Eigenvalues are $\lambda_1 \leq \ldots \leq \lambda_n$ with corresponding eigenvectors $v_1, \ldots, v_n$. If $S_k$ is a $k$-dimensional subspace then:

$$\exists x \in S_k \text{ s.t. } \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \geq \lambda_k$$

**Proof.** By a dimension counting argument, $S_k \cap \text{span} \{v_k, \ldots, v_n\} \neq \emptyset$ so there exists $x \in S_k$ with $x \in \text{span} \{v_k, \ldots, v_n\}$. For this $x$ it is easy to verify $\frac{\langle Ax, x \rangle}{\langle x, x \rangle} \geq \lambda_k$. $\square$

**Theorem 3.** *(The Min-Max Theorem)* Let $A$ be Hermitian and suppose its Eigenvalues are $\lambda_1 \leq \ldots \leq \lambda_n$:

$$\min_{\dim S_k = k} \max_{x \in S_k} \frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \lambda_k$$

**Proof.** By the above lemma, the LHS is $\geq \lambda_k$. Choosing $S_k = \text{span} \{v_k, \ldots, v_n\}$ gives LHS $\leq \lambda_k$. $\square$

**Definition 4.** A Hermitian matrix is called positive definite if all of its eigenvalues are $> 0$. Equivalently, if $\langle Ax, x \rangle > 0$ for all non-zero $x$.

**Theorem 5.** Say $M, L$ are Hermitian with eigenvalues $\mu_1 \leq \ldots \leq \mu_n$ and $\lambda_1 \leq \ldots \leq \lambda_n$ respectively. If $L - M$ is positive definite (i.e., $L > M$) then $\lambda_k > \mu_k$ for each $k$.

**Proof.** (Using Rayleigh Quotient) Write:

$$\min_{\dim S_k = k} \max_{x \in S_k} \frac{\langle Lx, x \rangle}{\langle x, x \rangle} = \lambda_k$$

$$\min_{\dim S_k = k} \max_{x \in S_k} \frac{\langle Mx, x \rangle}{\langle x, x \rangle} = \mu_k$$

Let $T$ be the $k$-dimensional subspace that achieves the minimum for the $\lambda_k$ above and let $x_0 \in T$ be the vector that achieves $\max_{x \in S_k} \frac{\langle Mx, x \rangle}{\langle x, x \rangle}$. Have then (using $\langle Lx, x \rangle > \langle Mx, x \rangle$ for every $x$):

$$\lambda_k = \min_{\dim S_k = k} \max_{x \in S_k} \frac{\langle Lx, x \rangle}{\langle x, x \rangle}$$

$$= \max_{x \in T} \frac{\langle Lx, x \rangle}{\langle x, x \rangle}$$

$$> \max_{x \in T} \frac{\langle Mx, x \rangle}{\langle x, x \rangle}$$

$$= \frac{\langle Mx_0, x_0 \rangle}{\langle x_0, x_0 \rangle}$$

On the other hand:

$$\mu_k = \min_{\dim S_k = k} \max_{x \in S_k} \frac{\langle Mx, x \rangle}{\langle x, x \rangle}$$

$$\leq \max_{x \in T} \frac{\langle Mx, x \rangle}{\langle x, x \rangle}$$

$$= \frac{\langle Mx_0, x_0 \rangle}{\langle x_0, x_0 \rangle}$$

So finally we have $\lambda_k \geq \frac{\langle Mx_0, x_0 \rangle}{\langle x_0, x_0 \rangle} \geq \mu_k$. $\square$