The Fibonacci Sequence

A sequence is a list of numbers that never ends (e.g. $1, 3, 5, 7, 9, 13, \ldots$) The Fibonacci Sequence is an exciting sequence of numbers we will talk about today. The first few numbers are:

<table>
<thead>
<tr>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
<th>$f_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
</tr>
</tbody>
</table>

The rule that makes the Fibonacci Sequence is “the next number is the sum of the previous two”. This kind of rule is sometimes called a recurrence relation. Mathematically, this is written as:

$$f_n = f_{n-1} + f_{n-2}$$

One strange fact about Fibonacci numbers is that they can be used to convert kilometers to miles:

$$3 \text{ mi} \approx 5 \text{ km}$$
$$5 \text{ mi} \approx 8 \text{ km}$$
$$8 \text{ mi} \approx 13 \text{ km}$$
$$13 \text{ mi} \approx 21 \text{ km}$$
$$21 \text{ mi} \approx 34 \text{ km}$$

We will explain how this works by using a really powerful idea called “generating functions” which let us attack these problems. Generating functions involve using algebra to solve infinite sums. Before we jump into Fibonacci, we will start with some warm up problems to get the hang of it.

**Exercise 0:** $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = ?$

**Exercise 1:** $1 + x + x^2 + x^3 + x^4 + \ldots = ?$

*TECHNICAL NOTE: Technically this sum only makes sense when $-1 < x < 1$, but don’t worry about that!*

**Exercise 2:** $1 + 2x + 4x^2 + 8x^3 + 16x^4 + \ldots = ?$

**Exercise 3:** $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = ?$
Generating Functions

For any sequence of numbers, there is a *generating function* associated with that sequence. (By a function, I mean an expression that depends on \( x \).) The rule for the generating function is to multiply each term of the sequence by \( x^n \), and finally do the infinite sum of all these terms. This is written mathematically as:

**Definition:** The generating function for the sequence \( a_0, a_1, a_2, a_3, \ldots \) is \( S = a_0 \cdot x^0 + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \ldots \)

Here are some examples. We already did these as exercises before!

**Example 1:** The generating function for the sequence 1,1,1,1,1,\ldots is \( S = \frac{1}{1-x} \).

We must evaluate the infinite sum \( S = 1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + \ldots \). We use the "shifting trick":

\[
S = 1 + x + x^2 + x^3 + \ldots \quad \text{and} \quad xS = x + x^2 + x^3 + \ldots
\]

\[
\therefore (1-x)S = 1 + 0 + 0 + 0 + \ldots
\]

\[
\therefore S = \frac{1}{1-x}
\]

**Example 2:** The generating function for the sequence 1,2,4,8,16\ldots is \( S = \frac{1}{1-2x} \).

We must evaluate the infinite sum \( S = 1 \cdot x^0 + 2 \cdot x^1 + 4 \cdot x^2 + 8 \cdot x^3 + \ldots \)

Notice that we can rewrite each term as a power of 2\( x \), namely \( S = 1 \cdot x^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots \)

Now we can use the same shifting trick as before (Alternatively, you can "plug in 2\( x \) into Example 1):

\[
S = 1 + (2x) + (2x)^2 + \ldots
\]

\[
- 2xS = 2x + (2x)^2 + \ldots
\]

\[
\therefore (1 - 2x)S = 1 + 0 + 0 + 0 + \ldots
\]

\[
\therefore S = \frac{1}{1-2x}
\]

**Example 3:** The generating function for the sequence 1,2,3,4,5,6\ldots is \( S = \frac{1}{(1-x)^2} \).

We must evaluate the infinite sum \( S = 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + \ldots \). We can use the same shifting trick to reduce this to the sum we already computed in example 1:

\[
S = 1 + 2x + 3x^2 + 4x^3 + \ldots
\]

\[
- xS = x + 2x^2 + 3x^3 + \ldots
\]

\[
\therefore (1-x)S = 1 + x + x^2 + x^3 + \ldots
\]

\[
= \frac{1}{1-x}
\]

\[
\therefore S = \frac{1}{(1-x)^2}
\]

**Fact 1:** The generating function for the Fibonacci sequence 0,1,1,2,3,5,8\ldots is \( S = \frac{x}{1-x-x^2} \).

We must evaluate the infinite sum \( S = 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 + \ldots \). Since the recurrence relation for the Fibonacci numbers involves the last two numbers, we must use a "double" shifting trick:

\[
S = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \ldots
\]

\[
- xS = 0 + x^2 + x^3 + 2x^4 + 3x^5 + \ldots
\]

\[
- x^2S = 0 + x^3 + x^4 + 2x^5 + \ldots
\]

\[
\therefore (1 - x - x^2)S = 0 + x + 0 + 0 + 0 + \ldots
\]

\[
\therefore S = \frac{x}{(1-x-x^2)}
\]

The reason all the 0’s appear is from the recurrence relation for the Fibonacci numbers; in other words the fact that the next Fibonacci number is the sum of the previous two.
The Golden Ratio

Define two numbers \( \varphi \) and \( \beta \) to be the roots of the quadratic equation \( x^2 - x - 1 \). (This quadratic equation appeared "in reverse" in the denominator for the generating function of the Fibonacci numbers).

By the quadratic equation, these are:

\[
\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887 \ldots
\]

\[
\beta = \frac{1 - \sqrt{5}}{2} \approx -0.6180339887 \ldots
\]

The number \( \varphi \) is called the Golden Ratio and has a number of exciting properties (go see the Wikipedia page for more info!). Both \( \varphi \) and \( \beta \) are intimately related to the Fibonacci sequence because they appear in the generating function!

**Fact 2:** \( 1 - x - x^2 = (1 - \varphi x)(1 - \beta x) \)

You can check this fact by expanding it out using "FOIL" and the fact that \( \varphi \beta = -1 \) and \( \varphi + \beta = 1 \). Try it here!

**Fact 3:** Let \( f_n \) be the \( n \)-th Fibonacci number. Then we have the explicit formula:

\[
f_n = \frac{1}{\sqrt{5}} (\varphi^n - \beta^n)
\]

The trick we use to get this result is to rewrite the generating function for the Fibonacci numbers \( S = \frac{x}{1 - x - x^2} \) we found in Fact 1 using the factorization \( 1 - x - x^2 = (1 - \varphi x)(1 - \beta x) \) we found in Fact 2. By some algebraic manipulations, (This is called the partial fractions trick!) we find that:

\[
\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \frac{1}{1 - \varphi x} + \frac{1}{\sqrt{5}} \frac{1}{1 - \beta x}
\]

Now we recognize that \( \frac{1}{1 - \varphi x} \) and \( \frac{1}{1 - \beta x} \) look a lot like the answer from Example 2. These are infinite sums!

\[
\frac{1}{1 - \beta x} = 1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots
\]

\[
\frac{1}{1 - \varphi x} = 1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots
\]

Putting this all together gives:

\[
f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \ldots
= \frac{x}{1 - x - x^2}
= \frac{1}{\sqrt{5}} \frac{1}{1 - \varphi x} - \frac{1}{\sqrt{5}} \frac{1}{1 - \beta x}
= \frac{1}{\sqrt{5}} (1 + \varphi x + \varphi^2 x^2 + \varphi^3 x^3 + \ldots) - \frac{1}{\sqrt{5}} (1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \ldots)
= 0 + \frac{1}{\sqrt{5}} (\varphi - \beta) x + \frac{1}{\sqrt{5}} (\varphi^2 - \beta^2) x + \frac{1}{\sqrt{5}} (\varphi^3 - \beta^3) x^3 + \ldots
\]

Since these two "polynomials" are equal for every value of \( x \), each term must individual by equal. Reading off the coefficients of the power \( x^n \) gives the final result,

\[
f_n = \frac{1}{\sqrt{5}} (\varphi^n - \beta^n)
\]
Converting Miles to Kilometers

**Factoid 1:** Fibonacci numbers can be used to convert miles to kilometers by: \( f_n \text{ km} \approx f_{n-1} \text{ mi} \)

The secret of this factoid is an amazing coincidence between the numerical value of \( \varphi \) and the number of kilometers in a mile, and the fact that \( |\beta| < 1 \). Firstly, notice that:

\[
\varphi = 1.6180...
\]

\[
\frac{1 \text{ mi}}{1 \text{ km}} = 1.6093...
\]

Because these two values are close, the approximation \( 1 \text{ mi} \approx \varphi \text{ km} \) is pretty good (to about 1%). Now notice that since \( \beta < 1 \), that \( \beta^n \) is really small as \( n \) gets larger; \( \beta^n \approx 0 \). So we have some more approximations:

\[
f_n = \frac{1}{\sqrt{5}} (\varphi^n - \beta^n) \approx \frac{1}{\sqrt{5}} \varphi^n
\]

\[
f_{n-1} = \frac{1}{\sqrt{5}} (\varphi^{n-1} - \beta^{n-1}) \approx \frac{1}{\sqrt{5}} \varphi^{n-1}
\]

\[
\therefore f_n \approx \varphi f_{n-1}
\]

Along with \( 1 \text{ mi} \approx \varphi \text{ km} \), this means that \( f_{n-1} \text{ km} \approx f_n \text{ mi} \). This works best if \( n \) is not-too-small, because when \( n \) is large, our approximation that \( \beta^n \approx 0 \) becomes more accurate. \( n = 5 \) is already quite a good approximation (\( \beta^4 \approx 0.0002 \)). The first couple listed for you, starting at \( n = 5 \), so that you can travel to Canada without fear!

\[
3 \text{ mi} \approx 5 \text{ km}
\]

\[
5 \text{ mi} \approx 8 \text{ km}
\]

\[
8 \text{ mi} \approx 13 \text{ km}
\]

If you want to covert numbers not on this list, you can bootstrap from the above approximations. For example, starting from \( 5 \text{ mi} \approx 8 \text{ km} \), you can do:

\[
100 \text{ mi} = 20 \cdot 5 \text{ mi} \approx 20 \cdot 8 \text{ km} = 160 \text{ km}
\]

**If you found this interesting...**

Here are some great websites that you can check out where you can get more!

Wikipedia links:

- Fibonacci number
- Golden ratio
- Recurrence relation (advanced)
- Generating function (advanced)

Other links:

- “Exercise - Write a Fibonacci Function” by Salman Khan (has a computer science flavor) https://www.youtube.com/watch?v=Bdbc1ZC-vhw