Problems

April 29, 2014

Problem. (Jan 10 #3) Describe all meromorphic functions \( f(z) \) in the complex plane with a simple pole at \( z = 1 \), a simple zero at \( z = -1 \) and for which:

\[ |f(z)| \leq M |z|, |z| \geq 2 \]

for some \( M > 0 \)

Problem. (Jan 11 #3) The function \( f \) is analytic in the whole plane with positive imaginary part. What can it be? What if all you know is that the imaginary part of \( f \) tends to 0 at \( \infty \)?

Problem. (Jan 07 #4) What is the most general entire function that takes each complex value once and only once in \( \mathbb{C} \)? Give a complete proof for full marks.

Problem. (Sept 01 #4) Suppose that \( f(z) \) is entire and has \( n \) simple zeros at \( z_1, z_2, \ldots, z_n \).

Part 1 Suppose \( |f(z)| \leq k|z|^m + L \) for some \( m \). What is \( m \)?

Part 2 What is the most general such function \( f(z) \)?

Part 3 Suppose it is known that \( |f(z)| \leq A|z|^{3/2} \), what is \( f(z) \)?

Problem. (Jan 03 #4) Let \( f \) be an entire function and \( n \) a positive integer. Show that there is an entire function \( g \) such that \( g^n = f \) if and only if the orders of zeros of \( f \) are divisible by \( n \).

Problem. (Jan 11 #5) The picture shows what the function \( f : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \) does to the plane. The values 0 at 0, 1 at \( \pm 1 \), and \( \infty \) at \( \pm i \) are specified. The signatures +/− indicate that the regions so marked are mapped 1 to 1 onto the upper/lower half plane. What is \( f \)? Explain why it cannot be otherwise.

Problem. (Jan 01 #1) Explain why the function \( \sqrt{1 - z^2} \) can be thought of as single valued in a plane cut along \(-1 \leq z \leq 1\). Then the integral:

\[ I = \int \frac{dz}{\sqrt{1 - z^2}} \]

taken around the circle \( |z| = R > 1 \) makes sense. How is \( I \) related to \( \int_0^1 \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{2} \)? How does \( I \) change as \( R \to \infty \)? Compute \( I \) by “pure thought” in light of these remarks.