CONTOUR INTEGRALS

1. Theory

**Definition 1.** Let $\gamma : [a,b] \to \mathbb{C}$ be a $C^1$ curve and let $f : \mathbb{C} \to \mathbb{C}$. Define:

$$\int_{\gamma} f(z)dz = \int_{a}^{b} f(\gamma(t))\gamma'(t)dt$$

**Remark 2.** This is just like a line integral in $\mathbb{R}$, so we can use tools from there e.g. the fundamental theorem of calculus. One can check using the change of variables formula that if we re-parametrize the curve, then the value of the integral does not change. If you reverse the direction of the curve, you get a multiplicative factor of $-1$.

**Theorem 3.** [Cauchy-Goursat Theorem] If $f(z)$ is holomorphic, and $\gamma$ is the closed curve over a triangle, then $\int_{\gamma} f(z)dz = 0$.

**Proof.** (super rough sketch) Cut triangle into four pieces (midpoints), and integrate along each piece. There is one piece so that $|\int$ piece $| > 1/4|\int$ whole triangle$. Doing this recursively, gives a sequence of triangles $\Delta^k$ with $|\int \Delta^k| > \frac{1}{2^k}|\int$ original triangle$. But since triangle pieces are closed sets, there is a single point in the intersection $\Delta^k \to \{z_0\}$. Use the fact that $f$ is holomorphic to write a Taylor series approx near $z_0$, with error to 0 as $\Delta^k$ as $k \to \infty$. Then plug approx into $\int \Delta^k$ to get an upper bound estimate and conclude $\int$ original $= 0$. □

**Corollary 4.** Holomorphic function have antiderivatives in convex (or starlike) domains.

**Proof.** Define $F(z) = \int_{[z_0,z]} f(\zeta)d\zeta$. Show that $F$ is continuous by the Cauchy Goursat theorem. The check $F$ is differentiable by the Cauchy Riemann Equations (again, by C-G) and the derivative is $f$.

**Corollary 5.** If $f$ holomorphic in a convex (or starlike) domain then $\int_{\gamma} f(z)dz = 0$ for any closed curve $\gamma$.

**Proof.** By last corollary it has an antiderivative, so integral is zero by Fundamental Theorem of Calculus. □

**Corollary 6.** If $\gamma_1$ and $\gamma_2$ are homotopic (i.e. they can be continuously deformed into each other) in a region $\Omega$ where $f$ is holomorphic, then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$

**Proof.** Requires a little bit of work and set up... but not too hard from where we are. □

**Theorem 7.** [Cauchy Integral Formula] Suppose that $f$ is holomorphic in a disk of radius $R$ centered at some point $z_0$. Then for any $a$ in the disk and $r < R$:

$$f(a) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-a}dz$$

**Proof.** By the previous theorem, the integral on the RHS does not depend on $r$. By taking $r \to 0$ we can see that this value has to be $f(a)$. □

**Corollary 8.** Suppose $f$ is holomorphic in $A_{r,R} = \{z : r < z < R\}$. The $f$ has Laurent Series expansion, $f(a) = \sum_{n=-\infty}^{\infty} a^n c_n$ for $a \in A_{r,R}$. Moreover, the coefficients $c_n$ are given explicitly in terms of an integrals of $f(z)$.

**Proof.** Suppose WLOG we are working in $\mathbb{R}$. Draw a picture then use the homotopic curves theorem and the Cauchy Integral Formula to see that for any $a \in A_{r,R}$:

$$f(a) = \frac{1}{2\pi i} \int_{|z-a|=\epsilon} \frac{f(z)}{z-a}dz = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i} \int_{|z|=\epsilon} \frac{f(z)}{z-a}dz$$
Then do some manipulations to force a geometric sum out:

\[
\int_{|z|=R} \frac{f(z)}{z-a} \, dz = \int_{|z|=R} \frac{1}{1 - \frac{a}{z}} f(z) \, dz = \int_{|z|=R} \sum_{n=0}^{\infty} \left( \frac{a}{z} \right)^n f(z) \, dz = \sum_{n=0}^{\infty} a^n \left( \int_{|z|=R} f(z) z^n \, dz \right)
\]

Exchanging the sum with the integral here is ok because the sum is uniformly convergent as \(a \in A_r,R\) has \(|\frac{a}{z}| < 1\) for \(|z| = R\). We do the same thing on the inner circle of radius \(r\), where this time \(|\frac{z}{a}| < 1\) for \(|z| = r\):

\[
- \int_{|z|=r} \frac{f(z)}{z-a} \, dz = \int_{|z|=r} \frac{-1}{\frac{z}{a} - 1} \frac{f(z)}{a} \, dz = \int_{|z|=r} \sum_{n=0}^{\infty} \left( \frac{z}{a} \right)^n f(z) \, dz = \sum_{n=0}^{\infty} a^{-(n+1)} \left( \int_{|z|=r} f(z) z^n \, dz \right)
\]

So finally then:

\[
f(a) = \sum_{n=-\infty}^{\infty} a^n c_n
\]

\[
c_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} \, dz \text{ for } n \geq 0
\]

\[
c_{-n} = \frac{1}{2\pi i} \int_{|z|=r} f(z) z^{-n-1} \, dz \text{ for } n \geq 1
\]

\[\square\]

**Corollary 9.** In the above setting, \(\int_{\gamma} f(z) \, dz = 2\pi i c_{-1}\) is the coefficient of the exponent \(-1\) in the Laurent series expansion of \(f\).

**Definition 10.** The coefficient \(c_{-1}\) above is called the Residue of \(f\) at \(z = 0\). This is because the annulus \(A_{r,R}\) was centered at \(z = 0\). In general \(Res_{z=z_0} f := c_{-1}\) where \(c_{-1}\) is the Laurent series expansion for \(f\) in an annulus centered at \(z_0\). There are lots of different notations for the residue of \(f\).

**Theorem 11.** [Residue Theorem] \(\int_{\gamma} f(z) \, dz = 2\pi i \sum_{z_k} Wind_{z}(z_k) \cdot Res_{z=z_k} f\)

**Proof.** This is not very far from the corollary we just saw! \[\square\]

## 2. Computing Residues

**Method 1** Do some algebraic manipulations to get the Laurent series expansion of \(f\) and look at \(c_{-1}\). Usually, you do this by combining some power series you already know e.g. \((1+z)^a = 1 + az + \ldots\) or \(\sin(z) = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \ldots\)
Example: Show $\text{Res}_{z=0} \frac{1}{z - \sin(z)} = \frac{3!}{5 \cdot 4}$

\[
\frac{1}{z - \sin(z)} = \frac{1}{z} \left( 1 - \sin \frac{z}{z} \right)^{-1} = \frac{1}{z} \left( 1 - 1 + \frac{1}{3!} z^2 - \frac{1}{5!} z^4 \ldots \right)^{-1} = \frac{1}{z} \frac{1}{\left( 1 - \frac{1}{5 \cdot 4} z^2 \right)^{-1}} = \frac{3!}{z^3} \left( 1 + \frac{1}{5 \cdot 4} z^2 + \ldots \right) = \frac{3!}{z^3} + \frac{3!}{5 \cdot 4} \frac{1}{z} + \ldots
\]

Method 2: If you know the order of the singularity and it is finite, then you know that $f(z) = c_{-k} (z - z_0)^{-k} + \ldots$, so $(z - z_0)^k f(z) = c_{-k} + \ldots$ is an ordinary power series. The coefficient $c_{-1}$ is the coefficient of $(z - z_0)^{k-1}$ in $f(z)$, so it's not hard to see that:

\[
\text{Res}_{z=z_0} f = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[ (z - z_0)^k f(z) \right]
\]

For a simple pole, this is:

\[
\text{Res}_{z=0} f = \lim_{z \to 0} zf(z)
\]

If $f(z) = \frac{g(z)}{h(z)}$ and $h(0) = 0, h'(0) \neq 0$ then:

\[
\text{Res}_{z=0} f = \lim_{z \to 0} \frac{g(z)}{h(z)} = \lim_{z \to 0} \frac{g(z)}{h(z)/z} = \frac{g(0)}{h'(0)}
\]

3. Making Estimates

It often comes up that you want estimates on $|\int_{\gamma} f(z)dz|$ to show that certain contour integrals are small. Here are two that are good.

Theorem 12. [Estimation Lemma] Let $M = \sup_{z \in \gamma} |f(z)|$ and let $L = \int |\gamma'(t)|dt$ be the length of the curve $\gamma$. Then $|\int_{\gamma} f(z)dz| \leq ML$

Proof. From the definition, $|\int_{\gamma} f(z)dz| = |\int f(\gamma(t))\gamma'(t)dt| \leq \int |f(\gamma(t))||\gamma'(t)|dt \leq \int M |\gamma'(t)|dt = ML \quad \Box$

Theorem 13. [Jordan’s Lemma] Suppose $f(z)$ can be written $f(z) = e^{iaz} g(z)$. Let $C_R = \{ z : z = R e^{i\theta}, \theta \in [0, \pi] \}$ be a curve in the upper half plane. Then:

\[
\left| \int_{C_R} f(z)dz \right| \leq \frac{\pi}{a} \max_{\theta \in [0, \pi]} \left| g(R e^{i\theta}) \right|
\]

Proof. (From Wikipedia) Let $I_R = \left| \int_{C_R} f(z)dz \right|$ and $M_R = \max_{\theta \in [0, \pi]} \left| g(R e^{i\theta}) \right|$ for convenience. Have:

\[
\int_{C_R} f(z)dz = \int_{0}^{\pi} g(R e^{i\theta}) e^{iaR(cos\theta+i\sin\theta)} iRe^{i\theta} d\theta = R \int_{0}^{\pi} g(R e^{i\theta}) e^{aR(i\cos\theta-i\sin\theta)} i e^{i\theta} d\theta.
\]
So then:

\[ I_R = \left| \int_{C_R} f(z) \, dz \right| \leq R \int_0^\pi \left| g(R e^{i\theta}) \right| e^{a R (i \cos \theta - \sin \theta)} \, e^{i \theta} \, d\theta \]

\[ = R \int_0^\pi \left| g(R e^{i\theta}) \right| e^{-a R \sin \theta} \, d\theta \]

\[ \leq M_R R \int_0^\pi e^{-a R \sin \theta} \, d\theta \]

\[ = 2 M_R R \int_0^{\pi/2} e^{-a R \sin \theta} \, d\theta \]

Now use the inequality \( \sin \theta \geq \frac{2 \theta}{\pi} \) (This holds since \( \sin \) is concave down for \( \theta \in [0, \pi/2] \); draw a picture to see what the inequality is saying), to get:

\[ I_R \leq 2 M_R R \int_0^{\pi/2} e^{-a R R^\theta/\pi} \, d\theta \]

\[ = 2 M_R R \frac{\pi}{2 a R} (1 - e^{-a R}) \]

\[ \leq \frac{\pi}{a} M_R \]

\[ \square \]

Remark 14. Jordan’s lemma will work nicely for many applications. If you need a bit more power, it is possible to improve it. One way is to split the integral above into \( \int_0^{\theta_R} \) and then from \( \int_{\theta_R}^{\pi/2} \). If you choose \( \theta_R \) to be small (i.e. \( \theta_R \to 0 \) as \( R \to \infty \)) then the integral \( \int_0^{\theta_R} \) is controlled since \( \theta_R \) is small, while \( \int_{\theta_R}^{\pi/2} \) is controlled since \( y > \theta_R \) here. This works well for functions for which \( |f(x + iy)| \) is small for large \( y \).


Example 16. \( \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx =? \)

Proof. (From Wikipedia) To do this integral, it turns out it is useful to look at the function \( f(z) = \frac{\log(z)^2}{(1 + z^2)^2} \). (We use the branch of \( \log \) where \( \text{Arg}(z) \in (-\pi, \pi) \)) Draw a keyhole contour around this branch of logarithm. The outside and inside circular contours \( \to 0 \) by the estimation lemma. The keyhole bits hit the axis from above and below, and so converge to:

\[ \int_0^\infty \frac{\log x + i\pi}{(1 + x^2)^2} \, dx - \int_0^\infty \frac{\log x - i\pi}{(1 + x^2)^2} \, dx. \]

This is equal to:

\[ 4\pi i \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx \]

It remains only to calculate the residues at \( z = \pm i \). These are poles of order 2, so using method 2 above, we calculate:

\[ \lim_{z \to \pm i} \frac{d}{dz} \left[ (z \mp i)^2 \log(z)^2 \right] = \lim_{z \to \pm i} \frac{d}{dz} \left( \frac{\log(z)^2}{(z \mp i)^2} \right) \]

\[ = \lim_{z \to \pm i} \frac{2 \log(z)(z \pm i)^2 - 2(z \pm i) \log(z)}{(z \pm i)^4} \]

\[ = -\frac{\pi}{4} \pm \frac{1}{16} i \pi^2 \]

So we have by the residue theorem:

\[ 4\pi i \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx = 2\pi i \left( -\frac{\pi}{4} + \frac{1}{16} i - \frac{\pi}{4} - \frac{1}{16} i \right) \]

\[ \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx = -\frac{\pi}{4} \]

\[ \square \]