FUNCTIONAL ANALYSIS NOTES

THE BAIRE CATEGORY THEOREM AND RELATED

Definition 1. A set $A$ in a metric space is called nowhere dense if the closure of $A$ has empty interior. $(A)^\circ = \emptyset$.

Example 2. The rationals are NOT nowhere dense. A finite number of points is nowhere dense. Cantor sets are nowhere dense sets. Subsets of nowhere dense sets are nowhere dense.

Remark 3. Finite unions of nowhere dense sets are still nowhere dense. $\bigcup_{i=1}^n \overline{A_i}^\circ = (\bigcup_{i=1}^n A_i)^\circ = \bigcup_{i=1}^n A_i = \emptyset$.

Definition 4. A set $A$ which can be written as a countable union of nowhere dense sets is called 1st Category or Meagre. Sets which cannot be written in this way are called 2nd Category or Nonmeagre.

Example 5. The countable union of meagre sets is still meagre. Any subset of a meagre set is still meagre.

Lemma 6. Let $X$ be a complete metric space. If $\{U_n\}$ is a sequence of open dense sets, then $\cap_n U_n$ is also dense.

Proof. It is sufficient to show that for any open set $V$, the intersection $V \cap (\cap_n U_n) \neq \emptyset$. This is a simple recursive argument. Since $U_1$ is open and dense, $\exists B_{r_1}(x_1)$ so that $B_{r_1}(x_1) \subset V \cap U_1$. Since $U_2$ is open and dense, $\exists B_{r_2}(x_2)$ so that $B_{r_2}(x_2) \subset B_{r_1}(x_1) \cap U_2 \subset V \cap (U_1 \cap U_2)$. Continuing in this way, we get a nested sequence of balls. Since we are in a complete metric space, by the nested sets property for complete spaces, there is a point in the intersection of all the balls. (Another way to do this is to assume WLOG that $r_n \to 0$ and then the sequence $x_n$ will be forced to be Cauchy).

Theorem 7. A complete metric space is always second category or non-meager.

Proof. Suppose by contradiction that $X$ is 1st category and we can write $X = \cup_n E_n$ with $\overline{E_n}^\circ = \emptyset$ for each $n$. Let $A_n = \overline{E_n}^c$. This is a sequence of open sets. Moreover, $\overline{A_n} = (\overline{E_n}^c)^c = (\overline{E_n}^\circ)^c = \emptyset^c = X$. This means each $A_n$ is dense! But then by the lemma, the intersection $\cap_n A_n$ is dense too, so $\cap_n A_n \neq \emptyset$. But then, by taking complements, we have $\cup_n \overline{E_n} = \emptyset$ which is a contradiction. \qed

Important Consequences:

<table>
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<tr>
<td>Banach-Schauder Open Mapping Theorem</td>
<td>Let $X, Y$ be Banach spaces and let $T \in B(X,Y)$ be a bounded linear map. Suppose moreover that $T$ is onto. Then $T$ is an open map.</td>
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<tr>
<td>Corr</td>
<td>If $T$ is a continuous linear bijection from $X$ to $Y$ then $T^{-1}$ is continuous too.</td>
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<td>Corr</td>
<td>If $|\cdot|_1$ and $|\cdot|_2$ are two norms on a space $X$, and there is an $m$ so that $|\cdot|_1 \leq m |\cdot|_2$, then there exists $M$ so that $|\cdot|_1 \geq M |\cdot|_2$.</td>
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<tr>
<td>The Closed Graph Theorem</td>
<td>Let $X, Y$ be Banach spaces and let $T : X \to Y$ be linear. Let $\Gamma(T) = {(x, T(x)) : x \in X}$ be the graph of $T$. Then $T$ is continuous if and only if $\Gamma(T)$ is closed.</td>
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<tr>
<td>Banach-Steinhaus Uniform Boundedness Principle</td>
<td>Suppose $X, Y$ are Banach spaces and $(T_\alpha)<em>{\alpha \in A}$ is a collection of bounded linear maps. Let $E = {x \in X : \sup</em>{\alpha \in A} |T_\alpha x| &lt; \infty}$. If $E$ is 2nd category or nonmeager, then $\sup_{\alpha \in A} |T_\alpha x| &lt; \infty$. I.e. the $T_\alpha$’s are uniformly bounded. By the Baire category theorem, it is enough to show $E = X$.</td>
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<tr>
<td>(Slightly Stronger version)</td>
<td>(Same setup as above) Let $M = E^c = {x \in X : \sup_{\alpha \in A} |T_\alpha x| = \infty}$. Then either $M$ is empty or $M$ is a dense $G_\delta$ set.</td>
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Theorem 8. (Banach-Steinhaus Uniform Boundedness Principle) Suppose $X, Y$ are Banach spaces and $(T_\alpha)_{\alpha \in A}$ is a collection of bounded linear maps. Let $E = \{x \in X : \sup_{\alpha \in A} \|T_\alpha x\| < \infty\}$. If $E = X$ is all of $X$ then $\sup_{\alpha \in A} \|T_\alpha x\| < \infty$. I.e. the $T_\alpha$’s are uniformly bounded.

Proof. Let $E_n = \{x \in X : \sup_{\alpha \in A} \|T_\alpha x\| \leq n\}$ so that $E = \cup_n E_n$. Notice also that $E_n = \cap_{\alpha} \|T_\alpha (\cdot)\|^{-1} [0, n]$ is an intersection of closed sets (because $x \to \|T_\alpha x\|$ is continuous), so $E_n$ is closed. Since $E$ is not 1st category, we know
that $E$ cannot be written as a countable union of nowhere dense sets. Hence it must be the case that at least one $E_n$ is not nowhere dense. In other words, $\exists n_0$ so that $E_n^c \neq \emptyset$. Hence $\exists x_0, r$ so that $B_r(x_0) \subset E_{n_0}$.

For any $x$ with $\|x\| \leq r$ now, notice that $x_0 + x \in B_r(x_0) \subset E_{n_0}$. Hence for such $x$, we know by definition of $E_{n_0}$ that $\sup_\alpha \|T_\alpha(x_0 + x)\| \leq n_0$. Have then for any $\|x\| \leq r$:

$$\sup_\alpha \|T_\alpha x\| = \sup_\alpha \|T_\alpha(x_0 + x) - T_\alpha(x_0)\| \leq \sup_\alpha (\|T_\alpha(x_0 + x)\| + \|T_\alpha(x_0)\|) \leq n_0 + n_0 = 2n_0$$

So by scaling, we conclude that for any $x$ with $\|x\| \leq 1$ that $\sup_\alpha \|T_\alpha x\| \leq \frac{2n_0}{r}$. Have finally then that $\sup_\alpha \|T_\alpha\| = \sup_\alpha \sup_{\|x\|=1} \|T_\alpha x\| \leq \frac{2n_0}{r} < \infty$. $\square$

**Theorem 9.** *(The slightly stronger version) (Same setup as Banach-Steinhaus)* Let $M = E^c = \{x \in X : \sup_\alpha \|T_\alpha x\| < \infty\}$. Then either $M$ is empty or $M$ is a dense $G_\delta$ set.

Let $U_n = \{x \in X : \sup_\alpha \|T_\alpha x\| > n\}$ so that $M = \bigcap_n U_n$. Notice that we can write:

$$U_n = \bigcup_\alpha \{x \in X : \|T_\alpha x\| > n\} = \bigcup_\alpha \left(\|T_\alpha(\cdot)\|^{-1}(n, \infty)\right)$$

since the map $x \rightarrow \|T_\alpha x\|$ is continuous each set $\|T_\alpha(\cdot)\|^{-1}(n, \infty)$ is open and we see from this that $U_n$ is a union of open sets. Since $M = \bigcap_n U_n$, we see that $M$ is a $G_\delta$-set.

**Claim:** Either $M$ is empty or $U_n$ is dense set for every $n \in \mathbb{N}$.

**PF:** It suffices to show the following: if there is a single $n_0$ for which $U_{n_0}$ is not dense, then $M$ is empty. Suppose $U_{n_0}$ is not dense. Then, by definition of dense, $\overline{U_{n_0}} \neq X$. In other words this is $\overline{U_{n_0}} \neq \emptyset$. Now, $\overline{U_{n_0}}$ is a closed set, so we know that $\overline{U_{n_0}}$ is an open set. Hence, since this is a non-empty open set, we can find $x_0 \in X$ and $r > 0$ so that $B_r(x_0) \subset \overline{U_{n_0}}$.

Consider any $x \in X$ with $\|x\| \leq r$. Then $x_0 + x \in \overline{B_r(x_0)} \subset \overline{U_{n_0}}$. Hence $x_0 + x \notin U_{n_0}$. By definition of $U_n$, this means that $\sup_\alpha \|T_\alpha(x_0 + x)\| \leq n$. Using scaling and translation invariance, we have then that for any $x$ with $\|x\| \leq 1$ that:

$$\sup_\alpha \|T_\alpha x\| = \frac{1}{r} \sup_\alpha \|T_\alpha(rx)\|$$

$$= \frac{1}{r} \sup_\alpha \|T_\alpha(x_0 + rx) - T_\alpha(x_0)\|$$

$$\leq \frac{1}{r} \sup_\alpha (\|T_\alpha(x_0 + rx)\| + \|T_\alpha(x_0 + 0)\|)$$

$$\leq \frac{1}{r} (n_0 + n_0) \text{ since } \|rx\| \leq r \text{ and } \|0\| \leq r$$

$$= \frac{2n_0}{r}$$

Finally then we see that the $T_\alpha$ are uniformly bounded,

$$\sup_\alpha \|T_\alpha\| = \sup_\alpha \sup_{\|x\|=1} \|T_\alpha x\| \leq \frac{2n_0}{r} < \infty$$

This means that $M$ is the empty set, because for every $x \in X$ we have that $\|T_\alpha x\| \leq \sup_\alpha \|T_\alpha\| \|x\| < \infty$ so $x \notin M$. $\square$

Combining the initial remarks and the claim we see that $M$ is either empty or otherwise we have that $M = \bigcap_n U_n$ and every $U_n$ is dense. Since the countable intersection of open dense sets is dense (this was the main lemma in the pf of Baire’s theorem), in the latter case we see that $M$ is a dense $G_\delta$ set, as desired. ■
### The Hahn-Banach Theorem and Related

"H-B" = "Hahn-Banach" for the rest of this section.

In all the statements the set up is:

\[(X, \|\cdot\|) \equiv \text{A normed vector space over the field } \mathbb{F} \]
\[\mathbb{F} \equiv \text{The field that } X \text{ is over. Will either be } \mathbb{R} \text{ or } \mathbb{C} \]
\[M \equiv \text{A linear subspace of } X \]
\[p \equiv \text{A sub-linear functional } p : X \to \mathbb{R}, \text{i.e. } p \text{satisfies:} \]
\[p(x + y) \leq p(x) + p(y), \ p(ax) = ap(x) \forall a > 0. \]
\[q \equiv \text{A semi-norm } q : X \to \mathbb{R}, \text{i.e. } q \text{satisfies:} \]
\[q(x + y) \leq q(x) + q(y), \ q(\lambda x) = |\lambda| q(x) \forall \lambda \in \mathbb{C}. \]

(Rmk: semi-norm is stricter than sub-linear functional)

\[\ell_A \equiv \text{A linear functional } \ell_M : A \to \mathbb{F} \text{ where } A \text{ will be some subspace of } X \]
\[\ell_A \leq p \equiv \text{Shorthand for: } \ell_A(x) \leq p(x) \forall x \in A \]
\[\ell_A \leq q \equiv \text{Shorthand for: } \ell_A(x) \leq q(x) \forall x \in A \]
\[\ell_B \uparrow \ell_A \equiv \"\ell_B \text{ extends } \ell_A\", \text{ shorthand for } A \subset \text{Band } \ell_A(x) = \ell_B(x) \forall x \in A\]

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<tr>
<th>Name</th>
<th>(\mathbb{F})?</th>
<th>Hypothesis</th>
<th>Conclusion</th>
<th>“PT”</th>
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| Baby H-B Thm       | \(\mathbb{R}\)  | \(\ell_M \leq p; x_0 \in X - M. \)
Define \(M \oplus x_0\mathbb{R} = \{x + \lambda x_0 : x \in M, \lambda \in \mathbb{R}\}\) | \(\exists \ell_M \oplus x_0\mathbb{R} \uparrow \ell_M \text{ so that } \ell_M \oplus x_0\mathbb{R} \leq p\) | Can find the correct \(\ell_M \oplus x_0\mathbb{R}\) as long as we can define \(\ell_M \oplus x_0\mathbb{R}(x_0)\) to be in some particular interval. The fact \(\ell_M \leq p\) can be used to show that this interval is non-empty. |
| Real H-B Thm       | \(\mathbb{R}\)  | \(\ell_M \leq p\)                                                          | \(\exists \ell_X \uparrow \ell_M \text{ so that } \ell_X \leq p\)                                                        | Zorn’s Lemma is used with the partial ordering \(\uparrow\). Every totally ordered set has a maximal element by taking the union of all the subspaces. By Zorn’s Lemma, there is a maximal element. By the Baby H-B Thm, the maximal element cannot be a proper subspace of \(X\). |
| Complex H-B Thm    | \(\mathbb{C}\)  | \(|\ell_M| \leq q\)                                                        | \(\exists \ell_X \uparrow \ell_M \text{ so that } |\ell_X| \leq q\)                                                     | Use the Real H-B Thm to prove this one. Its a pretty unenlightening proof involving manipulations with complex numbers. |
| (Not named)        | \(\mathbb{F}\)  | \(x_0 \in X - \{0\}\)                                                      | \(\exists \ell \in X^* \text{s.t. } \|\ell\|_{X^*} = 1, \ell(x_0) = \|x\|\)                                     | Apply the H-B thm on the space \(M = \{\lambda x_0 : \lambda \in \mathbb{F}\}\) with the functional \(\ell_M(\lambda x_0) := \lambda \|x_0\|\) and seminorm \(q(x) = \|x\|\). The extension \(\ell_X \uparrow \ell_M\) is what we want. The ineq \(|\ell_X| \leq q\) gives that \(\|\ell\|_{X^*} \leq 1\) by linearity. The other inequality is clear by plugging in \(x_0\). |
| Analytic H-B Thm   | \(\mathbb{F}\)  | \(\ell_M \in M^*\)                                                         | \(\exists \ell_X \in X^*, \ell_X \uparrow \ell_M \text{ with } \|\ell_X\|_{X^*} = \|\ell_M\|_{M^*}\) | Let \(q(x) = \|\ell_M\|_{M^*} \|x\|\) be the seminorm. Then apply H-B thm. The ineq \(|\ell_X| \leq q\) gives that \(\|\ell_X\|_{X^*} \leq \|\ell_M\|_{M^*}\) by linearity. The other ineq is clear since \(M \subset X\). |
Theorem 10. (Baby H-B Thm) Suppose \( \ell_M \leq p \) and that \( x_0 \in X - M \). Define \( M \oplus x_0 \mathbb{R} = \{ x + \lambda x_0 : x \in M, \lambda \in \mathbb{R} \} \). Then \( \exists \ell_{M \oplus x_0} \geq \ell_M \) so that \( \ell_{M \oplus x_0} \leq p \).

**Proof.** Suppose we found a value \( \ell(x_0) \) that we liked a lot. Then we could define: \( \ell_{M \oplus x_0}(x + \lambda x_0) = \ell_M(x) + \lambda \ell(x_0) \) and we would have found the functional \( \ell_{M \oplus x_0} \geq \ell_M \) we want! Of course, since we want \( \ell_{M \oplus x} \leq p \), not just any value of \( \ell(x_0) \) will do. We need \( \ell(x_0) \) to obey the following inequalities:

**Claim 1:** For a fixed value \( \ell(x_0) \), define \( \ell_{M \oplus x_0}(x + \lambda x_0) = \ell_M(x) + \lambda \ell(x_0) \). Then:

\[
\ell_{M \oplus x_0} \leq p \iff \forall x \in M, \quad \ell_M(x) + \lambda \ell(x_0) \leq p(x + x_0) \quad \text{and} \quad \ell_M(x) - \lambda \ell(x_0) \leq p(x - x_0)
\]

**Pf:** \((\Rightarrow)\) Plug in \( x \pm x_0 \) into \( \ell_{M \oplus x_0} \leq p \), get \( \ell_{M \oplus x_0}(x \pm x_0) \leq p(x \pm x_0) \). By using the definition of \( \ell_{M \oplus x_0} \) on the LHS, we get the desired inequalities.

\((\Leftarrow)\) Let \( x + \lambda x_0 \in M \oplus x_0 \mathbb{R} \) be arbitrary. There are two cases, one where \( \lambda > 0 \) and one where \( \lambda \leq 0 \). We handle both cases simultaneously by using the \( \pm \) sign abusively and writing \( \lambda = \pm |\lambda| \).

Write:

\[
\ell_{M \oplus x_0}(x + \lambda x_0) = \ell_{M \oplus x_0}(x \pm |\lambda| x_0)
\]

\[
= |\lambda| \left( \frac{x}{|\lambda|} \pm x_0 \right)
\]

\[
= |\lambda| \left( \ell_M \left( \frac{x}{|\lambda|} \right) \pm \ell(x_0) \right) \quad \text{by def'n of } \ell_{M \oplus x_0}
\]

\[
\leq |\lambda| \left( p \left( \frac{x}{|\lambda|} \right) + p(x_0) \right) \quad \text{by the hypothesis inequalities}
\]

\[
= p(x \pm |\lambda| x_0) \quad \text{since } p \text{ is a sublinear functional}
\]

\[
= p(x + \lambda x_0)
\]

So indeed, \( \ell_{M \oplus x_0} \leq p \)

To show that a value of \( \ell(x_0) \) exists which satisfies the inequalities from Claim 1, we need the following to hold for all \( x \in M \):

\[
\ell_M(x) - p(x - x_0) \leq \ell(x_0) \leq p(x + x_0) - \ell_M(x)
\]

It suffices then to show that \( \forall x_1, x_2 \in M \) that \( \ell_M(x_1) - p(x_1 - x_0) \leq p(x_2 + x_0) - \ell_M(x_2) \). Indeed, this is a consequence of \( \ell_M \leq p \). Pluggin in \( x_1 + x_2 \) into \( \ell_M \leq p \), we have:

\[
\ell_M(x_1) + \ell_M(x_2) = \ell_M(x_1 + x_2)
\]

\[
\leq p(x_1 + x_2)
\]

\[
= p((x_1 - x_0) + (x_0 + x_2))
\]

\[
\leq p(x_1 - x_0) + p(x_0 + x_2)
\]

Rearranging now gives the desired inequality.

**Lemma 11.** (Zorn’s Lemma) A partial ordering on a set \( P \) is a relation “ \( \preceq \) ” that is reflexive (\( a \preceq a \)), antasymmetric (\( a \preceq b, b \preceq a \implies a = b \)), and transitive (\( a \preceq b, b \preceq c \implies a \preceq c \)). Suppose that every totally ordered subset (i.e. a set in which for each pair \( a, b \) either \( a \preceq b \) or \( b \preceq a \)), \( \{a_\alpha \}_\alpha \) has an upper bound in \( P \) (i.e. an element \( a_\ast \in P \) so that \( a_\alpha \preceq a_\ast \) for all \( \alpha \in \Lambda \)). Then \( P \) contains at least one maximal element (i.e. an \( a_\ast \) so that \( a \preceq a_\ast \) for all \( a \in P \)).

**Remark 12.** This is equivalent to the axiom of choice, but the proof is non-trivial!

**Theorem 13.** (Real H-B Theorem) Let \( X \) be a normed vector space over \( \mathbb{R} \), \( p \) a sublinear functional on \( X \), \( M \) a subspace, and \( \ell_M : M \to \mathbb{R} \) a linear functional such that \( \ell_M \leq p \) (i.e. \( \ell_M(x) \leq p(x) \forall x \in M \)). Then \( \exists \chi \) a linear functional that extends \( \ell_X \) (i.e. \( \ell_X(x) = \ell_X(x) \forall x \in M \)) and \( \ell_X \leq p \) (i.e. \( \ell_X(x) \leq p(x) \forall x \in X \))
Proof. Let $P = \left\{ \ell_A : A \to \mathbb{R} : \ell_A \preceq \ell_M \right\}$ be the space of all linear functions which are defined on subspaces $A$ of $X$. Then $\preceq$ is a partial ordering on $P$ (Rmk: one way to see this is to notice that $\preceq$ is inclusion of the graphs, that is $\ell_A \preceq \ell_B$ Graph $(\ell_A) \supset \text{Graph} (\ell_B)$ where Graph $(f) = \{(x, f(x)), x \in \text{Domain} (f)\}$. Moreover, every totally ordered subset has a maximum element in $P$. Namely, if $\{\ell_{A_\alpha}\}_{\alpha \in A}$ is a totally ordered set, then define $A_\ast = \bigcup_{\alpha \in A} A_\alpha$ and $\ell_{A_\ast, \alpha}(x) = \ell_{A_\alpha}(x)$ for $x \in A_\alpha$. (This is well defined because $\{\ell_{A_\alpha}\}_{\alpha \in A}$ is a totally ordered set).

Now by the conclusion of Zorn’s lemma, there is a maximal element $\ell_{A_\ast}$ for all of $P$. Now we claim that $A_\ast = X$. Indeed, if by contradiction, $A_\ast \neq X$, then there is at least one element $x_0 \in X - A_\ast$. But now by the Baby H-B Thm, we can get an extension $\ell_{A_\ast \oplus x_0 \mathbb{R}}$. But this contradicts the maximality of $\ell_{A_\ast}$ in $P$! So it must be that $A_\ast = X$. □

Theorem 14. (Complex H-B Theorem) Let $X$ be a normed vector space over $\mathbb{C}$, $q$ a seminorm on $X$, $M$ a subspace, and $\ell_M : M \to \mathbb{C}$ a linear functional such that $|\ell_M| \leq q$ (i.e. $|\ell_M(x)| \leq q(x) \forall x \in M$). Then $\exists \ell_X$ a linear functional that extends $\ell_M$, (i.e. $\ell_M(x) = \ell_X(x) \forall x \in M$) and $|\ell_X| \leq q$ (i.e. $|\ell_X(x)| \leq q(x) \forall x \in X$)

Proof. (By manipulations using the Real H-B Thm) Let $u_M(x) = \text{Re}(\ell_M(x))$ and $v(x) = \text{Im}(\ell_M(x))$ so that $\ell_M = u_M + iv_M$. $u_M$ and $v_M$ are seen to be $\mathbb{R}$–linear functionals, because $\ell_M$ is $\mathbb{R}$-linear. ($\ell_M$ is more than $\mathbb{R}$–linear actually!) Since $\ell_M$ is actually $\mathbb{C}$-linear, we have that:

$$v_M(x) = \text{Im}(\ell_M(x)) = \text{Re}(-i\ell_M(x)) = \text{Re}(\ell_M(ix)) = u_M(ix)$$

So then $\ell_M(x) = u_M(x) + iv_M(ix)$ can be entirely reconstructed from $u_M$. Now, $q$ being a semi-norm, is also a sublinear map (which is a slightly looser condition), and $u_M(x) \leq |\ell_M(x)| \leq q(x)$. So applying the Real H-B Thm we get a $u_X \preceq u_M$ and $u_X \preceq q(x)$. Now let $\ell_X(x) = u_X(x) + iv_X(ix)$. One now verifies that $\ell_X \preceq \ell_M$ (our calculation early basically did this). Finally to check that $|\ell_X| \leq q$ have:

$$|\ell_X(x)| = e^{i\theta} \ell_X(x) \text{ for some } \theta$$
$$= \ell_X(e^{i\theta} x)$$
$$= \text{Re}(\ell_X(e^{i\theta} x)) \text{ since the LHS is real}$$
$$= u_X(e^{i\theta} x)$$
$$\leq q(e^{i\theta} x) \text{ since } |\ell_X| \leq q$$
$$= |e^{i\theta}|q(x) = q(x) \text{ since } q \text{ is a seminorm.}$$

□