Linear algebra reviews exercises

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BE CAREFUL, THE SOLUTION IN THE FOLLOWING ARE QUICK ANSWERS WITHOUT JUSTIFICATION AND DO NOT CONTAIN ALWAYS ALL THE DETAILS AS THEY HAVE ALREADY BEEN MENTIONED IN CLASS AND IN HOMEWORK MANY TIMES... For justification and redaction for the exams refer to homework or/and class notes.

**Problem 1:**
Find a bases for $\text{Col}(A)$ and $\text{Nul}(A)$ and then deduce the rank of $A$ and dim of $\text{Nul}(A)$, where

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 & 6 \\ 5 & 1 & -9 & 2 & 10 \\ 4 & 6 & -9 & 12 & 15 \\ 3 & 4 & -5 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 8 & 4 & -6 \\ 0 & 2 & 3 & 4 & -1 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Solution :** Basis for $\text{Col}(A)$:

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ -9 \\ -5 \end{pmatrix} \right\}$$

So, $\text{rank}(A) = \text{dim}(\text{Col}(A)) = 3$. Basis for $\text{Nul}(A)$:

$$\left\{ \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

So $\text{dim}(\text{Nul}(A)) = 2$.

**Problem 2:**
Use the cofactor expansion to compute

$$\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$$
Solution:
\[
\begin{vmatrix}
5 & -2 & 4 \\
0 & 3 & -5 \\
2 & -4 & 7 \\
\end{vmatrix} = 1
\]

Problem 3:
Use row reduction to echelon form to compute the following determinant
\[
\begin{vmatrix}
1 & a & a_2 \\
1 & b & b^2 \\
1 & c & c^2 \\
\end{vmatrix}
\]
Solution:
\[
\begin{vmatrix}
1 & a & a_2 \\
1 & b & b^2 \\
1 & c & c^2 \\
\end{vmatrix} = \begin{vmatrix}
1 & a & a_2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & c^2-a^2 \\
\end{vmatrix} = (b-a)(c-a)(c-a)
\]

Problem 4:
Using the Cramer’s rule, determine the values of the parameter for which the system has a unique solution, and describe the solution.
\[
\begin{cases}
 sx_1 - 2sx_2 = 1 \\
 3x_1 + 6sx_2 = 4
\end{cases}
\]
Solution: The system is equivalent to \( Ax = b \), where
\[
A = \begin{pmatrix}
 s & -2s \\
 3 & 6 \\
\end{pmatrix}
\]
and
\[
b = \begin{pmatrix}
-1 \\
4
\end{pmatrix} \text{.}
\]
We compute
\[
A_1 = \begin{pmatrix}
-1 & -2s \\
4 & 6 \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
s & -1 \\
3 & 4 \\
\end{pmatrix}.
\]
\[
x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{7}{3(s+1)} \text{ and } x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{4s-3}{6s(s+1)} \text{ with } s \neq 0, -1.
\]

Problem 5:
Let \( R \) be the triangle with vertices at \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\). Show that
\[
\text{area of triangle} = \frac{1}{2} \det \begin{pmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1 \\
\end{pmatrix}
\]
Solution: Translate \( R \) to a new triangle of equal area by subtracting \((x_3, y_3)\) from each vertex. The new triangle has vertices \((0,0), (x_1 - x_3, y_1 - y_3), (x_2 - x_3, y_2 - y_3)\).
$x_3, y_1 - y_3$) and $(x_2 - x_3 y_2 - y_3)$. The area of the triangle will be

$$\frac{1}{2} \begin{vmatrix} x_2 - x_3 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$$

Now consider using row operations and a cofactor expansion to compute the determinant in the formula:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

Since $\det(A^T) = \det(A)$,

$$\begin{vmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix} = \begin{vmatrix} x_2 - x_3 & x_2 - x_3 \\ y_1 - y_2 & y_2 - y_3 \end{vmatrix}$$

So the above observation allows us to state that the area of the triangle will be

$$\{\text{area of triangle}\} = \frac{1}{2} \det \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Problem 6:
Let $H$ and $K$ be subspaces of a vector space $V$. The intersection of $H$ and $K$, written $H \cap K$ is the set of $v$ in $V$ that belong to both $H$ and $K$. Show that $H \cap K$ is a subspace of $V$. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general a subspace.

Solution: Both $H$ and $K$ contain the zero vector of $V$ because they are subspaces of $V$. Thus the zero vector of $V$ is in $H \cap K$. Let $u$ and $v$ be in $H \cap K$. Then $u$ and $v$ are in $H$. Since $H$ is a subspace $u + v$ is in $H$. Likewise $u$ and $v$ are in $K$. Thus $u + v$ is in $H \cap K$. Let $u$ be in $H \cap K$. Then $u$ is in $H$. Since $K$ is a subspace $cu$ is in $H$. Likewise $u$ is in $K$. Since $K$ is a subspace $cu$ is in $K$. Thus $cu$ is in $H \cap K$ for any scalar $c$, and $H \cap K$ is a subspace of $V$.

The union of two subspaces is not in general a subspace. For, an example in $\mathbb{R}^2$ let $H$ be the $x$-axis and let $K$ be the $y$-axis. Then both $H$ and $K$ are
subspaces of $\mathbb{R}^2$, but $H \cup K$ is thus not a space in $\mathbb{R}^2$ since $(0,1)$ is in $K$ but not in $H$ and $(1,0)$ is in $H$ but not in $K$ so they are both in $H \cup K$ but $(1,1) = (1,0) + (0,1)$ is neither is $H$ nor in $K$ thus not in $H \cup K$ but this is supposed to be true if $H \cup K$ was a subspace.

**Problem 7:** Let $M_{2 \times 2}$ be the vector space of all $2 \times 2$ matrices and define $T : M_{2 \times 2} \rightarrow M_{2 \times 2}$ by $T(A) = A + A^T$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. Show that $T$ is a linear transformation.
2. Let $B$ be any element of $M_{2 \times 2}$ such that $B^T = B$. Find an $A$ in $M_{2 \times 2}$ such that $T(A) = B$.
3. Show that the range of $T$ is the set of $B$ in $M_{2 \times 2}$ with the property that $B^T = B$.
4. Describe the kernel of $T$

**Solution:**

1. For any $A$ and $B$ in $M_{2 \times 2}$, and for any scalar $c$,

$$T(A + B) = (A + B) + (A + B)^T = (A + A^T) + (B + B^T) = T(A) + T(B)$$

and

$$T(cA) = (cA) + (cA)^T = c(A + A^T) = cT(A)$$

So $T$ is a linear transformation.

2. Let $B$ be an element of $M_{2 \times 2}$ with $B^T = B$, and let $A = 1/2B$. Then

$$T(A) = A + A^T = 1/2B + 1/2B^T = B$$

3. Note that $B$ in $\text{Range}(T)$, then there is $A \in M_{2 \times 2}$, $T(A) = B$ since then $B = A + A^T$ then

$$B^T = (A + A^T)^T = A^T + A = B$$

Thus the $\text{Range}(A)$ is included on the symmetric matrix $M_{2 \times 2}$. The reverse inclusion is obtained by the previous question. Thus, the $\text{Range}(T)$ is the $2 \times 2$ symmetric matrices.

4. $\text{Ker}(T) = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, b \in \mathbb{R} \right\}$
Problem 7:
Use coordinate vectors to test the linear independence of the sets of polynomials, \( \{1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3\} \). Explain your work.

**Solution:** The coordinate mapping produces the coordinate vectors \((1, 0, 0, 2), (2, 1, -3, 0), \) and \((0, -1, 2, -1)\) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

\[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & -1 \\
0 & -3 & 2 \\
2 & 0 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

Since the matrix has a pivot in each column, its columns and thus the given polynomials, since the coordinate map is an isomorphism, are linearly independent.

Problem 8:
Explain why the space \( \mathbb{P} \) of all the polynomials is an infinite dimensional space.

**Solution:** Suppose that \( \text{dim}(\mathbb{P}) = k < \infty \). Now, \( \mathbb{P}_n \) is a subspace of \( \mathbb{P} \) for all \( n \), and \( \text{dim}(\mathbb{P}_k) = k, \) so \( \text{dim}(\mathbb{P}_{k-1}) = \text{dim}(\mathbb{P}) \). This would imply that \( \mathbb{P}_{k-1} = \mathbb{P} \), which is clearly untrue: for example, \( p(t) = t^k \) is in \( \mathbb{P} \) but not in \( \mathbb{P}_{k-1} \). Thus the dimension of \( \mathbb{P} \) cannot be finite.

Problem 9:
Verify that \( \text{rank}(uv^T) \leq 1 \) if \( u = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \) and \( v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \).

**Solution :** Compute that

\[
uv^T = \begin{pmatrix}
2a & 2b & 2c \\
-3a & -3b & -3c \\
5a & 5b & 5c
\end{pmatrix}
\]

. Each column of \( uv^T \) is a multiple of \( u \), so \( \text{dim}(|\text{Col}(uv^T)|) = 1 \), unless \( a = b = c = 0 \), in which case \( uv^T \) is the \( 3 \times 3 \) zero matrix and \( \text{dim}(|\text{Col}(uv^T)|) = 0 \). In any case, \( \text{rank}(uv^T) = \text{dim}(|\text{Col}(uv^T)|) \leq 1 \).
Problem 10:
Let \( D = \{d_1, d_2, d_3\} \) and \( F = \{f_1, f_2, f_3\} \) be bases for a vector space \( V \), and suppose that \( f_1 = 2d_1 - d_2 + d_3 \), \( f_2 = 3d_2 + d_3 \) and \( f_3 = -3d_1 + 2d_3 \).

1. Find the change-of-coordinates matrix from \( F \) to \( D \).
2. Find \([x]_D\) for \( x = f_1 - 2f_2 + 2f_3 \).

Solution:
1. \[
P_{D\rightarrow F} = [[f_1]_D, [f_2]_D, [f_3]_D] = \begin{pmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}
\]
2. \[
[x]_D = P_{D\rightarrow F}[x]_F = \begin{pmatrix} -4 \\ -7 \\ 3 \end{pmatrix}
\]

Problem 11:
Find the general solution of this difference equation.
\[
y_k = k - 2; y_{k+2} - 4y_k = 8 - 3k
\]

Solution: The general solution of the difference equation \( y_{k+2} - 4y_k = 8 - 3k \) is \( y_k = k - 2 + c_1 2^k + c_2 (-2)^k \).

Problem 12:
Show that every \( 2 \times 2 \) matrix has at least one steady-state vector. Any such matrix can be written in the form \( P = \begin{pmatrix} 1 - \sigma & \beta \\ \alpha & 1 - \beta \end{pmatrix} \), where \( \alpha \) and \( \beta \) are constant 0 and 1. (There are two linearly independent steady-state vectors if \( \alpha = \beta = 0 \). Otherwise there is only one.)

Solution: A steady state vector for \( P \) is a vector \( x \) such that \( Px = x \), that is such that \((P - I)x = 0\).

If \( \alpha = \beta = 0 \), then \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are two linearly independent steady vector.
If $\alpha$ or $\beta$ is not 0, then

\[ P - I = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \]

Row reducing the augmented matrix give one possible solution

\[ x = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \]

**Problem 13:**
Consider a matrix $A$ with the property that the rows sum all equal the same number $s$. Show that $s$ is an eigenvalue of $A$.

**Solution:** Let $v$ be the vector in $\mathbb{R}^n$ whose entries are all ones. Then $Av = sv$.

**Problem 14:** Orthogonally diagonalize if possible:

\[
\begin{pmatrix}
3 & 1 & 1 \\
1 & 3 & 1 \\
1 & 1 & 3
\end{pmatrix}
\]

**Solution:** Let

\[
P = \begin{pmatrix}
1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\
1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\
1/\sqrt{3} & 0 & 2/\sqrt{6}
\end{pmatrix}
\]

and

\[
D = \begin{pmatrix}
5 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

Then $P$ orthogonally diagonalizable $A$, and $A = PD P^{-1}$.

**Problem 15:** Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_4$ be the transformation that maps a polynomial $p(t)$ into a polynomial $p(t) + 2t^2 p(t)$.

1. Find the image of $p(t) = 3 - 2t^2$.
2. Show that $T$ is a linear transformation.
3. Find the matrix for $T$ relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$. 
Solution:

1. \( T(p) = 3 - 2t + 7t^2 - 4t^3 + 2t^4. \)

2. Let \( p \) and \( q \) be polynomials in \( \mathbb{P}_2 \) and \( c \) be any scalar. Then
   \[
   T(p + q) = (p + q) + 2t^2(p + q) = (p + 2t^2p) + (q + 2t^2q) = T(p) + T(q)
   \]
   and
   \[
   T(cp) = (cp) + 2t^2(cp) = c(p + 2t^2p) = cT(p)
   \]
   Thus \( T \) is a linear transformation.

3. ... The matrix for \( T \) relative to \( B \) and \( C \) is
   \[
   [\begin{bmatrix} [T(b_1)]_C & [T(b_2)]_C & [T(b_3)]_C \end{bmatrix} = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   2 & 0 & 1 \\
   0 & 2 & 0 \\
   0 & 0 & 2
   \end{pmatrix}
   \]

Problem 16: Classify the origin as an attractor, repeller or saddle point of the dynamical system \( x_{k+1} = Ax_k \). Find the direction of the greatest attraction and/or repulsion when

\[
A = \begin{pmatrix} 1.7 & 0.6 \\ -0.4 & 0.7 \end{pmatrix}
\]

Solution: The eigenvalue for \( A \) are \( \lambda = 1.1 \) and \( 1.3 \). The origin is a repeller because both eigenvalues are greater than 1 in magnitude. The direction of the greatest repulsion is through the origin and the eigenvector \( v_1 \) such that \( Av_1 = 1.3v_1 \), you can find \( v_1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \).

Problem 17: Let \( W = \text{Span}\{v_1, \ldots, v_p\} \). Show that if \( x \) is orthogonal to each \( v_j \), for \( 1 \leq j \leq p \), then \( x \) is orthogonal to every vector in \( W \).

Solution: A typical vector in \( W \) has the form \( w = c_1v_1 + \cdots + c_pv_p \). If \( x \) is orthogonal to each \( v_j \), then
\[
w \cdot x = (c_1w_1 + \cdots + c_pv_p) \cdot x = c_1(v_1 \cdot w_1) + \cdots + c_p(v_p \cdot w_p) = 0
\]
So \( x \) is orthogonal to each \( w \) in \( W \).
Problem 18:
Show that if the vector of an orthogonal set are normalized the new set
will still be orthogonal.

Solution: Just note that $v_1 \cdot v_2 = 0$ then
\[(c_1v_1) \cdot (c_2v_2) = c_1c_2v_1 \cdot v_2 = c_1c_20 = 0\]

Problem 19:
Find the orthogonal projection of $y$ onto $Span\{u_1, u_2\}$ where
\[y = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\]

Solution:
\[\hat{y} = \begin{pmatrix} -6 \\ 4 \\ 1 \end{pmatrix}\]

Problem 20:
Find the QR factorization for $A$ where
\[A = \begin{pmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}\]

Solution:
\[Q = \begin{pmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{pmatrix}\]
and
\[R = Q^T A = \begin{pmatrix} 6 & 12 \\ 0 & 6 \end{pmatrix}\]

Problem 21:
Find the least squares solution of $Ax = b$ where
\[A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}\]
Solution:
\[ \hat{x} = \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} \]

\( \hat{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \)

**Problem 22:**
Find the least squares line \( y = \beta_0 + \beta_1 x \) of the least-squares line that best fits data points:
\[ (2, 3), (3, 2), (5, 1), (6, 0) \]

**Solution:**
The least square line is \( y = 1.1 + 1.3x \).

**Problem 23:**
Let \( \mathbb{P}_3 \) have the inner product given by evaluation at \(-3, -1, 1\) and 3. Let \( p_0(t) = 1, p_1(t) = t \) and \( p_2(t) = t^2 \).

1. Compute the orthogonal projection of \( p_2 \) onto the subspace spanned by \( p_0 \) and \( p_1 \).
2. Find a polynomial \( q \) that is orthogonal to \( p_0 \) and \( p_1 \), such that \( \{p_0, p_1, q\} \) is an orthogonal basis for \( \text{Span}\{p_0, p_1, p_2\} \). Scale the polynomial \( q \) so that its vector of values at \((-3, -1, 1, 3)\) is \((1, -1, -1, 1)\).

**Solution:**
1. \( \hat{p}_2 = 5 \)
2. \( q = p_2 - \hat{p}_2 = t^2 - 5 \) will be orthogonal to both \( p_0 \) and \( p_1 \) and \( \{p_0, p_1, q\} \) will be an orthogonal basis for \( \text{Span}\{p_0, p_1, p_2\} \). The vector values for \( q \) at \((-3, -1, 1, 3)\) is \((4, -4, -4, 4)\), so scaling by \(1/4\) yields the new vectors \( q = (1/4)(t^2 - 5) \).

**Problem 24:**
Make a change of variable \( x = Py \) so that transforms the quadratic form into one with no cross-product term. Write the new quadratic form and determine if it is positive definite, negative definite or indefinite when the quadratic form is
\[ Q(x) = 8x_1^2 + 6x_1x_2 \]
Solution: The matrix of the quadratic form is

\[ A = \begin{pmatrix} 8 & 0 \\ 3 & 0 \end{pmatrix} \]

Orthogonally diagonalize \( A \) and get

\[ P = \begin{pmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} \]

and

\[ D = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix} \]

The desired based change is \( x = Py \) and the new quadratic is

\[ x^T Ax = y^T Dy = 9y_1^2 - y_2^2 \]

Problem 25:
Find a SVD for

\[ A = \begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} \]

Solution:

\[ A = U\sigma V^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \]